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Wigner rotations, Bargmann invariants and geometric phases

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Abstract

The concept of the 'Wigner rotation', familiar from the composition law of (pure) Lorentz transformations, is described in the general setting of Lie group coset spaces and the properties of coset representatives. Examples of Abelian and non-Abelian Wigner rotations are given. The Lorentz group Wigner rotation, occurring in the coset space $SL(2, R)/SO(2) \simeq SO(2, 1)/SO(2)$, is shown to be an analytic continuation of a Wigner rotation present in the behaviour of particles with nonzero helicity under spatial rotations, belonging to the coset space $SU(2)/U(1) \simeq SO(3)/SO(2)$. The possibility of interpreting these two Wigner rotations as geometric phases is shown in detail. Essential background material on geometric phases, Bargmann invariants and null phase curves, all of which are needed for this purpose, is provided.

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1. Introduction

It is a well-known kinematic fact in special relativity that two successive pure Lorentz transformations (Lorentz boosts) in noncollinear directions do not amount to a resultant pure Lorentz transformation, but to such a transformation either preceded or followed by a spatial rotation. This rotation is called a 'Wigner rotation', and it is closely related to the Thomas precession, another effect of relativistic kinematics [1].

The occurrence of the 'Wigner rotation' in the above context can be traced to the structure of the Lorentz group $SO(3, 1)$, and it is in fact an instance of very general properties of Lie group coset spaces and coset representatives. It has naturally been studied and commented upon in many contexts, wherever the mathematical properties of the Lorentz group play a role even outside special relativity [2]. Thus, the three-dimensional Lorentz group $SO(2, 1)$ shows up in the description of linear optical systems in Gaussian or first-order optics, as well as in the canonical transformation theory of the single mode quantized radiation field and the discussion of squeezing [3]. More recently, several papers have explored the connection between Wigner rotations and geometric phases [4].

The purpose of this paper, written from a pedagogical point of view, is to give a short account of all these themes bringing out their essential structural features and relationships. More specifically, our aim is to describe the general properties of Lie group coset spaces, coset representatives and the very general setting in which the concept of a 'Wigner rotation' arises. Lie group coset spaces have been used extensively in diverse physical applications, such as the theory of nonlinear realizations of symmetry groups in particle physics [5], systems of generalized coherent states [6] and the theory of classical relativistic particles with nontrivial internal structure [7]. We give several instructive examples of situations in which Wigner rotations naturally arise, including both Abelian cases where they amount to only phases, and non-Abelian cases. One of our aims is to show that the familiar Wigner rotation one comes across in the composition of pure Lorentz transformations is of the same nature as, and an analytic continuation of, a phase present in the description of particles with nonzero helicity, purely within the framework of the rotation group $SO(3)$.

We have tried to make this paper reasonably self-contained without making it excessively long. Some simple proofs of statements made in the text are omitted, as the interested reader can easily supply them with modest effort. On the other hand, some acquaintance with basic group theoretical notions would be helpful. The homomorphic relationships among $SU(2)$ and $SO(3)$, $SL(2, R)$ and $SO(2, 1)$, $SL(2, C)$ and $SO(3, 1)$ will be important in our account; these are recalled briefly, covering the essential points in each case. The kinematical approach to geometric phases [8], and the associated concepts of Bargmann invariants [9] and null phase curves [10], are very briefly described. One of our points will be to convince the reader that a definite bridge needs to be made between Wigner rotations and the associated group theory on the one hand, and the interpretation in terms of geometric phases on the other, and that this is by no means obvious.

The paper is organized as follows. Section 2 describes the basic concepts and properties of Lie group coset spaces, little groups and coset representatives. The natural manner in which the Wigner rotation emerges in any such set-up is brought out. Examples of Wigner rotations of both Abelian and non-Abelian types are given. The former examples correspond to the coset spaces $SU(2)/U(1) \simeq SO(3)/SO(2)$ and $SL(2, R)/SO(2) \simeq SO(2, 1)/SO(2)$, and here the Wigner rotations reduce to Wigner angles, while the latter non-Abelian examples correspond to the cases $SL(2, C)/SU(2) \simeq SO(3, 1)/SO(3)$ and $U(n)/U(n-1)$. Section 3 gives the details of the computations of the Wigner angles in the two Abelian examples, and shows that they are analytic continuations of one another. The relevance of the $SU(2)/U(1)$ Wigner angle in the rotational behaviour of nonzero helicity states is also sketched. The aim of section 4 is to review the basic definitions and formulae for geometric phases in the kinematic approach, and to relate these phases to the so-called Bargmann invariants of quantum mechanics and to the recently introduced concept of null phase curves. The details presented are chosen to provide the basis for the applications in section 5. Here the two Wigner angles in the Abelian cases are shown to be interpretable as geometric phases in particular situations. For the $SU(2)/U(1)$ case we work within a general spin j unitary irreducible representation of $SU(2)$. For the $SL(2, R)/SO(2)$ case we have to use the positive discrete class unitary irreducible representations of $SL(2, R)$. Section 6 contains some concluding comments.

2. Coset spaces, coset representatives and Wigner rotations

Let G be a Lie group of dimension n, and $H \subset G$ a Lie subgroup of dimension k. For simplicity we assume both to be connected. The coset space $M = G/H$ is then a space of dimension $(n - k)$, made up of right cosets in G with respect to H. We denote points of M by $q, q', \ldots$. A general $q \in M$ denotes a subset gH of G, for some $g \in G$, forming a right coset:

$$q = gH = \{gh \in G | g \in G \text{ fixed}, h \in H\}. \tag{2.1}$$

There is a distinguished point $q_0 \in M$, an origin, corresponding to the coset containing the identity element $e \in G$, namely H itself:

$$q_0 = eH = H. \tag{2.2}$$

The group G acts transitively on the space M. Given any $q = gh \in M$ and $g' \in G$, we write this action as

$$q' = g'q = g'gH. \tag{2.3}$$

At each $q \in M$, there is a little group or stability group $H_q \subset G$:

$$H_q = \{g' \in G | g'q = q\} \subset G. \tag{2.4}$$

For different points of M the corresponding little groups are clearly related to one another by conjugation:

$$q' = g'q \quad \Longrightarrow \quad H_{q'} = g' H_q g'^{-1}. \tag{2.5}$$

In particular, at q_0 we have

$$H_{q_0} = H \tag{2.6}$$

so all the other little groups are related to this fixed H by conjugation.

For each $q \in M$, we may choose a convenient coset representative, an element $\ell(q) \in G$ belonging to the coset represented by q, so that

$$q \in M \longrightarrow \ell(q) \in G : q = \ell(q) q_0. \tag{2.7}$$

It is of course convenient to choose $\ell(q)$ so that it varies continuously with q.[5] However, it is often the case that at some isolated points in M the definition of $\ell(q)$ runs into difficulties of discontinuity or nonuniqueness. This must be kept in mind, and some examples will be mentioned later.

With this background and notation, we are able to define the general concept of the 'Wigner rotation'. Given a point $q \in M$ and an element $g \in G$ such that $q' = gq$, we can see easily that the particular product of three elements

$$\ell(q')^{-1} g \ell(q) = h(q, g) \tag{2.8}$$

is always in the subgroup $H \subset G$. We call this $h(q, g)$ the 'Wigner rotation' corresponding to the coset space $M = G/H$ and the choice of coset representatives $\ell(q)$; clearly, as indicated, it depends on both $q \in M$ and $g \in G$. The use of the word 'rotation' here refers to the fact that in its original context H was the rotation subgroup $SO(3)$ of the Lorentz group $SO(3, 1)$. We will consider this as an example later.

We now present a few typical examples of coset spaces of physical interest, group actions on them and choices of coset representatives. The actual calculation of the corresponding Wigner rotation will be taken up later in selected cases. If H is the one-dimensional group

[5] The choice of coset representatives is of course not unique. We have the freedom to change $\ell(q)$ to $\ell'(q) = \ell(q)h(q)$, any $h(q) \in H$. This leads to the change $h(q, g) \to h'(q, g) = h(q')^{-1} h(q, g) h(q)$ in equation (2.8).

$U(1)$ or $SO(2)$, the Wigner rotation reduces to a Wigner phase or Wigner angle. However if H is larger and non-Abelian, the Wigner rotation will amount to more than just a phase.

In what follows we will use the well-known relationships (two-to-one homomorphisms) $SU(2) \to SO(3)$, $SL(2, R) \simeq SU(1,1) \to SO(2,1)$ and $SL(2, C) \to SO(3,1)$. This is especially convenient since many computations in these cases can be reduced to manipulations with 2×2 matrices which are easily carried out. Necessary details of these relationships will be recalled when needed.

2.1. Examples with Abelian H

2.1.1. $G/H = SO(3)/SO(2) \simeq SU(2)/U(1)$. For each 2×2 unitary unimodular matrix $u \in SU(2)$, there is a corresponding unique 3×3 real proper rotation matrix $R(u) \in SO(3)$ given by

$$u \in SU(2) \to R_{jk}(u) = \tfrac{1}{2}\,\mathrm{Tr}(\sigma_j u \sigma_k u^{-1}) \qquad j, k = 1, 2, 3 \tag{2.9}$$

and obeying

$$R(u')R(u) = R(u'u). \tag{2.10}$$

Here the σ_j are the triplet of Pauli matrices. In particular, if u is an element in the diagonal $U(1)$ subgroup of $SU(2)$, $R(u)$ is a rotation about the third axis:

$$\begin{aligned} u_3(\psi) &= \exp\left(-\frac{\mathrm{i}}{2}\psi\sigma_3\right) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\psi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\psi/2} \end{pmatrix} \in U(1) \subset SU(2) : \\ R(u_3(\psi)) &= \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(2) \subset SO(3) \qquad 0 \leqslant \psi \leqslant 4\pi. \end{aligned} \tag{2.11}$$

It may be noted that $R(-u) = R(u)$, $\forall u \in SU(2)$, and we have the two-to-one homomorphism $SU(2)/Z_2 = SO(3)$.

There are several familiar parametrizations for $SU(2)$ (and associated ones for $SO(3)$), namely, homogeneous Euler parameters, axis-angle parameters and Euler angle parameters. Here we use a variant of the third one as it is more convenient for our purposes. Given any $u \in SU(2)$, it can always be written as the product of an element belonging to a two-parameter family and dependent on two angles θ, ϕ, and an element of $U(1)$:

$$\begin{aligned} u \in SU(2) : u &= u(\theta, \phi, \psi) \\ &= u_\perp(\theta, \phi) u_3(\psi) \\ u_\perp(\theta, \phi) &= \exp\left\{-\frac{\mathrm{i}}{2}\theta(\sigma_2\cos\phi - \sigma_1\sin\phi)\right\} \\ &= \cos\theta/2 - \mathrm{i}\sin\theta/2(\sigma_2\cos\phi - \sigma_1\sin\phi) \qquad 0 \leqslant \theta \leqslant \pi \\ &\quad 0 \leqslant \phi < 2\pi \quad 0 \leqslant \psi < 4\pi. \end{aligned} \tag{2.12}$$

Moreover the values of the Euler parameters θ, ϕ, ψ are unique for almost all $u \in SU(2)$, nonuniqueness being present only for $\theta = 0, \pi$ (since all values of ϕ are identified at either pole of S^2). Within $SO(3)$, $R(u_\perp(\theta, \phi))$ is a right-handed rotation by angle θ about the axis $(-\sin\phi, \cos\phi, 0)$ lying in the x–y plane[6].

[6] The matrix

$$R(u_\perp(\theta, \phi)) = \begin{pmatrix} \cos\theta\cos^2\phi + \sin^2\phi & (\cos\theta - 1)\cos\phi\sin\phi & \sin\theta\cos\phi \\ (\cos\theta - 1)\cos\phi\sin\phi & \cos\theta\sin^2\phi + \cos^2\phi & \sin\theta\sin\phi \\ -\sin\theta\cos\phi & -\sin\theta\sin\phi & \cos\theta \end{pmatrix}.$$

The parametrization (2.12) makes explicit the well-known fact that the coset spaces $SU(2)/U(1)$ and $SO(3)/SO(2)$ are the same, namely the two-dimensional unit sphere $S^2 \subset \mathcal{R}^3$:

$$SU(2)/U(1) \simeq SO(3)/SO(2) = S^2. \tag{2.13}$$

The distinguished 'origin' in S^2 is the north pole $(0, 0, 1)$ as it represents the coset $U(1)$. Moreover both $SU(2)$ and $SO(3)$ act on S^2 via $SO(3)$ rotations in the expected and natural manner. We also see that we can regard $u_\perp(\theta, \phi)$ as a coset representative (for $SU(2)$ as well as for $SO(3)$):

$$\theta, \phi \in S^2 \longrightarrow \ell(\theta, \phi) = u_\perp(\theta, \phi) \in SU(2) : R(\ell(\theta, \phi)) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \in S^2. \tag{2.14}$$

This coset representative obeys a (partial) covariance condition with respect to $U(1)$ action:

$$\ell(\theta, \phi) = u_3(\phi)\ell(\theta, 0)u_3(\phi)^{-1} \qquad u_3(\psi)\ell(\theta, \phi)u_3(\psi)^{-1} = \ell(\theta, \phi + \psi). \tag{2.15}$$

We shall hereafter deal mainly with $SU(2)$, the accompanying $SO(3)$ statements and results being more or less obvious. The general Wigner rotation in this case reduces to a Wigner angle ψ' dependent on a point $(\theta, \phi) \in S^2$ and an element $u \in SU(2)$. It is determined by computing the right-hand side in

$$u\ell(\theta, \phi) = \ell(\theta', \phi')u_3(\psi') \qquad \text{i.e.} \quad \ell(\theta', \phi')^{-1}u\ell(\theta, \phi) = u_3(\psi') \tag{2.16}$$

where (θ', ϕ') is the point on S^2 resulting from the point (θ, ϕ) through the rotation $R(u)$. Clearly θ', ϕ', ψ' must all be expressed in terms of θ, ϕ, u. However, on account of equations (2.12) and (2.15) it is easy to convince oneself that the only nontrivial aspect of equation (2.16) is captured in the special case

$$\ell(\theta', \phi')\ell(\theta, 0) = \ell(\theta'', \phi'')u_3(\psi'') \tag{2.17}$$

and it is this equation whose solution will be developed and interpreted in section 3. An important physical situation where this Wigner angle ψ'' shows up will also be described.

2.1.2. $G/H = SU(1,1)/U(1) \simeq SL(2,R)/SO(2) \simeq SO(2,1)/SO(2)$. It is well known that the groups $SU(1,1)$ and $SL(2,R)$, made up of 2×2 complex pseudounitary unimodular matrices and 2×2 real unimodular matrices respectively, are isomorphic (and further $SL(2,R)$ coincides with the symplectic group $Sp(2,R)$). This is displayed by a fixed similarity transformation connecting these two groups of matrices:

$$\begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda^* \end{pmatrix} \in SU(1,1) \qquad |\lambda|^2 - |\mu|^2 = 1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R) \qquad ad - bc = 1 \quad a, b, c, d \text{ real} : \begin{pmatrix} \lambda & \mu \\ \mu^* & \lambda^* \end{pmatrix} = S_0 \begin{pmatrix} a & b \\ c & d \end{pmatrix} S_0^{-1}$$

$$S_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}. \tag{2.18}$$

In this isomorphism the $U(1)$ subgroup of $SU(1,1)$ goes into the $SO(2)$ subgroup of $SL(2,R)$:

$$\begin{aligned} u_3(\phi) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\phi/2} \end{pmatrix} \in SU(1,1) \longleftrightarrow r(\phi) = \exp\left(-\frac{\mathrm{i}}{2}\phi\sigma_2\right) \\ = \begin{pmatrix} \cos\phi/2 & -\sin\phi/2 \\ \sin\phi/2 & \cos\phi/2 \end{pmatrix} \in SL(2,R). \end{aligned} \tag{2.19}$$

It may be noted that S_0 is precisely the matrix which connects the (q, p) basis to the $(a, a^\dagger)$ basis, where $a \equiv (q + \mathrm{i}p)/\sqrt{2}$. Thus, $SL(2, R)$ and $SU(1, 1)$ are pictures of one and the same group viewed from the (q, p) basis and the $(a, a^\dagger)$ basis respectively.

We shall hereafter work with $SL(2, R)$. The two-to-one $SL(2, R) \to SO(2, 1)$ homomorphism, parallel to what appears in equations (2.9), is given in full detail by

$$\begin{aligned}
S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R) &\longrightarrow \Lambda(S) \\
&= \begin{pmatrix} \frac{1}{2}(a^2+b^2+c^2+d^2) & ab+cd & \frac{1}{2}(b^2+d^2-a^2-c^2) \\ ac+bd & ad+bc & bd-ac \\ \frac{1}{2}(c^2+d^2-a^2-b^2) & cd-ab & \frac{1}{2}(a^2-b^2-c^2+d^2) \end{pmatrix} \in SO(2, 1) \\
\Lambda(S)^T g \Lambda(S) = g &= \mathrm{diag}(1, -1, -1) \\
S', S \in SL(2, R) &\Longrightarrow \Lambda(S')\Lambda(S) = \Lambda(S'S).
\end{aligned} \tag{2.20}$$

The analogue of the $SU(2)$ parametrization (2.12) now expresses every $S \in SL(2, R)$ as the product of a 'pure Lorentz transformation' along a direction in the x–y plane, and a real rotation in that plane[7]:

$$\begin{aligned}
S \in SL(2, R) : S &= S(\beta, \phi, \psi) \\
&= \ell(\beta, \phi) r(\psi) \\
\ell(\beta, \phi) &= \exp\left(\frac{\beta}{2}(\sigma_1 \cos\phi - \sigma_3 \sin\phi)\right) \\
&= \cosh\beta/2 + \sinh\beta/2(\sigma_1\cos\phi - \sigma_3\sin\phi) \\
& 0 \leqslant \beta < \infty \quad 0 \leqslant \phi \leqslant 2\pi \quad 0 \leqslant \psi \leqslant 4\pi.
\end{aligned} \tag{2.21}$$

We see that the matrix $\ell(\beta, \phi)$ is real symmetric positive definite, while $r(\psi)$ is real orthogonal unimodular, so in fact (2.21) is the polar decomposition of S. That $\ell(\beta, \phi)$ belongs to the subset of $SL(2, R)$ mapping on to pure Lorentz transformations in $SO(2, 1)$ is seen by computing $\Lambda(\ell(\beta, \phi))$ using equation (2.20):

$$\Lambda(\ell(\beta, \phi)) = \begin{pmatrix} \cosh\beta & \sinh\beta\cos\phi & \sinh\beta\sin\phi \\ \sinh\beta\cos\phi & \sin^2\phi + \cosh\beta\cos^2\phi & \sinh\beta\sin\phi\cos\phi \\ \sinh\beta\sin\phi & \sinh\beta\sin\phi\cos\phi & \cos^2\phi + \cosh\beta\sin^2\phi \end{pmatrix}. \tag{2.22}$$

Similarly, we have the reassuring result

$$\Lambda(r(\psi)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}. \tag{2.23}$$

The parametrization (2.21) for $SL(2, R)$ shows explicitly the well-known fact that the coset spaces $SL(2, R)/SO(2)$ and $SO(2, 1)/SO(2)$ are the same, namely the two-dimensional unit time-like positive hyperboloid $M^{(2)} \subset M^{(2,1)}$ the three-dimensional Minkowski space:

$$\begin{aligned}
& SL(2, R)/SO(2) \simeq SO(2, 1)/SO(2) = M^{(2)} \\
& M^{(2)} = \{x = (x^0, x^1, x^2) \in M^{(2,1)} | (x^0)^2 - (x^1)^2 - (x^2)^2 = 1, x^0 \geqslant 1\} \subset M^{(2,1)}.
\end{aligned} \tag{2.24}$$

The distinguished 'origin' in $M^{(2)}$ is the point $(1, 0, 0)$ representing the coset $SO(2)$. The actions by both $SL(2, R)$ and $SO(2, 1)$ on $M^{(2)}$ are via proper orthochronous Lorentz

[7] The use of the same symbol ℓ in equation (2.14) and here should cause no confusion, as the significance is always clear from the context.

transformations in (2 + 1) dimensions, in a natural manner. We see that the factor $\ell(\beta,\phi)$ in equation (2.21) is a coset representative:

$$x(\beta,\phi) = (\cosh\beta, \sinh\beta\cos\phi, \sinh\beta\sin\phi) \in M^{(2)}$$
$$\to \ell(\beta,\phi) \in SL(2,R) : \Lambda(\ell(\beta,\phi)) \begin{pmatrix}1\\0\\0\end{pmatrix} = x(\beta,\phi) \in M^{(2)}. \qquad (2.25)$$

This coset representative obeys a covariance condition similar to $u_\perp(\theta,\phi)$ in equation (2.15):

$$\ell(\beta,\phi) = r(\phi)\ell(\beta,0)r(\phi)^{-1} \qquad r(\psi)\ell(\beta,\phi)r(\psi)^{-1} = \ell(\beta,\phi+\psi). \qquad (2.26)$$

Let us now make all statements in relation to $SL(2,R)$, the corresponding ones for $SO(2,1)$ being evident. The general Wigner rotation is a Wigner angle ψ' dependent on a point $(\beta,\phi) \in M^{(2)}$ and an element $S \in SL(2,R)$. It is found by computing the right-hand side in

$$S\ell(\beta,\phi) = \ell(\beta',\phi')r(\psi') \qquad \text{i.e.} \quad \ell(\beta',\phi')^{-1}S\ell(\beta,\phi) = r(\psi') \qquad (2.27)$$

and expressing β', ϕ', ψ' as functions of β, ϕ, S. As with equation (2.16), here again the point $x(\beta',\phi') \in M^{(2)}$ arises from the point $x(\beta,\phi) \in M^{(2)}$ by application of the $SO(2,1)$ transformation $\Lambda(S)$. On account of equations (2.21) and (2.26) however, it is easily seen that the only nontrivial aspect of equation (2.27) is contained in the special case[8]

$$\ell(\beta',\phi')\ell(\beta,0) = \ell(\beta'',\phi'')r(\psi''). \qquad (2.28)$$

We will develop and interpret the solution to this problem later in section 3.

In contrast to the $SU(2)$ case, with $SL(2,R)$ we have one other distinct coset space with respect to the Abelian noncompact $SO(1,1)$ subgroup. Its geometrical representation is as the two-dimensional single sheeted unit space-like hyperboloid in $M^{(2,1)}$. However since this case is not of particular physical interest, we omit a discussion of it.

2.2. Examples with non-Abelian H

2.2.1. $G/H = SL(2,C)/SU(2) \simeq SO(3,1)/SO(3)$. These coset spaces arise in the discussion of the quantum mechanics of massive relativistic particles, and essentially describe momentum space. Elements of $SL(2,C)$ are 2×2 complex unimodular matrices [11]:

$$S = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in SL(2,C) \qquad \alpha\delta - \beta\gamma = 1 \quad \alpha,\beta,\gamma,\delta \text{ complex}. \qquad (2.29)$$

(The use of the same symbol S for elements of $SL(2,R)$ and here of $SL(2,C)$ should cause no confusion, as the meaning will be clear from the context.) The two-to-one $SL(2,C) \to SO(3,1)$ homomorphism is given by an extension of equations (2.9) and (2.20) to four-dimensional Minkowski space $M^{(3,1)}$:

$$\begin{aligned}
&S \in SL(2,C) \to \Lambda(S) = \left(\Lambda^\mu{}_\nu(S)\right) \in SO(3,1)\\
&\Lambda^\mu{}_\nu(S) = \tfrac{1}{2}\operatorname{Tr}(\tilde\sigma^\mu S\sigma_\nu S^\dagger)\\
&\sigma_\mu = (1, \underline{\sigma}) \qquad \tilde\sigma_\mu = (1, -\underline{\sigma}) \qquad \mu,\nu = 0,1,2,3\\
&S', S \in SL(2,C) \Longrightarrow \Lambda(S')\Lambda(S) = \Lambda(S'S).
\end{aligned} \qquad (2.30)$$

Raising and lowering of indices is done using the diagonal Minkowski metric $g = \operatorname{diag}(1,-1,-1,-1)$. Thus for instance

$$\tilde\sigma^\mu = g^{\mu\nu}\tilde\sigma_\nu \qquad \Lambda_{\mu\nu} = g_{\mu\rho}\Lambda^\rho{}_\nu. \qquad (2.31)$$

[8] Again the use of the same symbol ψ'' for the Wigner angle in the $SU(2)$ and $SL(2,R)$ cases should cause no confusion.

Both the earlier homomorphisms $SU(2) \to SO(3)$, $SL(2, R) \to SO(2, 1)$ are subsumed in the present one. Hereafter we mainly deal with $SL(2, C)$.

Every element of $SL(2, C)$ is uniquely expressible, via polar decomposition, as the product of a Hermitian unimodular positive definite matrix and an $SU(2)$ matrix:

$$\begin{aligned}
&S \in SL(2, C) : S = S(\underline{\beta}, u) = \ell(\underline{\beta})u \\
&\ell(\underline{\beta}) = \exp\left(\tfrac{1}{2}\underline{\beta} \cdot \underline{\sigma}\right) = \cosh\beta/2 + \hat{\beta} \cdot \underline{\sigma} \sinh\beta/2 \\
&\beta = |\underline{\beta}| \qquad \hat{\beta} = \underline{\beta}/\beta \qquad \underline{\beta} \in \mathcal{R}^3 \\
&u \in SU(2).
\end{aligned} \tag{2.32}$$

Within $SO(3, 1)$, $\ell(\underline{\beta})$ is mapped onto a pure Lorentz transformation in the direction of $\underline{\beta}$ and with rapidity β,

$$\begin{aligned}
\ell(\underline{\beta}) \in SL(2, C) \to \Lambda(\ell(\underline{\beta})) = \left(\Lambda^{\mu}{}_{\nu}(\ell(\underline{\beta}))\right) \in SO(3, 1) : \Lambda_{00}(\ell(\underline{\beta})) = \cosh\beta \\
\Lambda_{0j}(\ell(\underline{\beta})) = -\Lambda_{j0}(\ell(\underline{\beta})) = \hat{\beta}_j \sinh\beta \\
\lambda_{jk}(\ell(\underline{\beta})) = \delta_{jk}\cosh\beta + \hat{\beta}_j\hat{\beta}_k(\cosh\beta - 1)
\end{aligned} \tag{2.33}$$

while for $u \in SU(2)$ we have

$$\Lambda(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R(u) & \\ 0 & & & \end{pmatrix}. \tag{2.34}$$

These considerations and the $SL(2, C)$ parametrization (2.32) show that the coset spaces $SL(2, C)/SU(2)$ and $SO(3, 1)/SO(3)$ are the same, namely the three-dimensional unit time-like positive hyperboloid $M^{(3)} \subset M^{(3,1)}$ the four-dimensional Minkowski space:

$$\begin{aligned}
&SL(2, C)/SU(2) \simeq SO(3, 1)/SO(3) = M^{(3)} \\
&M^{(3)} = \{x = (x^{\mu}) \in M^{(3,1)} | (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 1, x^0 \geqslant 1\} \subset M^{(3,1)}.
\end{aligned} \tag{2.35}$$

The distinguished 'origin' in $M^{(3)}$ is the point $(1, 0, 0, 0)$ representing the coset $SU(2)$ (or $SO(3)$). Both $SL(2, C)$ and $SO(3, 1)$ act on $M^{(3)}$ via proper orthochronous Lorentz transformations in (3 + 1) dimensions, in a natural manner. The factor $\ell(\underline{\beta})$ in equation (2.32) is a coset representative:

$$x(\underline{\beta}) = (\cosh\beta, \hat{\beta}\sinh\beta) \in M^{(3)} \longrightarrow \ell(\underline{\beta}) \in SL(2, C) : \Lambda(\ell(\underline{\beta})) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = x(\underline{\beta}). \tag{2.36}$$

This coset representative is $SU(2)$ covariant (compare with (2.15) and (2.26)):

$$u \in SU(2) : u\ell(\underline{\beta})u^{-1} = \ell(R(u)\underline{\beta}). \tag{2.37}$$

'Wigner rotations' were originally defined in this context. In general they are $SU(2)$ (or $SO(3)$) elements dependent on $\underline{\beta}$ and $S \in SL(2, C)$,

$$S\ell(\underline{\beta}) = \ell(\underline{\beta}')u'(\underline{\beta}, S) \qquad \text{i.e.} \quad \ell(\underline{\beta}')^{-1}S\ell(\underline{\beta}) = u'(\underline{\beta}, S) \tag{2.38}$$

with both $\underline{\beta}'$ and u' to be determined. Thus it is not just a phase anymore. However the covariance law (2.37) allows us to simplify equation (2.38) and reduce it essentially to the $SL(2, R)$ problem. This is because with no loss of generality we can limit ourselves to consideration of the product $\ell(\underline{\beta}')\ell(\underline{\beta})$ and moreover choose $\underline{\beta} = (\beta, 0, 0)$, $\underline{\beta}' = (\beta_1', \beta_2', 0)$. In this special case equation (2.38) becomes

$$\ell(\beta_1', \beta_2', 0)\ell(\beta, 0, 0) = \ell(\beta_1'', \beta_2'', 0)u_3(\psi'') \tag{2.39}$$

whose solution is already available. Thus in this example, while H is certainly non-Abelian and the Wigner rotation is an $SU(2)(SO(3))$ element, the core of the calculation can be greatly simplified. (In fact, for any n, the $SO(n, 1)$ Wigner rotation problem reduces to the $SO(2, 1)$ case; see for instance [12].)

With $SL(2, C)$ and $SO(3, 1)$ we have other examples of coset spaces, namely $SL(2, C)/SL(2, R) \simeq SO(3, 1)/SO(2, 1)$ and $SL(2, C)/E(2) \simeq SO(3, 1)/E(2)$, both of which are involved in the Wigner theory of unitary representations of the Poincaré group. (Some aspects of these are studied in [13].) However we do not consider them here.

2.2.2. $G/H = U(n)/U(n-1)$. Our final example has a quantum mechanical flavour. Consider an n-level quantum system, with associated n-dimensional complex Hilbert space $\mathcal{H}^{(n)}$. We define the complex unit sphere $\mathcal{B}^{(n)}$ in $\mathcal{H}^{(n)}$ by

$$\mathcal{B}^{(n)} = \{\psi \in \mathcal{H}^{(n)} \,|\, \|\psi\|^2 = \psi^\dagger\psi = 1\} \subset \mathcal{H}^{(n)}. \tag{2.40}$$

(Here the vector ψ is an n-component column vector with complex entries.) The defining representation of the group $U(n)$ consists of $n \times n$ complex unitary matrices:

$$U(n) = \{A = (A_{rs}) = n \times n \text{ complex matrix } | A^+A = 1\}. \tag{2.41}$$

The action on $\mathcal{B}^{(n)}$ is given by

$$\psi \in \mathcal{B}^{(n)} \qquad A \in U(n) \to \psi' = A\psi \in \mathcal{B}^{(n)}. \tag{2.42}$$

It is evident that this is a transitive action, as any ψ can certainly be carried to

$$e_n = (0, 0, \ldots, 0, 1)^T \tag{2.43}$$

by a suitable $A \in U(n)$. Moreover the subgroup of $U(n)$ leaving e_n invariant is $U(n-1)$ acting on the first $(n-1)$ dimensions of $\mathcal{H}^{(n)}$:

$$Ae_n = e_n \Longleftrightarrow A = \begin{pmatrix} & & & \vdots & 0 \\ & B & & \vdots & \\ & & & & \vdots \\ \ldots & & \cdot & \ldots & 0 \\ 0 & \cdot & 0 & \cdot & 1 \end{pmatrix} \qquad B = (B_{jk}) \in U(n-1). \tag{2.44}$$

(We use index conventions $r, s, \ldots = 1, 2, \ldots, n$ and $j, k, \ldots = 1, 2, \ldots, n-1$.) This shows that $\mathcal{B}^{(n)}$ is the coset space $U(n)/U(n-1)$, with the 'point' $e_n \in \mathcal{B}^{(n)}$ being the distinguished origin.

Each $\xi \in \mathcal{B}^{(n)}$ determines a right $U(n-1)$ coset in $U(n)$, containing all those $A \in U(n)$ which carry e_n to ξ; this means that the last column of A is ξ. We can now look for a convenient coset representative $L(\xi) \in U(n)$ determined completely by ξ: its last column must be ξ, and all its earlier columns must be chosen as suitable functions of ξ. We write

$\alpha^{(j)}(\xi)$, $j = 1, 2, \ldots, (n-1)$ for these columns, each being an n-component vector, and present a choice which works as long as $\xi_1 \neq 0$ [14]:

$$L(\xi) = (\alpha^{(1)}(\xi)\alpha^{(2)}(\xi)\cdots\alpha^{(n-1)}(\xi)\xi)$$
$$\alpha_r^{(j)}(\xi) = \begin{cases} -\xi_{j+1}^*\xi_r/\rho_j\rho_{j+1} & r \leqslant j \\ \rho_j/\rho_{j+1} & r = j+1 \\ 0 & r \geqslant j+2 \end{cases} \tag{2.45}$$
$$\rho_r = (|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_r|^2)^{1/2} \qquad r = 1, 2, \ldots, n.$$

It is a matter of simple algebra to verify that

$$\xi^\dagger\alpha^{(j)}(\xi) = 0 \qquad \alpha^{(j)}(\xi)^\dagger\alpha^{(k)}(\xi) = \delta_{jk} \tag{2.46}$$

which shows that $L(\xi) \in U(n)$.

Now let $A \in U(n)$, $\xi \in \mathcal{B}^{(n)}$ and $\xi' = A\xi$, and assume $\xi_1' \neq 0$ as well. Then the 'Wigner rotation' in this situation is the $U(n-1)$ element $B(\xi, A)$ determined by

$$AL(\xi) = L(\xi')\begin{pmatrix} & & & \vdots & 0 \\ & & B(\xi, A) & \vdots & \vdots \\ \ldots & \ldots & \ldots & \cdot & 0 \\ 0 & \ldots & 0 & \cdot & 1 \end{pmatrix} \quad \text{i.e.}$$
$$L(A\xi)^\dagger AL(\xi) = \begin{pmatrix} & & & \cdot & 0 \\ & B(\xi, A) & \cdot\cdot & & \\ \ldots & \ldots & \ldots & \cdot & 0 \\ 0 & \ldots & 0 & \cdot & 1 \end{pmatrix}. \tag{2.47}$$

Since we have identically

$$\xi'^\dagger A\alpha^{(j)}(\xi) = \alpha^{(j)}(\xi')^\dagger\xi' = 0 \tag{2.48}$$

we do indeed have an $U(n-1)$ element on the right in equation (2.47); and the matrix elements of $B(\xi, A)$ are

$$B_{jk}(\xi, A) = \alpha^{(j)}(\xi')^\dagger A\alpha^{(k)}(\xi). \tag{2.49}$$

This is an interesting example of a non-Abelian Wigner rotation!

3. The $SU(2)$ and $SL(2, R)$ Wigner angles

In this section we present the calculations of the Abelian $SU(2)$ and $SL(2, R)$ Wigner phases ψ'' defined in equations (2.17) and (2.28) respectively, and describe them geometrically. We will also show that they are related by analytic continuation. To begin with, we describe an interesting physical situation where the $SU(2)$ Wigner phase shows up.

3.1. Photons with circular polarization and the $SU(2)$ Wigner angle

Consider the space of states of a single photon with fixed frequency ω_0, various propagation directions, and with definite, say right, circular polarization. It is well known that such a polarization state is invariant under all proper Lorentz transformations. For this limited problem, the magnitude of the wave vector is fixed by ω_0, so the appropriate Hilbert space can be defined to be the space of wavefunctions on directions in wave-vector space:

$$\mathcal{H}^{(+)} = \left\{\Psi(\theta, \phi) \mid \|\Psi\|^2 = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \sin\theta\, \mathrm{d}\theta |\Psi(\theta, \phi)|^2\right\}. \tag{3.1}$$

The superscript reminds us of the choice of right circular polarization. We can introduce as usual an ideal basis of kets $|\theta,\phi\rangle$ such that

$$\begin{aligned}\Psi(\theta,\phi) &= \langle\theta,\phi|\Psi\rangle \\ \langle\theta',\phi'|\theta,\phi\rangle &= \delta(\cos\theta'-\cos\theta)\delta(\phi'-\phi).\end{aligned} \tag{3.2}$$

Evidently, the space $\mathcal{H}^{(+)}$ carries a unitary (irreducible) representation of the Euclidean group $E(3)$ of spatial translations and rotations generated by the linear and angular momentum operators of the photon respectively. Denote these operators as usual by $\underline{P}$ and $\underline{J}$; and for the finite unitary operators generated by the latter write $U(u)$, $u \in SU(2)$. Then we have

$$\begin{aligned}U(u_3(\psi))|0,0\rangle &= \mathrm{e}^{-\mathrm{i}\psi J_3}|0,0\rangle \\ &= \mathrm{e}^{-\mathrm{i}\psi}|0,0\rangle \\ \underline{P}|\theta,\phi\rangle &= \omega_0(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)|\theta,\phi\rangle.\end{aligned} \tag{3.3}$$

(We have set $\hbar = c = 1$.) The general basis ket $|\theta,\phi\rangle$ can be taken to be related to $|0,0\rangle$ by

$$\begin{aligned}|\theta,\phi\rangle &= U(u_\perp(\theta,\phi))|0,0\rangle \\ &= \exp(-\mathrm{i}\theta(J_2\cos\phi - J_1\sin\phi))|0,0\rangle.\end{aligned} \tag{3.4}$$

Now if u is a general $SU(2)$ element, we see that in the notation of equation (2.16) and remembering from equation (2.14) that $\ell(\theta,\phi) = u_\perp(\theta,\phi)$,

$$\begin{aligned}U(u)|\theta,\phi\rangle &= U(uu_\perp(\theta,\phi))|0,0\rangle \\ &= U(u_\perp(\theta',\phi')u_3(\psi'))|0,0\rangle \\ &= \mathrm{e}^{-\mathrm{i}\psi'}|\theta',\phi'\rangle = \mathrm{e}^{-\mathrm{i}\psi'}|R(u)(\theta,\phi)\rangle.\end{aligned} \tag{3.5}$$

Thus under a spatial rotation not only does the wave vector of the photon get rotated in the expected manner, in addition there is also a phase factor arising from the $SU(2)$ Wigner angle and reflecting the right circular polarization of the photon. More generally, we can say that this Wigner angle is involved in any representation of $E(3)$ with nonzero helicity.

3.2. Calculation of the $SU(2)$ Wigner angle

Now we turn to calculating the angles θ'', ϕ'', ψ'' that appear on the right-hand side in equation (2.17). As for θ'', ϕ'' we have already seen that they denote the point on S^2 arising from the point (θ,ϕ) via the rotation $R(u_\perp(\theta',\phi'))$, therefore

$$\begin{pmatrix}\sin\theta''\cos\phi'' \\ \sin\theta''\sin\phi'' \\ \cos\theta''\end{pmatrix} = R(u_\perp(\theta',\phi'))\begin{pmatrix}\sin\theta \\ 0 \\ \cos\theta\end{pmatrix}. \tag{3.6}$$

In particular, we have

$$\cos\theta'' = \cos\theta'\cos\theta - \sin\theta'\sin\theta\cos\phi' \tag{3.7}$$

which will find a nice geometrical interpretation.

To get the Wigner angle ψ'' we write equation (2.17) explicitly:

$$\begin{aligned}&(\cos\theta'/2 - \mathrm{i}\sin\theta'/2(\sigma_2\cos\phi' - \sigma_1\sin\phi'))(\cos\theta/2 - \mathrm{i}\sigma_2\sin\theta/2) \\ &\quad = (\cos\theta''/2 - \mathrm{i}\sin\theta''/2(\sigma_2\cos\phi'' - \sigma_1\sin\phi''))(\cos\psi''/2 - \mathrm{i}\sigma_3\sin\psi''/2).\end{aligned} \tag{3.8}$$

It is now convenient and illuminating to use the following argument. If equation (3.8) is obeyed, then so is the equation obtained by the cyclic change $\sigma_1 \to \sigma_3, \sigma_2 \to \sigma_1, \sigma_3 \to \sigma_2$ of the Pauli matrices. That equation reads

$$\begin{aligned}(\cos\theta'/2 - \mathrm{i}\sin\theta'/2(\sigma_1\cos\phi' - \sigma_3\sin\phi'))(\cos\theta/2 - \mathrm{i}\sigma_1\sin\theta/2)\\ = (\cos\theta''/2 - \mathrm{i}\sin\theta''/2(\sigma_1\cos\phi'' - \sigma_3\sin\phi''))(\cos\psi''/2 - \mathrm{i}\sigma_2\sin\psi''/2).\end{aligned} \tag{3.9}$$

The useful feature of this form is that the left-hand side is the product of two complex symmetric noncommuting matrices, so the result is not expected to be symmetric; one expects in advance the appearance of the ψ'' factor on the right, which is real orthogonal. Working with equation (3.9), let us write L and R for its two sides. We then have

$$\begin{aligned}\mathrm{Tr}\,L &= 2(\cos\theta'/2\cos\theta/2 - \sin\theta'/2\sin\theta/2\cos\phi')\\ \mathrm{Tr}(\mathrm{i}\sigma_2 L) &= -2\sin\theta'/2\sin\theta/2\sin\phi'\\ \mathrm{Tr}\,R &= 2\cos\theta''/2\cos\psi''/2\\ \mathrm{Tr}(\mathrm{i}\sigma_2 R) &= 2\cos\theta''/2\sin\psi''/2.\end{aligned} \tag{3.10}$$

We thus arrive at the result

$$\begin{aligned}\tan\psi''/2 &= \mathrm{Tr}(\mathrm{i}\sigma_2 R)/\mathrm{Tr}\,R\\ &= \mathrm{Tr}(\mathrm{i}\sigma_2 L)/\mathrm{Tr}\,L\\ &= -\sin\theta'/2\sin\theta/2\sin\phi'/(\cos\theta'/2\cos\theta/2 - \sin\theta'/2\sin\theta/2\cos\phi')\\ &= -\sin\theta'\sin\theta\sin\phi'/\{(1+\cos\theta')(1+\cos\theta) - \sin\theta'\sin\theta\cos\phi'\}.\end{aligned} \tag{3.11}$$

This completes the calculation of the $SU(2)$ Wigner angle.

The results (3.7) and (3.11) for θ'', ψ'' have simple geometrical interpretations in terms of a suitable spherical triangle on S^2. Namely if we construct a triangle as in figure 1, we have two adjacent sides of lengths θ, θ' enclosing an angle $\phi' - \pi$; and then θ'' is the length of the third side, while ψ'' is the area of the triangle. For the latter, we recall that for a spherical triangle on S^2 with vertices $\hat{a}, \hat{b}, \hat{c}$, the area $\Omega(\hat{a}, \hat{b}, \hat{c})$ is given by

$$\tan\Omega(\hat{a},\hat{b},\hat{c})/2 = \frac{\hat{a}\cdot\hat{b}\times\hat{c}}{1+\hat{b}\cdot\hat{c}+\hat{c}\cdot\hat{a}+\hat{a}\cdot\hat{b}}. \tag{3.12}$$

For the triangle of figure 1 we take $\hat{a} = (0, 0, 1)$, $\hat{b} = (\sin\theta, 0, \cos\theta)$, $\hat{c} = (-\sin\theta'\cos\phi', -\sin\theta'\sin\phi', \cos\theta')$, and then we immediately see from equations (3.11) and (3.12) that

$$\psi'' = \Omega(\hat{a}, \hat{b}, \hat{c}). \tag{3.13}$$

3.3. Calculation of the SL(2, R) Wigner angle

Now we take up the calculation of the parameters β'', ϕ'', ψ'' appearing in equation (2.28). The values of β'', ϕ'' are immediate: they determine the point on M^2 arising from $x(\beta, 0)$ via the Lorentz transformation $\Lambda(\ell(\beta', \phi'))$. Using equation (2.22) we have

$$\begin{pmatrix}\cosh\beta''\\ \sinh\beta''\cos\phi''\\ \sinh\beta''\sin\phi''\end{pmatrix} = \Lambda(\ell(\beta',\phi'))\begin{pmatrix}\cosh\beta\\ \sinh\beta\\ 0\end{pmatrix}. \tag{3.14}$$

In particular we find

$$\cosh\beta'' = \cosh\beta'\cosh\beta + \sinh\beta'\sinh\beta\cos\phi'. \tag{3.15}$$

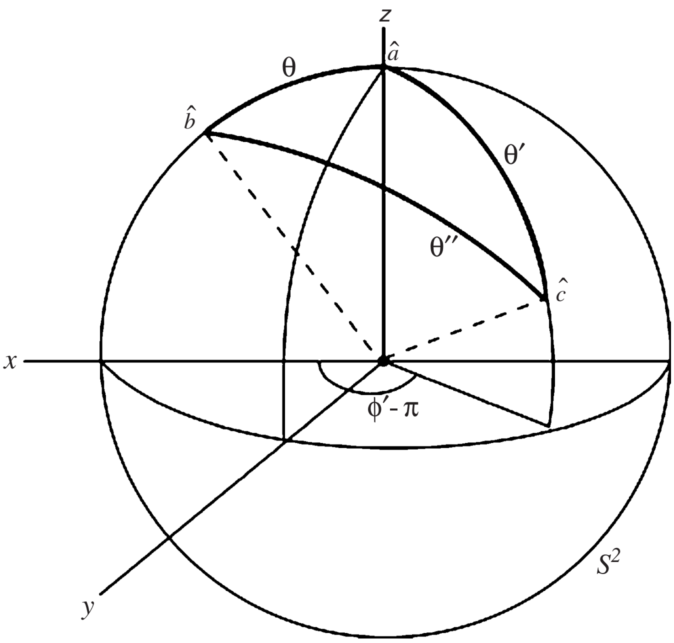

Figure 1. Computation of the $SU(2)$ Wigner angle. The plane of the paper is the x–z plane, with the y-axis coming out of this plane. This choice makes visualization easier. As drawn, the figure assumes $3\pi/2 < \phi' < 2\pi$.

To determine the Wigner angle ψ'', we write equation (2.28) in detail:

$$\begin{aligned}&(\cosh\beta'/2+\sinh\beta'/2(\sigma_1\cos\phi'-\sigma_3\sin\phi'))(\cosh\beta/2+\sigma_1\sinh\beta/2)\\&\qquad=(\cosh\beta''/2+\sinh\beta''/2(\sigma_1\cos\phi''-\sigma_3\sin\phi''))(\cos\psi''/2-\mathrm{i}\sigma_2\sin\psi''/2).\end{aligned}\tag{3.16}$$

This is already structurally similar to equation (3.9): the left-hand side is the product of two real symmetric positive definite noncommuting matrices, while the right-hand side is its polar decomposition. The Wigner angle factor is thus expected. Writing L' and R' for the two sides of equation (3.16) we have, in the same manner as equation (3.10),

$$\begin{aligned}\mathrm{Tr}\,L'&=2(\cosh\beta'/2\cosh\beta/2+\sinh\beta'/2\sinh\beta/2\cos\phi')\\\mathrm{Tr}(\mathrm{i}\sigma_2L')&=2\sinh\beta'/2\sinh\beta/2\sin\phi'\\\mathrm{Tr}\,R'&=2\cosh\beta''/2\cos\psi''/2\\\mathrm{Tr}(\mathrm{i}\sigma_2R')&=2\cosh\beta''/2\sin\psi''/2.\end{aligned}\tag{3.17}$$

This determines ψ'',

$$\begin{aligned}\tan\psi''/2&=\mathrm{Tr}(\mathrm{i}\sigma_2R')/\mathrm{Tr}\,R'\\&=\mathrm{Tr}(\mathrm{i}\sigma_2L')/\mathrm{Tr}\,L'\\&=\sinh\beta'/2\sinh\beta/2\sin\phi'/(\cosh\beta'/2\cosh\beta/2+\sinh\beta'/2\sinh\beta/2\cos\phi')\\&=\sinh\beta'\sinh\beta\sin\phi'/\{(1+\cosh\beta')(1+\cosh\beta)+\sinh\beta'\sinh\beta\cos\phi'\}.\end{aligned}\tag{3.18}$$

We see upon comparison of equations (3.11) and (3.18) that the two Wigner angle expressions are related by analytic continuation: to go from the $SU(2)$ result to the $SL(2,R)$

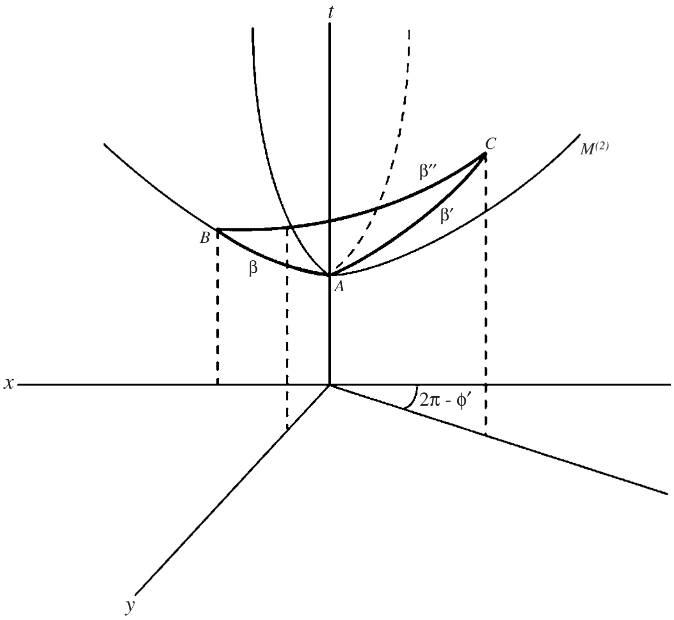

Figure 2. Computation of the $SL(2, R)$ Wigner angle. The plane of the paper is the x–t plane, with the y-axis coming out of this plane. The figure is drawn for the situation $3\pi/2 < \phi' < 2\pi$.

result we have to replace polar angles θ', θ by rapidities β', β according to $\theta' \to \mathrm{i}\beta'$, $\theta \to \mathrm{i}\beta$. This was one of the aims of the present calculation.

For the geometrical interpretation of these results, we now have to work with a suitable hyperbolic triangle drawn on the unit hyperboloid $M^{(2)}$ in Minkowski space $M^{(2,1)}$. This is shown in figure 2. We have two adjacent sides AB, AC of lengths β, β' enclosing an angle $\phi' - \pi$; then β'' is the length of the third side BC while ψ'' is the area of the hyperbolic triangle. The formula for the area is evidently an analytic continuation, or the hyperbolic version, of equation (3.12).

4. Geometric phases, null phase curves and Bargmann invariants

We wish to show that the Wigner angles in the $SU(2)$ and $SL(2, R)$ cases can be reinterpreted as geometric phases (GPs) in certain specific quantum mechanical situations [15]. It is well to bear in mind that basically both equations (2.17) and (2.28) are statements of a purely group theoretical nature, referring to specific coset spaces and coset representatives, and as they stand the Wigner angles ψ'' in them do not have obvious meanings as GPs. To arrive at such an interpretation necessary additional constructions have to be made. In this section we review briefly and in a simplified way the basic definitions connected with GPs. We use the kinematic approach and link GPs to the so-called Bargmann invariants (BIs) using as intermediary a class of Hilbert space curves called null phase curves (NPCs). We reassure the reader that all these concepts will be needed in the following section when we show that the Wigner angles computed in section 3 are instances of the GP.

Let the Hilbert space of states of some quantum system be denoted by $\mathcal{H}$; it may be of any dimension. The subset of unit vectors in $\mathcal{H}$, comprising the unit sphere, will be denoted by $\mathcal{B}$; this is similar to equation (2.40). There is a corresponding space of unit rays (pure state projection operators or density matrices) which is written as $\mathcal{R}$; and we have a natural projection map $\mathcal{B} \to \mathcal{R}$. Properly speaking, the GP is defined as a functional of a (sufficiently smooth—see below) parametrized open or closed curve C in $\mathcal{R}$. For practical calculations however it is convenient to 'lift' C to a curve $\mathcal{C}$ in $\mathcal{B}$, such that $\mathcal{C}$ projects onto C, and work with $\mathcal{C}$. This is because vectors are easier to deal with than projection operators. For simplicity, we shall avoid repeated reference to the ray space $\mathcal{R}$, and also limit ourselves to GPs for curves in $\mathcal{B}$ whose images in $\mathcal{R}$ are closed loops. These correspond in the usual terminology to cyclic evolutions. We repeat that these specializations are not essential but are made only in the interest of simplicity.

Let then $\mathcal{C}$ be a continuous parametrized piecewise once-differentiable curve in $\mathcal{B}$, whose end points differ at most by a phase. We describe $\mathcal{C}$ by

$$\mathcal{C} = \{\psi(s) \in \mathcal{B} | s_1 \leqslant s \leqslant s_2\} \subset \mathcal{B} \qquad \psi(s_2) = \mathrm{e}^{\mathrm{i}\varphi_{\mathrm{tot}}[\mathcal{C}]}\psi(s_1) \tag{4.1}$$

thereby introducing the total phase associated with $\mathcal{C}$:

$$\varphi_{\mathrm{tot}}[\mathcal{C}] = \arg(\psi(s_1), \psi(s_2)). \tag{4.2}$$

The dynamical phase $\varphi_{\mathrm{dyn}}[\mathcal{C}]$ is

$$\varphi_{\mathrm{dyn}}[\mathcal{C}] = \mathrm{Im} \int_{s_1}^{s_2} \mathrm{d}s \left(\psi(s), \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \right) \tag{4.3}$$

and the GP is the difference,

$$\varphi_{\mathrm{geom}}[\mathcal{C}] = \varphi_{\mathrm{tot}}[\mathcal{C}] - \varphi_{\mathrm{dyn}}[\mathcal{C}]. \tag{4.4}$$

In connection with equation (4.3) we note that on account of $\psi(s)$ being a unit vector for all s, the integrand $\left(\psi(s), \frac{\mathrm{d}\psi(s)}{\mathrm{d}s}\right)$ is pure imaginary.

We next define a special class of curves in $\mathcal{B}$ called NPCs. Let $\mathcal{C}$ be a parametrized continuous once-differentiable curve in $\mathcal{B}$ such that no two points on it (including the end points) are mutually orthogonal:

$$\mathcal{C} = \{\psi(s) \in \mathcal{B} | s_1 \leqslant s \leqslant s_2\} \qquad (\psi(s), \psi(s')) \neq 0 \qquad s_1 \leqslant s, \quad s' \leqslant s_2. \tag{4.5}$$

We say $\mathcal{C}$ is an NPC if in addition we have

$$\begin{aligned} \arg(\psi(s), \psi(s')) &= \text{separable} \\ &= \alpha(s') - \alpha(s). \end{aligned} \tag{4.6}$$

Such a curve has rather special properties which can be described quite easily. From equation (4.6) we have for all s and s'

$$(\mathrm{e}^{-\mathrm{i}\alpha(s)}\psi(s), \mathrm{e}^{-\mathrm{i}\alpha(s')}\psi(s')) = \text{real positive}. \tag{4.7}$$

Differentiating with respect to s' and then setting $s' = s$ we have

$$\left(\psi(s), \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \right) - \mathrm{i}\frac{\mathrm{d}\alpha(s)}{\mathrm{d}s} = \text{real} \tag{4.8}$$

but since both terms on the left are pure imaginary, we end up with

$$\left(\psi(s), \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \right) = \frac{\mathrm{i}\,\mathrm{d}\alpha(s)}{\mathrm{d}s}. \tag{4.9}$$

Integrating over the range s_1 to s_2 we obtain

$$\begin{aligned} \mathrm{Im} \int_{s_1}^{s_2} \left(\psi(s), \frac{\mathrm{d}\psi(s)}{\mathrm{d}s} \right) &= \alpha(s_2) - \alpha(s_1) \\ &= \arg(\psi(s_1), \psi(s_2)). \end{aligned} \tag{4.10}$$

It is this property of NPCs that we will exploit below. We may note incidentally that any connected subset of a NPC is also an NPC.

Next we turn to the BIs. Given any three vectors $\psi_1, \psi_2, \psi_3 \in \mathcal{B}$, no two of which are mutually orthogonal, the three-vertex or third-order BI is defined by

$$\Delta_3(\psi_1, \psi_2, \psi_3) = (\psi_1, \psi_2)(\psi_2, \psi_3)(\psi_3, \psi_1). \tag{4.11}$$

This is in general complex; in addition it is cyclically symmetric and invariant under independent phase changes in each of the vectors ψ_1, ψ_2, ψ_3. This definition can be generalized to higher order BIs. We will now show that the phase of the BI (4.11) can be interpreted as a GP for suitable cyclic evolutions.

To this end, suppose that we are able to connect ψ_1 to ψ_2, ψ_2 to ψ_3 and ψ_3 to ψ_1 by three NPCs $\mathcal{C}_{12}, \mathcal{C}_{23}, \mathcal{C}_{31}$ respectively. The union $\mathcal{C}_{12} \cup \mathcal{C}_{23} \cup \mathcal{C}_{31} = \mathcal{C}$, say, is a closed loop in $\mathcal{B}$ and satisfies the smoothness conditions for $\varphi_{\rm geom}[\mathcal{C}]$ to be defined. Let us parametrize $\mathcal{C}$ so that we begin at ψ_1 for $s = s_1$, reach ψ_2 for $s = s_2$, reach ψ_3 for $s = s_3$, and return to ψ_1 at $s = s_4$. Using definition (4.4), realizing that as $\mathcal{C}$ is closed we have $\varphi_{\rm tot}[\mathcal{C}] = 0$, and the property (4.10) for each stretch of $\mathcal{C}$, we find

$$\begin{aligned}\varphi_{\rm geom}[\mathcal{C}] &= -\varphi_{\rm dyn}[\mathcal{C}] \\ &= -\varphi_{\rm dyn}[\mathcal{C}_{12}] - \varphi_{\rm dyn}[\mathcal{C}_{23}] - \varphi_{\rm dyn}[\mathcal{C}_{31}] \\ &= -\arg(\psi_1, \psi_2) - \arg(\psi_2, \psi_3) - \arg(\psi_3, \psi_1) \\ &= -\arg \Delta_3(\psi_1, \psi_2, \psi_3).\end{aligned} \tag{4.12}$$

Thus phases of three-vertex BIs are, apart from a sign, GPs for 'triangles' with the same vertices, and with sides forming NPCs. We exploit this in the following section.

To avoid misunderstanding we repeat that the geometric phase defined in equation (4.4) is actually dependent only on the ray space image C of the curve $\mathcal{C}$ in $\mathcal{B}$, whereas the two individual terms on the right do depend on $\mathcal{C}$. Likewise the BI (4.11) is a ray space quantity. And lastly all the above can be set up also for noncyclic evolutions, i.e. for open curves $\mathcal{C}$ with open images C.

5. The Wigner angles as GPs

5.1. The SU(2) case

We will now show how the $SU(2)$ Wigner angle ψ'' in equation (2.17), evaluated in equation (3.11), can be interpreted as a GP. Let us consider a general spin j unitary irreducible representation (UIR) $U(u)$ of $SU(2)$, with generators J_1, J_2, J_3 in the standard form. Let us for simplicity write

$$\begin{aligned}U_\perp(\theta, \phi) &\equiv U(\ell(\theta, \phi) = u_\perp(\theta, \phi)) \\ &= \exp\{-{\rm i}\theta(J_2 \cos\phi - J_1 \sin\phi)\} \\ &= {\rm e}^{-{\rm i}\phi J_3}\, {\rm e}^{-{\rm i}\theta J_2}\, {\rm e}^{{\rm i}\phi J_3}.\end{aligned} \tag{5.1}$$

The general group theoretical result (2.17) appears in this UIR as

$$U_\perp(\theta', \phi') U_\perp(\theta, 0) = U_\perp(\theta'', \phi'')\, {\rm e}^{-{\rm i}\psi'' J_3} \tag{5.2}$$

with θ'', ϕ'', ψ'' being given by equations (3.6) and (3.11).

Since the generator J_3 plays a special role, we define $(2j + 1)$ families of states in the space of the spin j UIR by

$$\theta, \phi \in S^2 \to |j, m; \theta, \phi\rangle = U_\perp(\theta, \phi)|j, m\rangle \qquad m = j, j-1, \ldots, -j. \tag{5.3}$$

Here, $\{|j, m\rangle\}$ is the standard orthonormal basis of J_3 eigenstates, with the eigenvalue of J_3 being m. For fixed m, the states $\{|j, m; \theta, \phi\rangle\}$ are in correspondence with points on S^2; when $m = j$ they are the well-known spin-coherent states [16][9]. For each $(\theta, \phi) \in S^2$, we have an eigenstate of the component of $\underline{J}$ in that direction, with eigenvalue m:

$$(\sin\theta\cos\phi J_1 + \sin\theta\sin\phi J_2 + \cos\theta J_3)|j, m; \theta, \phi\rangle = m|j, m; \theta, \phi\rangle. \tag{5.4}$$

We also easily see that

$$\mathrm{e}^{-\mathrm{i}\psi J_3}|j, m; \theta, \phi\rangle = \mathrm{e}^{-\mathrm{i}m\psi}|j, m; \theta, \phi + \psi\rangle. \tag{5.5}$$

We now consider the inner product of two states of the form (5.3), for a common value of m (these are also dealt with in detail in [17]). We see from equation (5.5) that

$$\langle j, m; \theta', \phi'|j, m; \theta, \phi\rangle = \langle j, m; \theta', \phi' - \phi|j, m; \theta, 0\rangle \tag{5.6}$$

so the dependence on ϕ' and ϕ is only through their difference. We then see from definition (5.3) that

$$\begin{aligned}\langle j, m; \theta', \phi' - \phi|j, m; \theta, 0\rangle &= \langle j, m|U_\perp(\theta', \phi' - \phi)^{-1}U_\perp(\theta, 0)|j, m\rangle \\ &= \langle j, m|U_\perp(\theta', \phi' - \phi + \pi)U_\perp(\theta, 0)|j, m\rangle \\ &= \langle j, m|U\left(u_\perp(\theta', \phi' - \phi + \pi)u_\perp(\theta, 0)\right)|j, m\rangle.\end{aligned} \tag{5.7}$$

We can use here the Wigner angle result (2.17) once we make the change $\phi' \to \phi' - \phi + \pi$. This leads to

$$\begin{aligned}&u_\perp(\theta', \phi' - \phi + \pi)u_\perp(\theta, 0) = u_\perp(\theta''', \phi''')u_3(\psi''') \\ &\cos\theta''' = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\phi' - \phi) \\ &\tan\psi'''/2 = \frac{\sin\theta'\sin\theta\sin(\phi' - \phi)}{(1 + \cos\theta')(1 + \cos\theta) + \sin\theta'\sin\theta\cos(\phi' - \phi)}.\end{aligned} \tag{5.8}$$

(It will turn out that we do not need the value of ϕ'''.) Combining equations (5.6)–(5.8) we arrive at the result we are looking for,

$$\begin{aligned}\langle j, m; \theta', \phi'|j, m; \theta, \phi\rangle &= \langle j, m|\,\mathrm{e}^{-\mathrm{i}\theta''' J_2}|j, m\rangle\,\mathrm{e}^{-\mathrm{i}m\psi'''} \\ &= d^j_{mm}(\theta''')\,\mathrm{e}^{-\mathrm{i}m\psi'''}\end{aligned} \tag{5.9}$$

where $d^j_{mm}(\theta''')$ is the well-known d-function from the quantum theory of angular momentum (for more details on these functions, see the treatment of quantum theory of angular momentum in any standard text on quantum mechanics, or [18]). This function is known to be real positive for $0 \leqslant \theta''' < \pi$. In the maximal case $m = j$ we have

$$d^j_{jj}(\theta''') = (\cos\theta'''/2)^{2j}. \tag{5.10}$$

The meanings of θ''', ψ''' in equation (5.8) are clear in terms of a suitable spherical triangle drawn on S^2. Namely, as in figure 3, we draw a spherical triangle ABC with vertices A at the North pole $\theta = \phi = 0$, B at θ, ϕ and C at θ', ϕ'. Then θ''' is the arc length BC and ψ''' is the area of the triangle, as seen from equation (3.12):

$$\theta''' = BC \qquad \psi''' = \Omega(A, B, C). \tag{5.11}$$

From equation (5.9) we read off the relation

$$\arg\langle j, m; \theta', \phi'|j, m; \theta, \phi\rangle = -m\Omega(A, B, C). \tag{5.12}$$

[9] These are all instances of generalized coherent states in the sense of Perelomov [6] for $SU(2)$.

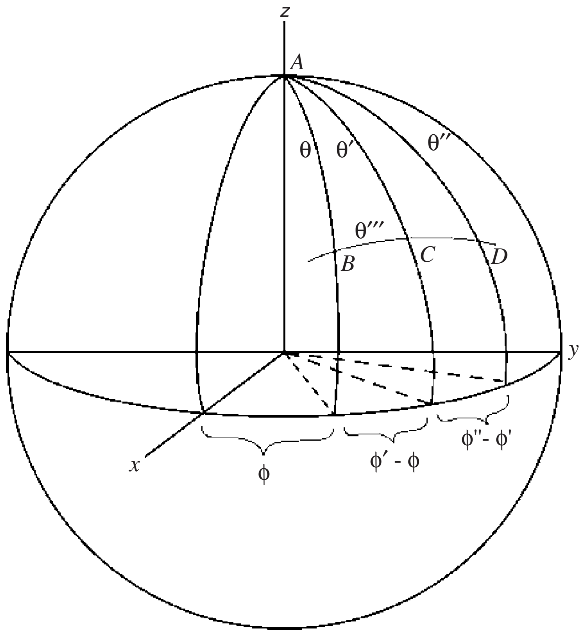

Figure 3. Spin-coherent states on a generic great circle arc on S^2 forming an NPC.

We now use this phase information in the following manner. Consider states $|j, m; \theta, \phi\rangle$, $|j, m; \theta', \phi'\rangle$, $|j, m; \theta'', \phi''\rangle \ldots$ corresponding to points $B, C, D \ldots$ lying on a generic great circle arc on S^2, as shown in figure 3. (By generic we mean other than a meridian $\phi =$ constant). From the additivity property of the areas of the indicated spherical triangles,

$$\Omega(A, B, D) = \Omega(A, B, C) + \Omega(A, C, D) \tag{5.13}$$

we learn using equation (5.12) that

$$\begin{aligned} \arg\langle j, m; \theta'', \phi''|j, m; \theta', \phi'\rangle &= -m\Omega(A, C, D) \\ &= -m\Omega(A, B, D) + m\Omega(A, B, C) \\ &= \arg\langle j, m; \theta''\phi''|j, m; \theta, \phi\rangle - \arg\langle j, m; \theta', \phi'|j, m; \theta, \phi\rangle. \end{aligned} \tag{5.14}$$

If we keep B fixed, and let C and D be independently variable points on the great circle, we find that the left-hand side in equation (5.14) is separable in C and D. Comparing this with definition (4.6) of an NPC we see that the family of states $\{|j, m; \theta, \phi\rangle\}$ lying on a generic great circle, and forming an arc of length less than π, forms an NPC. In the case we have a similar portion of a meridian $\phi =$ constant, the result (5.8) shows that $\psi''' = 0$ identically, so once again the NPC result holds. All in all, a family of states $\{|j, m; \theta, \phi\rangle\}$ lying on any great circle on S^2, and forming an arc of length less than π, is an NPC. The restriction to length less than π is to avoid antipodal points when these states become mutually orthogonal[10].

We can now appeal to the connection (4.12) between phases of BIs and GPs for triangles whose sides are NPCs to say: if P, Q, R are any three points on S^2 with the corresponding states $|j, m; P\rangle$, $|j, m; Q\rangle$, $|j, m; R\rangle$, and we join them by great circle arcs to form a spherical

[10] This remark regarding antipodal points is actually valid only for nonzero m.

triangle (with every side of length less than π), then

$$\varphi_{\rm geom}[\text{spherical triangle } PQR \text{ on } S^2] = -\arg \Delta_3(|j,m;\,P\rangle, |j,m;\,Q\rangle, |j,m;\,R\rangle)$$
$$= -\arg\langle j,m;\,P|j,m;\,Q\rangle\langle j,m;\,Q|j,m;\,R\rangle\langle j,m;\,R|j,m;\,P\rangle. \tag{5.15}$$

It is understood of course that on the left we have a triangle in the space of states $\{|j,m;\theta,\phi\rangle\}$ in Hilbert space, corresponding to the triangle traced out on S^2.

Now we can treat equation (5.2) involving the Wigner angle ψ''. Taking the expectation values of both sides in the state $|j,m\rangle$ gives

$$\langle j,m|U_\perp(\theta',\phi')U_\perp(\theta,0)|j,m\rangle = d^j_{mm}(\theta'')\,\mathrm{e}^{-\mathrm{i}m\psi''}. \tag{5.16}$$

To handle the left-hand side we use

$$\begin{aligned}\langle j,m|U_\perp(\theta',\phi') &= \langle j,m|U_\perp(\theta',\phi'-\pi)^{-1}\\ &= \langle j,m;\theta',\phi'-\pi|\end{aligned} \tag{5.17}$$

so equation (5.16) becomes

$$\langle j,m;\theta',\phi'-\pi|j,m;\theta,0\rangle = d^j_{mm}(\theta'')\,\mathrm{e}^{-\mathrm{i}m\psi''} \tag{5.18}$$

leading to

$$\arg\langle j,m;\theta,0|j,m;\theta',\phi'-\pi\rangle = m\psi''. \tag{5.19}$$

As both $\langle j,m;0,0|j,m;\theta,0\rangle$ and $\langle j,m;\theta',\phi'-\pi|j,m;0,0\rangle$ are real positive, we can include them on the left-hand side and rewrite equation (5.19) as

$$\arg\langle j,m;0,0|j,m;\theta,0\rangle\langle j,m;\theta,0|j,m;\theta',\phi'-\pi\rangle\langle j,m;\theta',\phi'-\pi|j,m;0,0\rangle = m\psi''. \tag{5.20}$$

Comparing this with equation (5.15) for a general configuration shows that $-m\psi''$ is indeed a GP:[11]

$$\varphi_{\rm geom}[\text{spherical triangle with vertices } |j,m;00\rangle, |j,m;\theta,0\rangle, |j,m;\theta',\phi'-\pi\rangle \text{ on } S^2]$$
$$= -m\psi''. \tag{5.21}$$

This matches exactly with figure 1. We are thus able to identify the $SU(2)$ Wigner angle ψ'', more precisely $-m\psi''$ for each m, with a specific GP indicated above. We appreciate the necessity of the steps taken to go from the group theoretical relation (5.2) to the GP interpretation (5.21) for ψ''; and also that equation (5.2) was used to obtain the phase relation (5.12).

5.2. The SL(2, R) case

We will next show that the $SL(2,R)$ Wigner angle ψ'' in equation (2.28), computed in equation (3.18), can be interpreted as a GP. The steps are very similar to the $SU(2)$ case, with hyperbolic geometry in place of spherical geometry. We have to work with the positive discrete class infinite-dimensional unitary irreducible representations $D^{(+)}_k$ of $SL(2,R)$, where $k = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ (see [19]). Within $D^{(+)}_k$ the compact generator J_3 has eigenvalues $m = k, k+1, k+2, \ldots$. In the UIR $D^{(+)}_k$, write $U(S)$ and J_3, K_1, K_2 for the operators representing $S \in SL(2,R)$, and the Hermitian generators, respectively[12]. For ease set

$$\begin{aligned}U(\beta,\theta) &\equiv U(\ell(\beta,\phi))\\ &= \exp\{-\mathrm{i}\beta(K_1\cos\phi + K_2\sin\phi)\}\\ &= \mathrm{e}^{-\mathrm{i}\phi J_3}\,\mathrm{e}^{-\mathrm{i}\beta K_1}\,\mathrm{e}^{\mathrm{i}\phi J_3}.\end{aligned} \tag{5.22}$$

[11] Here again, as in equation (5.15), the 'spherical triangle' is drawn in Hilbert space.

[12] The commutation relations among the generators are $[J_3, K_1] = \mathrm{i}K_2$, $[J_3, K_2] = -\mathrm{i}K_1$, $[K_1, K_2] = -\mathrm{i}J_3$.

In this UIR equation (2.28) involving the $SL(2,R)$ Wigner angle ψ'' is

$$U(\beta',\phi')U(\beta,0) = U(\beta'',\phi'')\,\mathrm{e}^{-\mathrm{i}\psi'' J_3} \tag{5.23}$$

where β'', ϕ'', ψ'' are given in equations (3.14) and (3.18).

Starting from the orthonormal basis vectors $|k,m\rangle$ made up of eigenvectors of J_3, we define an infinite sequence of families of states within $D_k^{(+)}$ by[13]

$$\beta,\phi\in M^{(2)} \to |k,m;\beta,\phi\rangle = U(\beta,\phi)|k,m\rangle \qquad m = k, k+1, \ldots. \tag{5.24}$$

For each m, the states $\{|k,m;\beta,\phi\rangle\}$ are in correspondence with points on $M^{(2)}$; and in place of equation (5.4) we now find

$$(\cosh\beta J_3 + \sinh\beta\sin\phi K_1 - \sinh\beta\cos\phi K_2)|k,m;\beta,\phi\rangle = m|k,m;\beta,\phi\rangle. \tag{5.25}$$

As with equation (5.5) we have

$$\mathrm{e}^{-\mathrm{i}\psi J_3}|k,m;\beta,\phi\rangle = \mathrm{e}^{-\mathrm{i}m\psi}|k,m:\beta,\phi+\psi\rangle. \tag{5.26}$$

We now use the Wigner angle formula (5.23) to evaluate the inner product of two states of the form (5.24) for the same m. From (5.26) it follows that

$$\langle k,m;\beta',\phi'|k,m;\beta,\phi\rangle = \langle k,m;\beta',\phi'-\phi|k,m;\beta,0\rangle. \tag{5.27}$$

This leads to

$$\begin{aligned}\langle k,m;\beta',\phi'-\phi|k,m;\beta,0\rangle &= \langle k,m|U(\beta',\phi'-\phi)^{-1}U(\beta,0)|k,m\rangle\\ &= \langle k,m|U(\beta',\phi'-\phi+\pi)U(\beta,0)|k,m\rangle\\ &= \langle k,m|U(\ell(\beta',\phi'-\phi+\pi)\ell(\beta,0))|k,m\rangle.\end{aligned} \tag{5.28}$$

Using equation (2.28) with the replacement $\phi'\to\phi'-\phi+\pi$ gives the result

$$\begin{aligned}&\ell(\beta',\phi'-\phi+\pi)\ell(\beta,0) = \ell(\beta''',\phi''')r(\psi''')\\ &\cosh\beta''' = \cosh\beta'\cosh\beta - \sinh\beta'\sinh\beta\cos(\phi'-\phi)\\ &\tan\psi'''/2 = \frac{-\sinh\beta'\sinh\beta\sin(\phi'-\phi)}{(1+\cosh\beta')(1+\cosh\beta)-\sinh\beta'\sinh\beta\cos(\phi'-\phi)}.\end{aligned} \tag{5.29}$$

From equations (5.27)–(5.29) we obtain the overlap in question:

$$\begin{aligned}\langle k,m;\beta',\phi'|k,m;\beta,\phi\rangle &= \langle k,m|\,\mathrm{e}^{-\mathrm{i}\beta''' K_1}|k,m\rangle\,\mathrm{e}^{-\mathrm{i}m\psi'''}\\ &= D^{(k)}_{mm}(\beta''')\,\mathrm{e}^{-\mathrm{i}m\psi'''}.\end{aligned} \tag{5.30}$$

Here $D^{(k)}_{mm}(\beta''')$ are the diagonal matrix elements of the infinite-dimensional D-matrices for $SL(2,R)$ in the UIR $D_k^{(+)}$ [19], and they are real positive in the standard form of the UIR since

$$\begin{aligned}\langle k,m|\,\mathrm{e}^{-\mathrm{i}\beta''' K_1}|k,m\rangle &= \langle k,m|\,\mathrm{e}^{+\mathrm{i}\beta''' K_1}|k,m\rangle\\ &= \langle k,m|\,\mathrm{e}^{\pm\mathrm{i}\beta''' K_2}|k,m\rangle.\end{aligned} \tag{5.31}$$

The geometrical meanings of β''', ψ''' in equation (5.29) are revealed by a suitable hyperbolic triangle on $M^{(2)}$, shown in figure 4. The vertices are at $A(\beta=\phi=0)$, $B(\beta,\phi)$ and $C(\beta',\phi')$. Then β''' is the length of the geodesic arc BC and ψ''' is (the negative of) the area of the triangle ABC:

$$\beta''' = BC \qquad \psi''' = -\Omega(A,B,C). \tag{5.32}$$

From equation (5.30) we get the phase information

$$\arg\langle k,m;\beta',\phi'|k,m;\beta,\phi\rangle = m\Omega(A,B,C). \tag{5.33}$$

[13] These are examples of $SL(2,R)$ generalized coherent states in the Perelomov sense.

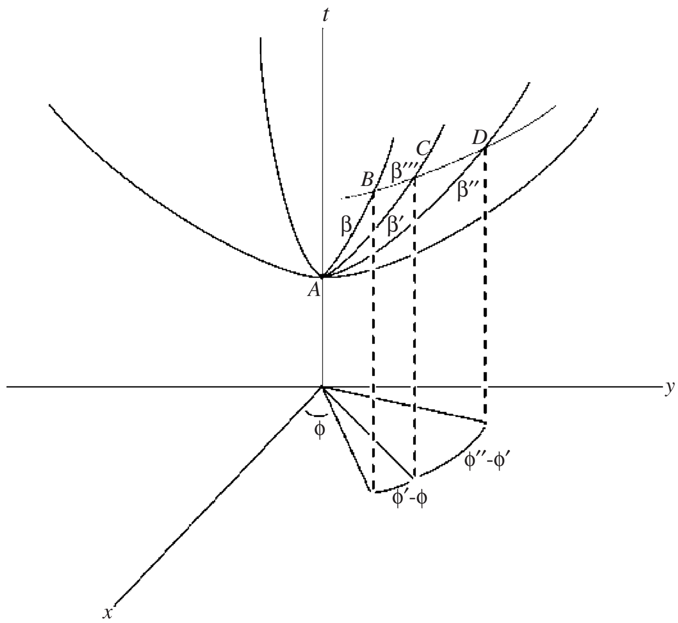

Figure 4. $SL(2, R)$ coherent states on a geodesic on $M^{(2)}$ forming an NPC.

We now use this phase information to show that the states $\{|k, m; \beta, \phi\rangle\}$ lying on a geodesic on $M^{(2)}$ form an NPC. The argument again uses additivity of areas. As in figure 4, consider points $B, C, D, \ldots$ on a geodesic as indicated, with D having parameters $\beta'', \phi'' \ldots$. Then as

$$\Omega(A, B, D) = \Omega(A, B, C) + \Omega(A, C, D) \tag{5.34}$$

we have from (5.33)

$$\begin{aligned} \arg\langle k, m; \beta'', \phi''|k, m; \beta', \phi'\rangle &= m\Omega(A, C, D) \\ &= m\Omega(A, B, D) - m\Omega(A, B, C) \\ &= \arg\langle k, m; \beta'', \phi''|k, m; \beta, \phi\rangle - \arg\langle k, m; \beta', \phi'|k, m; \beta, \phi\rangle. \end{aligned} \tag{5.35}$$

This is for a generic geodesic on $M^{(2)}$, and proves the stated NPC property. The same result follows for the states $|k, m; \beta, \phi\rangle$ on a geodesic $\phi = $ constant.

The connection (4.11) between BIs and GPs now gives us this result: if P, Q, R are any three points on $M^{(2)}$, and we form a hyperbolic triangle with them as vertices, then

$$\begin{aligned} \varphi_{\text{geom}}[\text{hyperbolic triangle } PQR \text{ on } M^{(2)}] &= -\arg\Delta_3(|k, m; P\rangle, |k, m; Q\rangle, |k, m; R\rangle) \\ &= -\arg\langle k, m; P|k, m; Q\rangle\langle k, m; Q|k, m; R\rangle\langle k, m; R|k, m; P\rangle. \end{aligned} \tag{5.36}$$

We now clinch the argument that the $SL(2, R)$ Wigner angle ψ'' in equation (5.23) is a GP. Take the expectation value of both sides of that equation in $|k, m\rangle$ to obtain

$$\langle k, m|U(\beta', \phi')U(\beta, 0)|k, m\rangle = D^{(k)}_{mm}(\beta'')\,\mathrm{e}^{-\mathrm{i}m\psi''}. \tag{5.37}$$

On the left we have the bra vector

$$\begin{aligned} \langle k, m|U(\beta', \phi') &= \langle k, m|U(\beta', \phi' - \pi)^{-1} \\ &= \langle k, m; \beta', \phi' - \pi| \end{aligned} \tag{5.38}$$

so taking the arguments in equation (5.37) gives

$$\arg\langle k,m;\beta,0|k,m;\beta',\phi'-\pi\rangle = m\psi''. \tag{5.39}$$

Since both $\langle k,m;0,0|k,m;\beta,0\rangle$ and $\langle k,m;\beta',\phi'-\pi|k,m;0,0\rangle$ are real positive, we can include them in (5.39) to get

$$\arg\Delta_3(|k,m;00\rangle,|k,m;\beta,0\rangle,|k,m;\beta',\phi'-\pi\rangle) = m\psi''. \tag{5.40}$$

Comparing with the more general equation (5.36), we reach the desired result

$$\varphi_{\text{geom}}[\text{hyperbolic triangle with vertices } |k,m;00\rangle,|k,m;\beta,0\rangle,|k,m;\beta',\phi'-\pi\rangle \text{ on } M^{(2)}] = -m\psi''. \tag{5.41}$$

Again, this matches properly with figure 2.

6. Concluding remarks

Against the background of the general coset space based definition of the concept of Wigner rotation, we have presented a detailed study of the two Abelian cases $SU(2)/U(1)$ and $SL(2,R)/SO(2)$. In both of these, the essential quantity is the Wigner angle, and we have brought out the fact that the two results are related by analytic continuation. This is to counteract the all too frequent implicit assumption that Wigner rotations are relevant only in the context of composing pure Lorentz transformations. The reinterpretation of these Wigner angles as geometric phases follows very similar lines based on spherical and hyperbolic geometries respectively.

It is interesting to point out that each of equations (2.17) and (2.28) for the Wigner angles is used in two ways. One is to compute the inner products among relevant $SU(2)$ generalized coherent states as in equations (5.9) and (5.12), and similarly among $SL(2,R)$ generalized coherent states as in equations (5.30) and (5.33). The other is to exploit the properties of phases of Bargmann invariants and so to establish that these Wigner angles are geometric phases, as shown by equations (5.20), (5.21), (5.36) and (5.40). Thus there is a certain economy in the arguments, with the same group theoretical results leading to two important consequences in each case.

The actions by $SU(2)$ [$SO(3)$] and $SL(2,R)$ [$SO(2,1)$] on the coset spaces S^2 and $M^{(2)}$ are easily visualized as $SO(3)$ rotations and $SO(2,1)$ Lorentz transformations respectively. Moreover these coset spaces carry natural invariant metrics and associated area elements, which coincide with the invariant symplectic two-forms because of the dimensionality being just two. Ultimately geometric phases turn out to be symplectic areas in both cases. The chain of arguments is thus that the (symplectic) area interpretations of the Wigner angles lead to additivity of areas of adjacent triangles bounded by a common geodesic, which leads to geodesics being null phase curves, and hence via Bargmann invariants the link to geometric phases.

The context of generalized coherent states helps to illustrate in a somewhat different manner the three-way connection among geometric phase, Wigner rotation and Bargmann invariants, at least in the Abelian case. In the Perelomov framework of coherent states associated with Lie group representations, each coherent state corresponds to a coset, and choice of phase convention to choice of coset representative. The Perelomov prescription chooses the phases in such a way that every coherent state is in phase with the fiducial state. Then two coherent states are not in phase in general, but differ by a geometric phase or Wigner angle determined, entirely, by the area of the triangle formed by the two coherent states under consideration and the fiducial state, and given by the phase of the three-vertex Bargmann invariant associated with this triplet.

It would be interesting to examine the non-Abelian Wigner rotations from similar points of view. In particular, the $U(n)/U(n-1)$ case is likely to be important in the context of finite-dimensional quantum systems, which are of much current interest in the context of the emerging quantum information theory, though for general n the problem of visualization may not be easy.

Geodesics were traditionally believed to play an important role in computations related to geometric phase, for 'being in phase' in the Pancharatnam sense is an equivalence relation on these curves. It is only recently that it has been realized that curves on which 'being in phase' is an equivalence relation form a much larger family, namely the NPCs: while every geodesic is an NPC, the converse is not true. Much of the detailed geometric phase analysis in the literature has been carried out in the lowest dimensional cases of $SU(2)$, $SL(2,R) \sim SU(1,1)$ and the Heisenberg–Weyl group, but this distinction cannot be expected to show up in these cases wherein every NPC is a geodesic. In *all* higher dimensional cases, however, the set of all NPCs is a much larger family than the set of all geodesics. It will be interesting to see what role this distinction plays on non-Abelian geometric phase or Wigner rotation.

We plan to return to these and related questions elsewhere.

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