TASK 7.1.2: THE CONE AND THE INTERSECTING PLANE

Solutions

1. What is the equation of a cone in the 3-dimensional coordinate system?

   $$x^2 + y^2 = z^2$$

2. Describe the different ways that a plane could intersect the cone. Draw a sketch or use your model to show the intersections.

   parallel to the base
   not parallel to the base
   parallel to the axis of the cone
   not parallel to the axis of the cone
   parallel to the generating line of the cone
   not parallel to the generating line of the cone

3. Complete the following table to relate the intersection of a plane and a cone to the algebraic representations of conic sections.

The plane is	Equation of the plane	Resulting Conic
parallel to the x-y plane	$z = k$	*Circle with radius k* $x^2 + y^2 = k^2$
perpendicular to the x-y plane	$y = k$	*Hyperbola* $z^2 - x^2 = k^2$
parallel to the generating line	$z = y + k$	*Parabola* $y = x^2/(2k) - k/2$ $y = ax^2 + c$ *where a = 1/(2k) and c = -k/2*
not parallel to the generating line	$z = my + b$ $\lvert m \rvert < 1$	*Ellipse* $x^2 + (1 - m^2)y^2 - 2bmy = b^2$ $(1 - m^2) > 0$
not parallel to the generating line	$z = my + b$ $\lvert m \rvert > 1$	*Hyperbola* $x^2 + (1 - m^2)y^2 - 2bmy = b^2$ $(1 - m^2) < 0$
not parallel to the generating line	$z = my$	*Degenerate Ellipse or Hyperbola* *Lines intersecting at (0, 0, 0)* $x^2 + (1 - m^2)y^2 = 0$

Math notes
Participants move toward developing equations for the conics by considering algebraic relations in three variables. This provides an opportunity for mathematical discussions that parallel those that participants have with their students in determining relations among two variables.

Teaching notes
Be sure to have copies of transparencies 1, 2, 3, and 4 ready for the session.
Distribute the task and allow some thinking time for questions 1 and 2 before starting the discussion.

Typically, participants will not have thought about 3-dimensional coordinate systems since they themselves were in college. It is important to review the common conventions when setting up a 3-dimensional coordinate system (e.g. the positive z axis points upward on the page, positive *x*-axis points "out of the page" and the positive y-axis points to the right of the page, all axes meeting at right angles, etc.).

Show transparency #1 and discuss how to generate a cone as the rotation of a line through the origin. A triangle cut from a paper honeycomb holiday decoration can be opened to show the cone that is generated. This model however will not be double napped.

For example, when the line $z = y$ is rotated about the *z*-axis the cone pictured on transparency #1 results. This cone makes a 45° angle with the *z-y* and *z-x* planes. This line generates "two" cones; one cone is above the *x-y* plane and one is below the *x-y* plane.

Discuss the equation of the cone.
Ask participants to think about a cross section of the cone parallel to the *x-y* plane. That is, for a given value of *z*, have them determine that the *x* and *y* values lie on a circle with radius *z*. Again, the honeycomb model in its closed position can be used to show this relationship. Since $z = y$, the equation of the cone is $x^2 + y^2 = z^2$.

$$z \quad \overline{\begin{array}{c} \underline{y} \quad (x,y) \\ \diagup \quad y = z \end{array}}$$
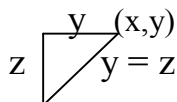
Participants may want to use the models that they created in Task #1 to answer question #2. Transparency #2 gives some examples of the answers to question 2 as well as examples of the first 4 situations described in part 3 of the task.

Display Transparency #3 to start the remainder of the task. Help participants write the equation of the plane. Then give them time to find the intersection of the cone and plane algebraically. To shorten the time needed to complete the task the facilitator may want to assign the systems to be solved to different groups. If so, groups should record their answers on Transparency #3 and be prepared to display their work for the other participants. They may want to record their solution on chart paper for display. Discuss transparency #4 to remind participants of the

standard forms of the equations they are seeking in the task. Participants may need to reference transparencies #2 and #4 throughout their work.

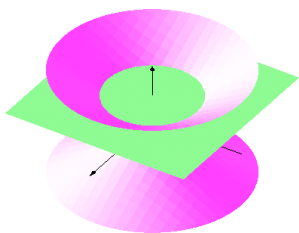*Algebra II: Strand 7. Conic Sections; Topic 1. Intersection of a Plane and a Cone; Task 7.1.2*

The Cone and the Intersecting Plane – Transparency 1
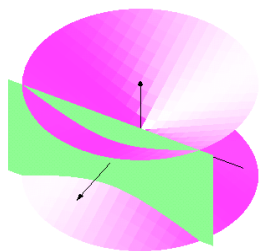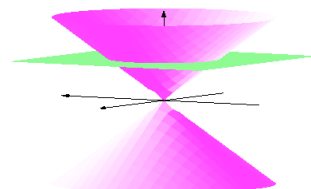
What is the equation of a cone?
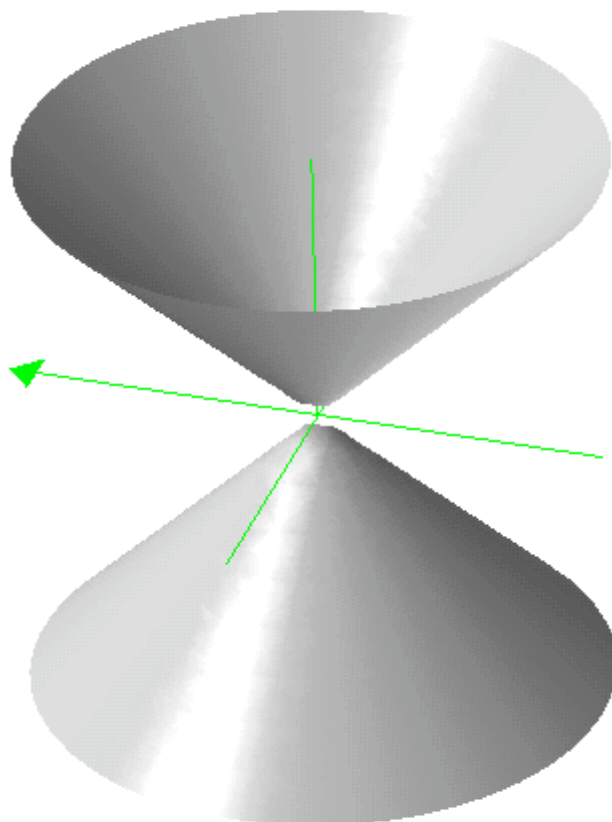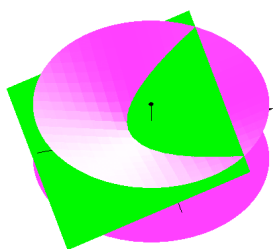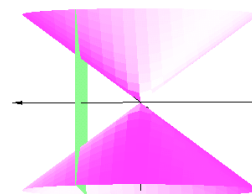
The Cone and the Intersecting Plane – Transparency 2
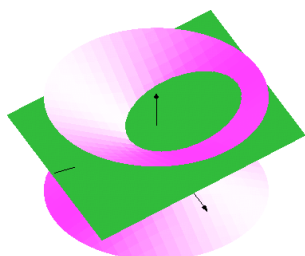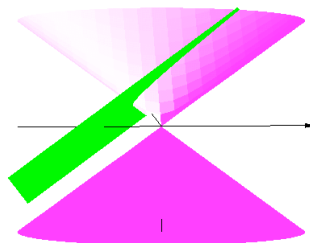
Examples of planes intersecting cones
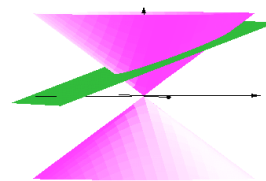
Circle: plane parallel to base of cone

Hyperbola: plane perpendicular to base of cone

Parabola: plane parallel to generating line

Ellipse: plane with slope less than slope of generating line

The Cone and the Intersecting Plane – Transparency 3

The equation of the cone is $x^2 + y^2 = z^2$.

What is the conic that is the intersection of the cone and the plane?

The plane is	Equation of the plane	Resulting Conic
parallel to the *x-y* plane		
perpendicular to the *x-y* plane		
parallel to the generating line		
not parallel to the generating line	$z = my + b$ $|m| < 1$	
not parallel to the generating line	$z = my + b$ $|m| > 1$	
not parallel to the generating line	$z = my$	

The Cone and the Intersecting Plane – Transparency 4
Algebraic Definitions
For conics with axis of symmetry parallel to the *x* or *y* axis.

Ellipse:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

Hyperbola:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Parabola:

$$y = a(x-h)^2 + k$$
or
$$x = a(y-k)^2 + h$$

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3.   Complete the following table to relate the intersection of a plane and a cone to the algebraic representations of conic sections.

The plane is	Equation of the plane	Resulting Conic
parallel to the *x-y* plane		
perpendicular to the *x-y* plane		
parallel to the generating line		
not parallel to the generating line	$z = my + b$ $\|m\| < 1$	
not parallel to the generating line	$z = my + b$ $\|m\| > 1$	
not parallel to the generating line	$z = my$	