Actions and Reality

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A basic constraint on a Minkowski space action is that it be real. An action $\mathcal{S}_m$ is the integral of a Lagrangian (density) L_m over Minkowski space M:

$$S_m = \int_M L_m. \tag{1}$$

Choose a time t on M. Then we (Wick) rotate to Euclidean space E by introducing imaginary time $\tau = it$. By convention the Euclidean action is $\frac{1}{i}$ times the rotated Minkowski action:

$$\frac{1}{i} S_m = S_e = \int_E L_e. \tag{2}$$

None that $e^{iS_m} = e^{-S_e}$. Also, S_e is not real in general.

We describe the continuation to Euclidean space more precisely for a σ-model. The field is a map $\phi : M \longrightarrow Y$ into some Riemannian manifold. The complexification of the space of maps $M \longrightarrow Y$ is the space of *holomorphic* maps $M_{\mathbb{C}} \longrightarrow Y_{\mathbb{C}}$ between the complexified spaces (see Deligne's notes *Real versus complex*). The Lagrangian extends to a holomorphic function on this space, and the Euclidean action is the restriction of this continuation to maps $E \longrightarrow Y$. (Note that $E_{\mathbb{C}} = M_{\mathbb{C}}$ so $E \subset M_{\mathbb{C}}$.) There is a similar picture for other types of fields.

We consider four types of terms which typically occur in an action: kinetic terms for bosons, potential terms, topological terms, and kinetic terms for fermions. For simplicity we discuss these terms in mechanics (the one dimensional case) and then indicate the generalization to higher dimensions.

Let $t, x^1, \dots, x^{n-1}$ be coordinate on M. We use the metric

$$dt^2 - (dx^1)^2 - \dots - (dx^{n-1})^2 \tag{3}$$

on M and the positive definite metric

$$d\tau^2 + (dx^1)^2 + \dots + (dx^{n-1})^2 \tag{4}$$

on E.

Kinetic Terms for Bosons

Consider a particle of mass m moving in some Riemannian manifold Y. It is described by a map $x : \mathbb{R} \longrightarrow Y$. Then the kinetic energy density is

$$(5) \qquad L_m = \frac{m}{2} \left| \frac{dx}{dt} \right|^2 dt.$$

The continuation to imaginary time – after dividing by i – is

$$(6) \qquad L_e = \frac{m}{2} \left| \frac{dx}{d\tau} \right|^2 d\tau.$$

In higher dimensions we might consider a real scalar field on Minkowski space, which is described by a real function $\phi : M \longrightarrow \mathbb{R}$. The kinetic lagrangian is

$$(7) \qquad L_m = \frac{1}{2} \, |d\phi|_M^2 \; dt dx^1 \cdots dx^{n-1},$$

where $|\cdot|_M$ is the norm (3) on M. The continuation to E is

$$(8) \qquad L_e = \frac{1}{2} \, |d\phi|_E^2 \; d\tau dx^1 \cdots dx^{n-1},$$

where $|\cdot|_E$ is the Euclidean norm (4).

Potential Terms

For the particle $x : \mathbb{R} \longrightarrow Y$, the potential energy is described by a function $V : Y \longrightarrow \mathbb{R}$. The corresponding term in the Lagrangian is

$$(9) \qquad L_m = -\, V(x(t)) \, dt.$$

The continuation to imaginary time is

$$(10) \qquad L_e = V(x(\tau)) \, d\tau.$$

The extension to higher dimensions is the same: Potential terms appear with a $-$ sign in Minkowski actions and with a $+$ sign in Euclidean actions.

Topological Terms

Let A be a *real* 1-form on Y and consider the Lagrangian (for $x\colon \mathbb{R}\to Y$)

$$L_m = -x^*A. \tag{11}$$

The corresponding action is invariant under orientation-preserving diffeomorphisms of $\mathbb{R}$, hence the appellation 'topological'. The continuation to imaginary time is innocuous except for the conventional division by i:

$$L_e = ix^*A. \tag{12}$$

Hence in the Euclidean (imaginary time) Lagrangian the topological term is imaginary.

The topological term (11) appears in the description of a charged particle moving in an electromagnetic field. Then A is the "vector potential". (This explains why we write a $-$ sign in (11): the term is part of the potential energy.) In a more geometric formulation, we consider the electromagnetic field to be a connection (on Y = Minkowski space) with gauge group $U(1)$. Relative to a trivialization this is an *imaginary* 1-form α on Y. Here is a difference between most physicists and mathematicians: Physicists write formulas in terms of $A=\pm i\alpha$ whereas mathematicians write formulas in terms of α. In either case the reality condition is clear. The role of the trivialization is a more interesting story ... for another time.

In higher dimensions there is a wide variety of topological terms which appear. Typically they are n–forms $\omega(a)$ constructed from some field(s) a. In Minkowski space $\omega(a)$ is real, and the continuation to Euclidean space is exactly as in (11) and (12).

Kinetic Terms for Fermions

By 'fermions' here we understand any anticommuting variables, i.e., elements of an odd vector space. To understand reality we begin with a general discussion of real structures.

Let A be an ungraded algebra over $\mathbb{C}$. Then a *real structure* on A is a real linear map $a\longmapsto a^*$ which satisfies

$$\begin{aligned} (\lambda a)^* &= \overline{\lambda} a^* &&(\lambda\in\mathbb{C},\quad a\in A),\\ (ab)^* &= b^*a^* &&(a,b\in A),\\ (a^*)^* &= a &&(a\in A). \end{aligned} \tag{13}$$

A familiar example is the quaternion algebra. The algebra of complex $n\times n$ matrices is another example, where $*$ is the conjugate transpose. Notice that simple conjugation of matrix elements does not satisfy $(ab)^*=b^*a^*$. Usually there is only one real structure relevant to a given problem. For example, the "real" matrices—those that satisfy $a^*=a$—have the nice property that their eigenvalues are real and in quantum mechanics they are the operators which correspond to real physical quantities. In general, notice that the real elements form a real vector space $A_{\mathbb{R}}$ which is *not* a subalgebra, though it is closed under anticommutators. Similarly, the imaginary elements form a real vector space closed under brackets, i.e., a real Lie algebra. The space of derivations $\mathrm{Der}(A)$ inherits a real structure from that on A by the rule

$$D^* a = (D a^*)^*. \tag{14}$$

It satisfies

$$[D_1, D_2]^* = [D_1^*, D_2^*]. \tag{15}$$

Now suppose A is a super ($\mathbb{Z}/2$–graded) algebra over $\mathbb{C}$. Denote the parity of a homogeneous element a by $p(a)$. Then a real structure satisfies (13) modified by the sign rule:

$$\begin{aligned} (\lambda a)^* &= \overline{\lambda} a^* && (\lambda \in \mathbb{C}, \quad a \in A), \\ (ab)^* &= (-1)^{p(a)p(b)} b^* a^* && (a, b \in A), \\ (a^*)^* &= a && (a \in A). \end{aligned}$$

Notice in the commutative case that

$$(ab)^* = a^* b^* \qquad (A \textit{ commutative}) \tag{17}$$

The super Lie algebra of derivations $\mathrm{Der}(A)$ inherits a real structure defined as before by (14), and it again satisfies (15).

Many physicists use a convention which omits the sign in (16). This leads to a complication which is explained in a footnote[1] (so as not to confuse the main text with inconsistent formulas).

[1] More explicitly, this alternative convention postulates

$$(ab)^* = b^* a^*, \tag{A}$$

and this differs from (16) if both a and b are odd. As a consequence (14) must be modified to

$$D^* a = (a^* \overleftarrow{D})^*, \tag{B}$$

where $\overleftarrow{D}$ denotes D thought of as acting on the right, which is accomplished via the formula

$$b \overleftarrow{D} = (-1)^{|D|\,|b|} D b. \tag{C}$$

Taken together, (A) and (C) are an inconsistent use of the sign rule. One strange consequence is that for D odd, $D = D^*$ if and only if D maps real even elements to *real* odd elements and D maps real odd elements to *imaginary*

As an example, let V be a real odd vector space and $A = \mathrm{Sym}(V)$ algebra of complex functions on V. (Since V is odd the symmetric algebra is finite dimensional.) Let $\phi^1, \dots, \phi^n$ be a basis of V^*. Then any product $\phi^{\alpha_1} \cdots \phi^{\alpha_k}$ is real, as is the derivation $\partial/\partial\phi^\alpha$.

As another example, consider the superspace $\mathbb{R}^{4|4}$. We usually use a complex basis $\theta^\alpha, \overline{\theta}^{\dot\alpha}$ ($\alpha = 1, 2$; $\dot\alpha = 1, 2$) for the odd coordinates. The conjugate of θ^α is $\overline{\theta}^{\dot\alpha}$. With our conventions, then, the operator

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - \overline{\theta}^{\dot\alpha} \frac{\partial}{\partial x^{\alpha\dot\alpha}}. \tag{18}$$

has conjugate

$$\overline{D}_{\dot\alpha} = \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - \theta^\alpha \frac{\partial}{\partial x^{\alpha\dot\alpha}}. \tag{19}$$

Here $x^{\alpha\dot\alpha}$ are the complex even coordinates on $\mathbb{R}^{4|4}$ induced from the product of spinors in the usual way.

After these preliminaries we return to the particle $x : \mathbb{R} \longrightarrow Y$ and add an odd field ψ which is a section of $x^*\Pi TY$, the parity-reversed pullback of the tangent bundle. The field ψ should be thought of as a spinor on $\mathbb{R}$, but of course the spin bundle on $\mathbb{R}$ is trivial. In any case ψ is real and in real time its kinetic term in the Lagrangian is

$$L_m = \frac{m}{2} \left(\psi, \frac{d\psi}{dt}\right) dt. \tag{20}$$

Rotating to imaginary time and dividing by i, we obtain

$$L_e = -\, i\, \frac{m}{2} \left(\psi \frac{d\psi}{d\tau}\right) d\tau. \tag{21}$$

In higher dimensions suppose ψ is a complex spinor field on Minkowski space. We use a Clifford algebra

$$\text{Minkowski:} \quad \begin{aligned} (\gamma^0)^2 &= 1 \\ (\gamma^i)^2 &= -1 \end{aligned} \tag{22}$$

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even elements.

This convention seems to be in force in most texts and papers on (four dimensional) supersymmetry. Compare (18) and (19) below with the usual definitions in those texts:

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - i\overline{\theta}^{\dot\alpha} \frac{\partial}{\partial x^{\alpha\dot\alpha}}, \\ \overline{D}_{\dot\alpha} &= -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + i\theta^\alpha \frac{\partial}{\partial x^{\alpha\dot\alpha}} \end{aligned}$$

where γ^0 is associated with time and γ^i with space. Let

$$\not{D}_m = \gamma^0 \frac{\partial}{\partial t} + \gamma^i \frac{\partial}{\partial x^i} \tag{23}$$

be the associated Dirac operator. Then $\not{D}_m$ is *skew-adjoint*. Let $\overline{\psi}$ be the conjugate spinor to ψ. Then the Lagrangian

$$L_m = \overline{\psi} \cdot \not{D}_m \psi \, dt \, dx^1 \cdots dx^{n-1} \tag{24}$$

is real, where we use a bilinear pairing on the spinor fields. In Euclidean space we use a Clifford algebra

$$\text{Euclidean:} \quad (\Gamma^\mu)^2 = -1 \tag{25}$$

and *self–adjoint* Dirac operator

$$\not{D}_e = \Gamma^0 \frac{\partial}{\partial \tau} + \Gamma^i \frac{\partial}{\partial x^i}. \tag{26}$$

Then the continuation of (24) to Euclidean space is

$$L_e = \overline{\psi} \cdot \not{D}_e \, \psi. \tag{27}$$

One should only take (24) and (27) as general guidelines; the particulars of spinors in each dimension should be considered.

Summary

For reference we collect the various types of terms in the real and imaginary time mechanics lagrangians:

$$L_m = \left\{ \frac{m}{2} \left| \frac{dx}{dt} \right|^2 + \frac{m}{2} \left(\psi, \frac{d\psi}{dt} \right) - V(x) \right\} dt - x^* A, \tag{28}$$

$$L_e = \left\{ \frac{m}{2} \left| \frac{dx}{d\tau} \right|^2 - i \, \frac{m}{2} \left(\psi, \frac{d\psi}{d\tau} \right) + V(x) \right\} d\tau + i \, x^* A. \tag{29}$$