Generic warp drives violate the null energy condition

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Abstract:

Three very recent articles have claimed that it is possible to, at least in theory, either set up positive energy warp drives satisfying the weak energy condition (WEC), or at the very least, to minimize the WEC violations. These claims are at best incomplete, since the arguments presented only demonstrate the existence of *one* set of timelike observers, the co-moving Eulerian observers, who see "nice" physics. While these observers might see a positive energy density, the WEC requires *all* timelike observers to see positive energy density. Therefore, one should revisit this issue. A more careful analysis shows that the situation is actually much grimmer than advertised — all physically reasonable warp drives will violate the null energy condition, and so also automatically violate the WEC, and both the strong and dominant energy conditions. While warp drives are certainly interesting examples of speculative physics, the violation of the energy conditions, at least within the framework of standard general relativity, is unavoidable. Even in modified gravity, physically reasonable warp drives will still violate the purely geometrical null convergence condition and the timelike convergence condition which, in turn, will place very strong constraints on any modified-gravity warp drive.

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Contents

1 Introduction

Three very recent articles [1–3] have argued for the existence of "physically reasonable" positive-energy warp drive configurations, either satisfying the weak energy condition (WEC), or minimizing violations thereof. This is in sharp contrast to the 25-year-old consensus opinion that at least some energy condition violations are necessary for the generation of warp fields [4–24]. Specifically, known results already include:

- Alcubierre warp drives are known to violate the WEC [4], and have more recently been shown to violate the null energy condition (NEC) [24].
- Natário zero-expansion warp drives are known to violate the WEC [5].
- Generic Natário warp drives are known to violate the dominant energy condition (DEC) [10], and have also been shown to violate either the strong energy condition (SEC) or the WEC [5].

We shall strengthen these results below, first showing that the Natário zero-expansion warp drives also violate the SEC, and the NEC, and then ultimately showing that generic Natário warp drives (including the variants discussed in references [1–3]) violate the NEC, and consequently also violate the WEC. How, then, is this result compatible with the various bold claims made in references [1–3]? (And, for that matter, various parts of the popular press.) The key observation is that WEC requires *all* timelike observers to see positive energy density, whereas the analyses of references [1–3] only investigate the energy density as seen by *one class* of timelike observers (the co-moving Eulerian observers). Thus the claims made in references [1–3] are at best incomplete, and in many key specific details, wrong.

To set the stage, in section 2 we shall first discuss the kinematics of generic Natário warp drive spacetimes, and then calculate the relevant spacetime curvature (Riemann, Ricci, and Einstein tensors) in section 3. Assuming standard general relativity, we calculate in section 4 all the components of the generic warp-field stress-energy tensor. In section 5, we then provide a number of general results regarding the implications of the point-wise energy conditions, and relate them to the null and timelike convergence conditions in section 6. With those preliminaries out of the way, the actual proof of energy condition violations in warp-field spacetimes will be straightforward (section 7). We conclude with some comments on the possibility of either moving beyond Einstein gravity, or the possibility of further generalizing the notion of warp-field spacetimes. Nevertheless, within the context of standard Einstein gravity, we emphasize that in physically reasonable warp field spacetimes energy condition violations are utterly unavoidable, and must be squarely faced.

2 Warp drive kinematics

The generic Natário warp drive line element we shall consider is this [5]:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \delta_{ij}\,\left(\mathrm{d}x^i - v^i(x,y,z,t)\,\mathrm{d}t\right)\,\left(\mathrm{d}x^j - v^j(x,y,z,t)\,\mathrm{d}t\right). \tag{2.1}$$

This line element is sufficiently general to cover well over 99% of the relevant literature, and the very few exceptions will be explicitly discussed later on in appendix B.

The line element represents an ADM-like (3+1) decomposition of the metric [25–31], with unit lapse $N \to 1$; a flow vector $v^i = -$(shift vector); and a flat spatial 3-metric $g_{ij} = \delta_{ij}$. The sign-flip on the shift vector is traditional in a warp drive context, and inspired by the notion of "flow" rather than "shift". Accordingly we shall speak of the "flow vector" rather than the "shift vector". (See also the traditional usage in the "analogue spacetime" programme [32–36].) The metric components are

$$g_{ab} = \left[\begin{array}{c|c} -(1-v^2) & -v_j \\ \hline -v_i & \delta_{ij} \end{array}\right], \qquad g^{ab} = \left[\begin{array}{c|c} -1 & -v^j \\ \hline -v^i & \delta^{ij} - v^i v^j \end{array}\right]. \tag{2.2}$$

The metric signature is manifestly $-+++$, while spacetime indices such as a, b, ... run from 0 to 3, and spatial indices such as i, j... run from 1 to 3. At large spatial distances we want the spacetime to be asymptotically flat, (that is, asymptotically Minkowski or at worst asymptotically Schwarzschild), and the most natural boundary conditions to impose are that $v^i(x,y,z,t) \to 0$ at spatial infinity. One could alternatively impose $v^i(x,y,z,t) \to v^i_*(t)$ at spatial infinity, with $v^i_*(t)$ some function of time only. This would still be asymptotically Minkowski, but with new spatial coordinates $x^i \to \bar{x}^i = x^i - \int v^i_*(t)\,\mathrm{d}t$. This would be useful, for instance, if one wishes to adopt a coordinate system moving with the warp bubble. On the other hand there is certainly no loss of generality in enforcing $v^i(x,y,z,t) \to 0$ at spatial infinity.

Because the spatial 3-slices are intrinsically flat, and the lapse is unity, the only non-trivial physics hides in the extrinsic curvature of the spatial 3-slices, for which we have the particularly simple result

$$K_{ij} = v_{(i,j)}. \tag{2.3}$$

In order to prevent the warp field being trivial (Minkowski space) we will demand that there is at least one point where the extrinsic curvature is nonzero.

Let us define $n_a = -\nabla_a t = (-1;0,0,0)_a$. Then we have $n^a = g^{ab} n_b = -g^{at} = (1, v^i)$. Thus n^a is future-pointing. It is actually a future-pointing 4-velocity normal to the spatial 3-slices: $g^{ab}\,n_a n_b = -1 = g_{ab}\,n^a n^b$.

Observers with 4-velocity n^a are often called "Eulerian". In some sense (see discussion below) they "go with the flow", they are "co-moving". Furthermore, the Eulerian observers $n^a = -g^{ab}\,\nabla_b t$ are in fact timelike *geodesics*. This is a simple consequence of the fact that the warp spacetimes are unit-lapse:

$$n^b \nabla_b n^a = g^{ac} n^b \nabla_b \nabla_c t = g^{ac} n^b \nabla_c \nabla_b t = g^{ac} n^b \nabla_c n_b = \frac{1}{2} g^{ac} \nabla_c (n^b n_b) = \frac{1}{2} g^{ac} \nabla_c (-1) = 0. \tag{2.4}$$

The warp drive spacetime is by construction globally hyperbolic, and at each and every instant in time "t" the Eulerian observers are 4-orthogonal to the flat spatial slices — so the Eulerian observers define a zero-vorticity congruence of timelike geodesics that by construction *cannot* have any focussing points.[1]

From the metric one can easily read off a suitable choice of co-tetrad: The timelike covector is simply

$$(e^{\hat{0}})_a = (1; 0, 0, 0)_a = n_a, \tag{2.5}$$

while the spatial co-triad is

$$(e^{\hat{1}})_a = (-v_x; 1, 0, 0)_a, \quad (e^{\hat{2}})_a = (-v_y; 0, 1, 0)_a, \quad (e^{\hat{3}})_a = (-v_z; 0, 0, 1)_a. \tag{2.6}$$

The corresponding tetrad is then easily determined: The timelike leg is

$$(e_{\hat{0}})^a = (1; v^x, v^y, v^z)^a = n^a, \tag{2.7}$$

while now the spatial triad is particularly simple

$$(e_{\hat{1}})^a = (0; 1, 0, 0)^a, \quad (e_{\hat{2}})^a = (0; 0, 1, 0)^a, \quad (e_{\hat{3}})^a = (0; 0, 0, 1)^a. \tag{2.8}$$

This implies that, in the co-moving orthonormal basis, for any T^0_2 tensor:

$$X_{\hat{a}\hat{b}} = e_{\hat{a}}{}^a \; e_{\hat{b}}{}^b \; X_{ab} = \left[\begin{array}{c|c} X_{\hat{0}\hat{0}} & X_{\hat{0}\hat{j}} \\ \hline X_{\hat{i}\hat{0}} & X_{\hat{i}\hat{j}} \end{array}\right] = \left[\begin{array}{c|c} X_{nn} & X_{nj} \\ \hline X_{in} & X_{ij} \end{array}\right] = \left[\begin{array}{c|c} X_{ab} n^a n^b & X_{aj} n^a \\ \hline X_{ia} n^a & X_{ij} \end{array}\right]. \tag{2.9}$$

This is why objects such as $X_{nn} = X_{ab}\, n^a\, n^b = X_{\hat{0}\hat{0}}$ and $X_{ni} = X_{ai}\, n^a = X_{\hat{0}\hat{i}}$ are so important. Subject to this choice of coordinates and tetrad, for the covariant spatial components we have the particularly simple result $X_{\hat{i}\hat{j}} = X_{ij}$. We will often use this observation to simplify formulae by suppressing the "hats" when they are not critical to understanding.

[1] Everett [6] takes the point of view that one might take this warp drive metric as a local notion only, thus avoiding the global hyperbolicity conditions of the original Alcubierre space-time, and uses that to construct closed timelike curves (CTCs). This introduces a whole new level of complexity related to global causality and "chronology protection", of which more anon.

Similarly, for any fully covariant T^0_4 tensor one has $X_{\hat{a}\hat{b}\hat{c}\hat{d}} = e_{\hat{a}}{}^a\, e_{\hat{b}}{}^b\, e_{\hat{c}}{}^c\, e_{\hat{d}}{}^d\, X_{abdc}$. Then for any tensor that has the same symmetries as the Riemann tensor it suffices to calculate

$$X_{\hat{0}\hat{i}\hat{0}\hat{j}} = X_{ninj} = n^a n^b X_{aibj}; \qquad X_{\hat{0}\hat{i}\hat{j}\hat{k}} = X_{nijk} = n^a X_{aijk}; \qquad X_{\hat{i}\hat{j}\hat{k}\hat{l}} = X_{ijkl}. \tag{2.10}$$

Some specific examples of the generic warp drive spacetime are:

Alcubierre warp field [4]:

The original Alcubierre warp field, taken to be moving in the z direction with constant velocity v_*, is given by

$$v^i(x,y,z,t) = (0,0,1)^i \;\; v_* \; f\left(\sqrt{x^2+y^2+(z-v_*t)^2}\right). \tag{2.11}$$

Here, $f(0) = 1$, and $f(\infty) = 0$.

This was rapidly generalized to a time-dependent velocity for the warp bubble

$$v^i(x,y,z,t) = (0,0,1)^i \;\; v_*(t) \; f\left(\sqrt{x^2+y^2+\left(z-\int v_*(t)\,\mathrm{d}t\right)^2}\right). \tag{2.12}$$

Note these specific models are "spherically symmetric". More on this point below. Despite assertions in reference [2] there is absolutely no difficulty in making the velocity of the warp bubble time-dependent.

Natário zero-expansion warp field [5]:

The Natário zero-expansion warp drive simply sets $\boldsymbol{\nabla}\cdot\vec{v} = 0$.

Lentz/Fell–Heisenberg zero-vorticity warp field [1, 3]:

The Lentz/Fell–Heisenberg zero-vorticity warp drive uses a purely gradient flow $\vec{v} = \boldsymbol{\nabla}\Phi$, implying $\vec{\omega} = \boldsymbol{\nabla}\times\vec{v} = 0$.

Thus the generic warp drive line element (2.1), introduced by Natário in [5], covers all three of these specific cases, and many more, and is sufficiently general to cover almost all of the relevant literature.

3 Warp-field curvature

The spacetime curvature for warp drive spacetimes is relatively easily determined via a specific application of the ADM formalism [25–31].

3.1 Riemann tensor

The Gauss–Codazzi equations would in general imply[2]:

$$R_{\hat{i}\hat{j}\hat{k}\hat{l}} = {}^{(3)}R_{\hat{i}\hat{j}\hat{k}\hat{l}} + K_{\hat{i}\hat{k}}K_{\hat{j}\hat{l}} - K_{\hat{i}\hat{l}}K_{\hat{j}\hat{k}}. \tag{3.1}$$

However, since our 3-geometry is flat, we have ${}^{(3)}R_{\hat{i}\hat{j}\hat{k}\hat{l}} \to 0$, and because of our choice of spatial triad we can dispense with the "hats". So for warp drive spacetimes we simply have

$$R_{ijkl} = K_{ik}K_{jl} - K_{il}K_{jk}. \tag{3.2}$$

The Gauss–Mainardi equations imply[3]:

$$R_{nijk} = R_{aijk}n^a = K_{ij,k} - K_{ik,j} = v_{(i,j),k} - v_{(i,k),j} = v_{[j,k],i}. \tag{3.3}$$

The remaining components of the Riemann tensor do not (to our knowledge) seem to have a special name, and are a little trickier to calculate. In the present context:

$$R_{ninj} = R_{aibj}\, n^a\, n^b = -\mathcal{L}_n K_{ij} + (K^2)_{ij}. \tag{3.4}$$

Here $\mathcal{L}_n K_{ij}$ is the Lie derivative, where $\dot{K}_{ij} = \partial_t K_{ij}$, and

$$\mathcal{L}_n K_{ij} = \dot{K}_{ij} + v^k \partial_k K_{ij} + \partial_i v^k K_{kj} + \partial_j v^k K_{ik}. \tag{3.5}$$

Also

$$(K^2)_{ij} = K_{ik}\, \delta^{kl}\, K_{lj}. \tag{3.6}$$

To explicitly calculate R_{ninj} one could either use brute force (Maple), or appeal to a simplification of the ADM formalism,[4] or use the general commutator identity

$$[\nabla_a, \nabla_b] n_c = -R^d{}_{cab}\, n_d, \tag{3.7}$$

suitably projected onto spatial and normal n directions.

[2]See for instance equation (2.92) of Gourgoulhon [26].

[3]See for instance equation (2.101) of Gourgoulhon [26], specializing to $K_{ij} = v_{(i,j)}$.

[4]For instance, use equation (3.43) of Gourgoulhon [26], specializing to $N \to 1$ and flipping the sign of the Lie derivative term to account for (flow) = −(shift).

3.2 Ricci tensor

Performing suitable contractions, for the Ricci tensor we find

$$R_{nn} = -\mathcal{L}_n K - \mathrm{tr}\big(K^2\big), \tag{3.8}$$

where $\mathcal{L}_n K$ is the Lie derivative of $K = K_{ij}\,\delta^{ij} = \mathrm{tr}(K)$, the trace of the extrinsic curvature. That is, $\mathcal{L}_n K = \dot{K} + v^i \partial_i K = n^a\,\partial_a K$. Furthermore,

$$R_{ni} = K_{ij,k}\,\delta^{jk} - K_{,i}. \tag{3.9}$$

In view of the fact that in this context spatial indices are always raised and lowered using Kronecker deltas, we can and shall often simplify notation by contracting over two indices down and writing

$$R_{ni} = K_{ij,j} - K_{,i}. \tag{3.10}$$

Finally,

$$R_{ij} = \mathcal{L}_n K_{ij} + K K_{ij} - 2(K^2)_{ij}. \tag{3.11}$$

For the Ricci scalar

$$R = 2\mathcal{L}_n K + K^2 + \mathrm{tr}\big(K^2\big). \tag{3.12}$$

Here, in full, $\mathrm{tr}(K^2) = (K^2)_{ij}\,\delta^{ij} = K_{ij}\,\delta^{ik}\,\delta^{jl}\,K_{kl}$, though we can and shall often simplify notation by writing $\mathrm{tr}(K^2) = K_{ij}\,K_{ij}$, implying contraction on the repeated down indices.

3.3 Einstein tensor

Using $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ we find

$$G_{nn} = \frac{1}{2}\left(K^2 - \mathrm{tr}\big(K^2\big)\right), \tag{3.13}$$

$$G_{ni} = K_{ij,j} - K_{,i}, \tag{3.14}$$

and the somewhat messier result that

$$G_{ij} = \mathcal{L}_n K_{ij} + K K_{ij} - 2(K^2)_{ij} - \left(\mathcal{L}_n K + \frac{1}{2} K^2 + \frac{1}{2}\,\mathrm{tr}\big(K^2\big)\right)\delta_{ij}. \tag{3.15}$$

4 Warp-field stress-energy tensor

In this section, we use the standard Einstein equations $G_{ab} = 8\pi T_{ab}$ and analyse the density, flux, and spatial components of the stress-energy.

4.1 Density

In the Eulerian orthonormal basis, using (3.13), for the density we have:

$$\rho = \frac{G_{nn}}{8\pi} = \frac{1}{16\pi}\left(K^2 - \mathrm{tr}\big(K^2\big)\right). \tag{4.1}$$

(Everyone agrees with this. This is the quantity that Lentz [1] and Fell–Heisenberg [3] eventually try to force to be positive. This is the only quantity Bobrick–Martire [2] calculate.) In the present context

$$\rho = \frac{1}{16\pi}\left((v_{i,i})^2 - v_{(i,j)}\, v_{(i,j)}\right). \tag{4.2}$$

It is straightforward to rewrite this as a spatial divergence plus a negative semi-definite contribution:

$$\rho = \frac{1}{16\pi}\left\{\partial_i(v_i\, v_{j,j} - v_j\, v_{i,j}) - v_{[i,j]}v_{[i,j]}\right\}. \tag{4.3}$$

In terms of the vorticity $\omega_i = \epsilon_{ijk}\, v_{[j,k]}$ this becomes

$$\rho = \frac{1}{16\pi}\left\{\partial_i(v_i\, v_{j,j} - v_j\, v_{i,j}) - \frac{1}{2}\,(\omega_i\,\omega_i)\right\}, \tag{4.4}$$

or even

$$\rho = \frac{1}{16\pi}\left\{\boldsymbol{\nabla}\cdot\{\vec{v}\,K - (\vec{v}\cdot\boldsymbol{\nabla})\vec{v}\} - \frac{1}{2}\,(\vec{\omega}\cdot\vec{\omega})\right\}. \tag{4.5}$$

We will have occasion to use this result in subsequent discussion.

4.2 Flux

For the flux, in the Eulerian orthonormal basis, using (3.14) we have:

$$f_i = \frac{G_{ni}}{8\pi} = \frac{1}{8\pi}\,(K_{ij,j} - K_{,i}) = \frac{1}{16\pi}\,(v_{i,jj} - v_{j,ji}) = \frac{1}{16\pi}\left(\boldsymbol{\nabla}^2 v_i - \boldsymbol{\nabla}_i(\boldsymbol{\nabla}\cdot\vec{v})\right) \tag{4.6}$$

That is

$$f_i = \frac{1}{16\pi}\,(\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\vec{v}))_i \tag{4.7}$$

(Everyone agrees with this. Note that in the Lentz/Fell-Heisenberg [1, 3] framework $\vec{v} = \boldsymbol{\nabla}\Phi$ is a gradient, so this co-moving Eulerian flux is identically zero.)

4.3 Spatial stresses

Finally, using (3.15), the spatial stresses are given by the 3×3 matrix:

$$T_{ij} = \frac{G_{ij}}{8\pi} = \frac{1}{8\pi}\left(\mathcal{L}_n K_{ij} + K K_{ij} - 2(K^2)_{ij} - \left(\mathcal{L}_n K + \frac{1}{2}K^2 + \frac{1}{2}\,\mathrm{tr}\big(K^2\big)\right)\delta_{ij}\right). \quad (4.8)$$

So for the average pressure, which we define to be $\bar{p} = \frac{1}{3}T_{ij}\,\delta^{ij}$, and noting that

$$\big(\mathcal{L}_n K_{ij} + K K_{ij} - 2(K^2)_{ij}\big)\,\delta^{ij} = \mathcal{L}_n K + K^2 - 2\,\mathrm{tr}\big(K^2\big) - K_{ij}\,\mathcal{L}_n \delta^{ij} = \mathcal{L}_n K + K^2, \quad (4.9)$$

we find

$$\bar{p} = \frac{1}{3}\,T_{ij}\,\delta^{ij} = \frac{1}{24\pi}\left(-2\mathcal{L}_n K - \frac{1}{2}K^2 - \frac{3}{2}\,\mathrm{tr}\big(K^2\big)\right). \quad (4.10)$$

It is furthermore useful to note that

$$\rho + 3\bar{p} = -\frac{1}{4\pi}\left(\mathcal{L}_n K + \mathrm{tr}\big(K^2\big)\right), \quad (4.11)$$

and that

$$\rho + \bar{p} = \frac{1}{24\pi}\left(-2\mathcal{L}_n K + K^2 - 3\,\mathrm{tr}\big(K^2\big)\right). \quad (4.12)$$

These quantities will soon be seen to be useful when investigating violations of the SEC and NEC, respectively.

It is sometimes useful to also note that

$$\nabla_a n^a = \partial_a n^a = K, \quad (4.13)$$

and so write

$$\mathcal{L}_n K = n^a\,\nabla_a K = \nabla_a(K n^a) - K^2, \quad (4.14)$$

whence

$$\bar{p} = -\frac{1}{24\pi}\left(2\nabla_a(K n^a) - \frac{3}{2}K^2 + \frac{3}{2}\,\mathrm{tr}\big(K^2\big)\right). \quad (4.15)$$

Equivalently

$$\bar{p} = \rho - \frac{1}{12\pi}\,\nabla_a(K n^a). \quad (4.16)$$

5 Energy conditions

The (classical point-wise) energy conditions are constraints one places on the stress-energy as a way of keeping unusual physics somewhat under control. The energy conditions all correspond, in some sense, to demanding that for some *class* of observers the energy density be non-negative. Standard definitions are [26–28, 37–43]:

NEC: For all null vectors k^a we demand $T_{ab}k^a k^b \geq 0$.

WEC: For all timelike vectors V^a we demand $T_{ab}V^a V^b \geq 0$.

SEC: For all timelike vectors V^a we demand $(T_{ab} - \frac{1}{2}T g_{ab})V^a V^b \geq 0$.

DEC: For all future-pointing timelike vectors V^a and W^a we demand $T_{ab}V^a W^b \geq 0$. (The DEC in particular has a number of equivalent formulations [40].)

The standard linkages between energy conditions are [26–28, 37–43]:

$$\text{DEC} \implies \text{WEC} \implies \text{NEC} \tag{5.1}$$

$$\text{DEC} \implies \text{SEC} \implies \text{NEC} \tag{5.2}$$

But

$$\text{WEC} \not\Longleftrightarrow \text{SEC} \tag{5.3}$$

Despite claims made in [3] the WEC is not the weakest of the energy conditions.

Despite the claim in [3] it is not possible to satisfy the DEC while violating the SEC.

Some examples of the subtleties involved are:

- A positive cosmological constant (positive vacuum energy density) satisfies the NEC and WEC but not the SEC.
- A negative cosmological constant (negative vacuum energy density) satisfies the NEC and SEC but not the WEC.
- Massive scalar fields, (classical, minimally coupled, with positive kinetic energy, and positive mass squared so that they are not tachyonic), satisfy the NEC and WEC but can violate the SEC. For example the standard inflaton of cosmological inflation satisfies the NEC and WEC but violates the SEC.

These examples suggest some caution when interpreting the physical significance of the energy conditions [44–53].

There are indications that NEC and WEC can be violated on microscopic (quantum) scales [54–72], though they seem to be satisfied by (reasonable) matter on macroscopic scales. In contrast SEC and DEC appear to be observationally violated on the largest cosmological scales [46–52].

While for the purposes of this paper we will be focussing on applying these point-wise energy conditions specifically to warp drive spacetimes, it should be noted that these energy conditions also have direct applications to singularity theorems, positive-mass theorems, traversable wormholes [73–82], and chronology protection [84–88].

Other less commonly used energy conditions are:

- TEC — trace energy condition: $\rho - 3p \geq 0$. Mainly of historic interest [44, 45]. Believed to be *violated* deep in the cores of neutron stars. Definitely violated by (hypothetical) "stiff matter" $\rho = p$, where (speed of sound) = (speed of light).
- FEC — flux energy condition: for all timelike observers, V^a, the flux $F^a = T^{ab}V_b$ is either timelike or null [54, 55].
  (FEC is a weakening of DEC; that is, DEC $\implies$ FEC.)
- Averaged energy conditions (typically averaged along timelike or null geodesics) are weaker than their corresponding point-wise counterparts. The most useful of the averaged energy conditions is typically the ANEC (averaged null energy condition) where one averages the NEC along timelike geodesics *using an affine parameterization* [38, 66–72].

As a cautionary example regarding application of the energy conditions, note that all three of the recent Lentz [1], Bobrick–Martire [2], and Fell–Heisenberg [3] articles merely find *one* sub-class of timelike observers for which the energy density is positive. This is *not* enough to show that the WEC is satisfied. To explicitly see a specific example of this behaviour, let us work in an orthonormal basis. Take $\rho_0 > 0$ and $\Gamma > 1$, and consider:

$$T_{\hat{a}\hat{b}} = \rho_0 \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & -\Gamma^2 & 0 & 0 \\ 0 & 0 & -\Gamma^2 & 0 \\ 0 & 0 & 0 & -\Gamma^2 \end{array}\right]_{\hat{a}\hat{b}} \tag{5.4}$$

Then in the natural rest frame $V^{\hat{a}} = (1; 0, 0, 0)^{\hat{a}}$ we have $\rho = T_{\hat{a}\hat{b}}V^{\hat{a}}V^{\hat{b}} = \rho_0 > 0$. But a moving observer, with 4-velocity $V^{\hat{a}} = \gamma(1; v\, n^i)^{\hat{a}}$, where n^i is any unit spatial 3-vector,

will see an energy density

$$\rho = T_{\hat{a}\hat{b}}\, V^{\hat{a}} V^{\hat{b}} = \rho_0 \gamma^2 (1 - \Gamma^2 v^2). \tag{5.5}$$

For a sufficiently rapidly moving but still subluminal observer, with $|v| > 1/\Gamma$, the energy density will be seen to be negative. So in this example the WEC is *violated*. What Lentz [1], and Bobrick–Martire [2], and Fell–Heisenberg [3] *should* be doing is to also calculate all of the stress components T_{ij}. It is not enough for them to just focus on T_{nn} and T_{ni}.

One way of proceeding is to assume the stress-energy tensor is of Hawking–Ellis type I, see references [37–39, 55, 89–93], and work with the Lorentz-invariant eigenvalues $\{\rho, p_i\}$, the Lorentz-invariant eigen-energy-density and principal pressures, respectively.

Then it is a standard result that in terms of the Lorentz-invariant eigenvalues of a type I stress-energy tensor one has the two-way implications [37–39, 55, 89–93]

$$\text{NEC} \iff \rho + p_i \geq 0. \tag{5.6}$$

$$\text{WEC} \iff \rho + p_i \geq 0 \quad \& \quad \rho > 0. \tag{5.7}$$

$$\text{SEC} \iff \rho + p_i \geq 0 \quad \& \quad \rho + \textstyle\sum_i p_i > 0. \tag{5.8}$$

$$\text{DEC} \iff |p_i| < \rho. \tag{5.9}$$

Unfortunately, we do not know *a priori* whether or not the stress-energy for the warp drive is Hawking–Ellis type I, nor do the Eulerian observers typically diagonalize the stress-energy. So one has to be a little more indirect. What the Eulerian observers do implement is a natural orthonormal frame in which

$$T_{\hat{a}\hat{b}} = \left[\begin{array}{c|c} \rho & f_j \\ \hline f_i & T_{ij} \end{array}\right]. \tag{5.10}$$

NEC: Let us first investigate what we can say about the NEC. Let us take any two oppositely oriented null vectors $\ell_+^{\hat{a}} = (1, +\ell^i)^{\hat{a}}$, and $\ell_-^{\hat{a}} = (1, -\ell^i)^{\hat{a}}$, where ℓ^i is any arbitrary unit spatial 3-vector. Then the NEC would imply *both*

$$T_{\hat{a}\hat{b}}\, \ell_+^{\hat{a}} \ell_\pm^{\hat{b}} = T_{\hat{a}\hat{b}}\, (1, +\ell^i)^{\hat{a}} (1, +\ell^j)^{\hat{b}} = \rho + 2 f_i \ell^i + T_{ij} \ell^i \ell^j \geq 0, \tag{5.11}$$

and

$$T_{\hat{a}\hat{b}}\, \ell_-^{\hat{a}} \ell_-^{\hat{b}} = T_{\hat{a}\hat{b}}\, (1, -\ell^i)^{\hat{a}} (1, -\ell^j)^{\hat{b}} = \rho - 2 f_i \ell^i + T_{ij} \ell^i \ell^j \geq 0. \tag{5.12}$$

Averaging over these two equations, for any unit spatial 3-vector

$$\text{NEC} \implies \rho + T_{ij}\ell^i\ell^j \geq 0. \tag{5.13}$$

Note the implication is one-way; effectively one throws away all information contained in the flux f^i. Now pick a triad ℓ^i_A of three mutually orthogonal unit vectors. Then for each member of the triad the NEC implies

$$\rho + T_{ij}\,\ell^i_A\,\ell^j_A \geq 0. \tag{5.14}$$

Now average over the three members of the triad

$$\rho + T_{ij}\left(\frac{1}{3}\sum_A \ell^i_A\ell^j_A\right) \geq 0. \tag{5.15}$$

Thence, noting that by construction $\sum_A \ell^i_A\ell^j_A = \delta^{ij}$, we see

$$\rho + T_{ij}\left(\frac{1}{3}\delta^{ij}\right) \geq 0. \tag{5.16}$$

Defining, as usual, the average pressure as $\bar{p} = \frac{1}{3}T_{ij}\,\delta^{ij}$, (this works even if the 3-stress T_{ij} is not diagonal), we see that

$$\text{NEC} \implies \rho + \bar{p} \geq 0. \tag{5.17}$$

The implication is again one-way. Note also that this argument has nothing specific to do with warp drives, it is purely a statement about how the NEC implies an easily checked inequality that depends only on the existence of some orthonormal basis. For our current purposes the point is that it is relatively easy to pick a specific direction and calculate $T_{ij}\,\ell^i\ell^j$, or to average over all directions and calculate $\bar{p}$.

WEC: Similar things can be said about the WEC, although now one takes some $0 \leq \beta < 1$ and considers two oppositely oriented timelike vectors $V^a_\pm = \gamma(1, \pm\beta\ell^i)^a$. Then for any unit 3-vector ℓ^i, after averaging over the two orientations:

$$\text{WEC} \implies \rho + \beta^2\,T_{ij}\,\ell^i\ell^j \geq 0. \tag{5.18}$$

Then averaging over a triad of unit vectors, for all $0 \leq \beta < 1$ one has

$$\text{WEC} \implies \rho + \beta^2\,\bar{p} \geq 0. \tag{5.19}$$

By considering $\beta = 0$ and the limit $\beta \to 1$ one has

$$\text{WEC} \implies \rho \geq 0 \quad \& \quad \rho + \bar{p} \geq 0. \tag{5.20}$$

The implication is again one-way. And the point is that $T_{ij}\,\ell^i \ell^j$ and $\bar{p}$ are relatively easy to calculate.

SEC: The SEC can be phrased in terms of the so-called "trace-reversed" stress-energy tensor $T_{ab} - \frac{1}{2} T g_{ab}$, as the condition $(T_{ab} - \frac{1}{2} T g_{ab}) V^a V^b \geq 0$, which in turn is equivalent to enforcing $T_{ab} V^a V^b \geq \frac{1}{2}(\rho - 3\bar{p})$. Then the SEC implies

$$\rho + \beta^2\, T_{ij}\, \ell^i \ell^j \geq (1 - \beta^2)\, \frac{1}{2}\, (\rho - 3\bar{p}). \tag{5.21}$$

This can be rearranged to

$$\text{SEC} \implies (1 + \beta^2)\rho + 3(1 - \beta^2)\bar{p} + 2\beta^2\, T_{ij}\, \ell^i \ell^j \geq 0. \tag{5.22}$$

Averaging over the unit directions ℓ^i, for all $0 \leq \beta < 1$ one has

$$\text{SEC} \implies (1 + \beta^2)\rho + (3 - \beta^2)\bar{p} \geq 0. \tag{5.23}$$

By considering $\beta = 0$ and the limit $\beta \to 1$ one has

$$\text{SEC} \implies \rho + 3\bar{p} \geq 0 \quad \& \quad \rho + \bar{p} \geq 0. \tag{5.24}$$

The implication is again one-way. And the point is that $T_{ij}\,\ell^i \ell^j$ and $\bar{p}$ are relatively easy to calculate.

DEC: One version of the DEC, as reported by Hawking and Ellis [37, page 91], is formulated as the requirement that in any orthonormal frame the energy density dominates all the other components of the stress-energy tensor:

$$|T^{\hat{a}\hat{b}}| \leq T^{\hat{t}\hat{t}}. \tag{5.25}$$

In our language this would be

$$|f_i| \leq \rho \quad \& \quad |T_{ij}| \leq \rho. \tag{5.26}$$

But in particular this implies

$$|\bar{p}| = \left| \frac{1}{3} \sum_i T_{ii} \right| \leq \frac{1}{3} \sum_i |T_{ii}| \leq \rho. \tag{5.27}$$

That is

$$\text{DEC} \quad \Longrightarrow \quad |\bar{p}| \leq \rho. \tag{5.28}$$

The implication is one-way, but the inequality is particularly clean and easy to work with.

6 Timelike and null convergence conditions

In applications the various energy conditions are, using the Einstein equations, typically immediately converted into purely geometrical convergence conditions such as the null convergence condition (NCC) and the timelike convergence condition (TCC) [37–40]. Within standard general relativity the NCC is equivalent to the NEC, and the TCC is equivalent to the SEC.

NCC: The NCC is the statement that for all null vectors $R_{ab}\,\ell^a \ell^b \geq 0$.

TCC: The TCC is the statement that for all timelike vectors $R_{ab}\,V^a V^b \geq 0$.

If one wishes to step outside the framework of standard general relativity then this adds a new level of speculative physics to the mix, and then the distinction between geometrical convergence conditions and dynamical energy conditions might become important. However it should be emphasized that in many (but not all) modified theories of gravity the equations of motion can be rearranged into the form

$$(\text{Einstein tensor}) = (\text{some “effective” stress-energy tensor}). \tag{6.1}$$

Whenever this can be done, statements about the usual energy conditions in Einstein gravity can be carried over to statements about “effective” energy conditions in modified gravity.

7 Violation of the energy conditions

Now consider the energy conditions in the warp drive spacetimes. We shall provide a number of results, ultimately demonstrating the generic violation of the NEC (thereby automatically implying violations of the WEC, SEC and DEC)

7.1 Alcubierre warp drive

The key defining characteristic of the Alcubierre warp drive is that $v_i(x, y, z, t)$ is always pointing in some fixed direction, which can without loss of generality be taken to be the z-direction $\hat{z}_i = (0, 0, 1)_i$. That is, take

$$v_i(x, y, z, t) = v(x, y, z, t)\ \hat{z}_i. \tag{7.1}$$

(This is slightly more general than what Alcubierre actually did [4]; but it is the best way of summarizing the key aspects of the physics.) Then the extrinsic curvature is simply

$$K_{ij} = \partial_{(i} v\ \hat{z}_{j)} = \left[\begin{array}{cc|c} 0 & 0 & \frac{1}{2}\partial_x v \\ 0 & 0 & \frac{1}{2}\partial_y v \\ \hline \frac{1}{2}\partial_x v & \frac{1}{2}\partial_x v & \partial_z v \end{array}\right]. \tag{7.2}$$

Consequently

$$K = \hat{z}^i\ \partial_i v = \partial_z v; \tag{7.3}$$

while

$$(K^2)_{ij} = \frac{1}{2} K K_{ij} + \frac{1}{4}(\partial v)^2 \hat{z}_i \hat{z}_j + \frac{1}{4}\partial_i v \partial_j v; \tag{7.4}$$

and

$$\mathrm{tr}\big(K^2\big) = \frac{1}{2}K^2 + \frac{1}{2}(\partial v)^2 = \frac{1}{2}(\partial_x v)^2 + \frac{1}{2}(\partial_y v)^2 + (\partial_z v)^2. \tag{7.5}$$

Then

$$\rho = \frac{1}{16\pi}\left(K^2 - \mathrm{tr}\big(K^2\big)\right) = -\frac{1}{32\pi}\left((\partial_x v)^2 + (\partial_y v)^2\right) \leq 0. \tag{7.6}$$

But to have a localized warp bubble, and avoid the triviality of everywhere flat Minkowski space, we need $((\partial_x v)^2 + (\partial_y v)^2) > 0$ somewhere in the spacetime, and so $\rho < 0$ at those locations, and is at best zero everywhere else. So the Alcubierre warp bubble definitely violates the WEC. In his original article [4] Alcubierre also claims (without proof): "In a similar way one can show that the strong energy condition is also violated." However in a more recent article by Alcubierre and Lobo [24], they explicitly prove the stronger result that the Alcubierre warp drive violates the NEC. (So SEC, WEC, and DEC are all definitely violated, because NEC is.)

Their proof is slightly tricky and depends (for a warp bubble moving in the z direction) on the identity $G_{zz} = 3G_{nn} < 0$. This identity is established by explicit computation, there does not seem to be an obvious geometrical reason for it. From this geometrical identity we have $T_{zz} = 3\rho < 0$ in the comoving frame. Hence $\rho + T_{zz} = 4\rho < 0$ and the NEC is violated.

7.2 Natário zero-expansion warp drive

The key defining characteristic of the Natario zero-expansion warp drive is that the flow $v_i(x, y, z, t)$ is zero divergence: $\boldsymbol{\nabla} \cdot \vec{v} = 0$. (This is slightly more general than what Natário actually did [5]; but it is the best way of summarizing the relevant physics.) Then $K = 0$ and from (4.1) we have:

$$\rho = \frac{1}{16\pi} \left(K^2 - \operatorname{tr}\left(K^2\right)\right) = -\frac{1}{16\pi} \operatorname{tr}\left(K^2\right) \leq 0. \tag{7.7}$$

But to avoid a trivial warp drive the extrinsic curvature K_{ij} must be nonzero somewhere, and wherever $K_{ij} \neq 0$ we have $\rho < 0$. So the Natário zero-expansion warp drive definitely violates the WEC [5].

But, now extending Natário's argument, we note that in this situation, from equation (4.11), we also have

$$\rho + 3\bar{p} = -\frac{1}{4\pi} \operatorname{tr}\left(K^2\right) \leq 0. \tag{7.8}$$

And the same non-triviality argument as immediately above now shows that any Natário zero-expansion warp drive also violates the SEC.

Furthermore, again extending Natário's argument, from equation (4.12) we note

$$\rho + \bar{p} = -\frac{1}{8\pi} \operatorname{tr}\left(K^2\right) \leq 0. \tag{7.9}$$

And the same non-triviality argument again shows that any Natário zero-expansion warp drive also violates the NEC.

7.3 Zero-vorticity warp drive

The key defining characteristic of the Lentz/Fell–Heisenberg zero-vorticity warp drive is that the flow $v_i(x, y, z, t)$ is taken to be a gradient $v_i(x, y, z, t) = \partial_i \Phi(x, y, z, t)$ [1, 3]. (This is slightly more general than what Lentz and Fell–Heisenberg actually did; but it is the best way of summarizing the relevant physics.)

Then

$$K_{ij} = \Phi_{,ij}, \qquad K = \boldsymbol{\nabla}^2\Phi. \tag{7.10}$$

From (4.1) the co-moving energy density is

$$\rho = \frac{1}{16\pi}\left(K^2 - \mathrm{tr}\big(K^2\big)\right) = \frac{1}{16\pi}\left((\boldsymbol{\nabla}^2\Phi)^2 - \Phi_{,ij}\Phi_{,ij}\right). \tag{7.11}$$

This no longer *obviously* violates the WEC. However, in view of equation (4.4), the energy density is now a pure divergence

$$\rho = \frac{1}{16\pi}\left\{\partial_i(\Phi_{,i}\Phi_{,jj} - \Phi_{,j}\,\Phi_{,ij})\right\}. \tag{7.12}$$

With suitable falloff conditions, (effectively, a negligible mass spacecraft), $\int \rho\,\mathrm{d}^3x \to 0$. But then, if the warp field has positive energy density anywhere, it must have negative energy density somewhere else. This implies violations of the WEC somewhere on each spatial slice. We shall subsequently be more specific by invoking stronger arguments.

In this class of zero-vorticity warp drives the comoving flux is identically zero, $f^i = 0$. Unfortunately, the spatial parts of the stress-energy T_{ij} are still a bit of a mess, they do not really simplify appreciably. (And one really needs to know something about the spatial stresses T_{ij} in order to say anything more precise about the energy conditions.) The best way of proceeding seems to be to adapt, modify, and extend a general argument that Natário applied to his general class of warp drive spacetimes [5]. Let us do this now.

7.4 Natário's generic warp drive

WEC: We have already established that from equation (4.5) it follows in general that

$$\rho = \frac{1}{16\pi}\left\{\boldsymbol{\nabla}\cdot\{\vec{v}\,K - (\vec{v}\cdot\boldsymbol{\nabla})\vec{v}\} - \frac{1}{2}\,(\vec{\omega}\cdot\vec{\omega})\right\}. \tag{7.13}$$

Hence, with suitable falloff conditions (effectively, a negligible mass spacecraft),

$$\int \rho\,\mathrm{d}^3x = -\frac{1}{32\pi}\int (\vec{\omega}\cdot\vec{\omega})\,\mathrm{d}^3x \leq 0. \tag{7.14}$$

Again, if the warp field has positive energy density anywhere, it must have negative energy density somewhere else. This implies violations of the WEC somewhere on each spatial slice.

SEC: To improve on this result, note that in reference [5] Natário presents a quite general argument to the effect that in *any* generic warp spacetime, (Alcubierre, Natário zero-expansion, and, yes, it even applies to Lentz/Fell–Heisenberg zero-vorticity warp drives), there *must* be violations of either the SEC or the WEC. (Or you have the trivial case of Minkowski space.) See his theorem 1.7.

This result can already be slightly improved in view of our comments above: In *any* generic warp spacetime, (Alcubierre, Natário, and, yes, it even applies to Lentz/Fell–Heisenberg), there *must* be violations of the SEC. (Or you have the trivial case of Minkowski space.) We shall go into some detail in order to localise *where* and *when* the SEC violations take place.

The key point is that Eulerian observers define a zero-vorticity congruence of timelike geodesics, that by construction *cannot* have any focussing points. Now apply a variant of the Raychaudhuri equation (timelike focussing theorem) [37, 94–100].

Explicit calculation has shown us, see equation (4.11), that for *any* warp drive spacetime:

$$\rho + 3\bar{p} = -\frac{1}{4\pi}\left(\mathcal{L}_n K + \mathrm{tr}\big(K^2\big)\right). \tag{7.15}$$

Thus if the SEC holds (implying $\rho + 3\bar{p} \geq 0$) we have

$$\mathcal{L}_n K + \mathrm{tr}\big(K^2\big) \leq 0. \tag{7.16}$$

So

$$\mathcal{L}_n K \leq -\,\mathrm{tr}\big(K^2\big). \tag{7.17}$$

Now split the extrinsic curvature into trace-free part $K^{\mathrm{tf}}_{ij} = K_{ij} - \frac{1}{3}K\delta_{ij}$ and trace. Then

$$\mathrm{tr}\big(K^2\big) = \mathrm{tr}\left(\left[K^{\mathrm{tf}}_{ij} + \frac{1}{3}K\delta_{ij}\right]^2\right) = \mathrm{tr}\big([K^{\mathrm{tf}}]^2\big) + \frac{1}{3}K^2 \geq \frac{1}{3}K^2 \tag{7.18}$$

Consequently the SEC would imply

$$\mathcal{L}_n K \leq -\frac{1}{3}K^2. \tag{7.19}$$

So if $K < 0$, then it will be even more negative in the future. Similarly if $K > 0$, then it must have been even more positive in the past. Noting that, acting on scalars, $\mathcal{L}_n K = V^a \partial_a K = \mathrm{d}K/\mathrm{d}\tau$, we see that the Eulerian observers satisfy

$$\frac{\mathrm{d}K}{\mathrm{d}\tau} \leq -\frac{1}{3}K^2. \tag{7.20}$$

Therefore

$$-\frac{1}{K^2}\frac{\mathrm{d}K}{\mathrm{d}\tau} \geq \frac{1}{3}. \tag{7.21}$$

So

$$\frac{\mathrm{d}K^{-1}}{\mathrm{d}\tau} \geq \frac{1}{3}. \tag{7.22}$$

Pick any Eulerian observer, and pick some point τ_0 on that world-line. Integrating upwards, from $\tau = \tau_0$ to some $\tau > \tau_0$ we have

$$K^{-1}(\tau) \geq K(\tau_0)^{-1} + \frac{1}{3}(\tau - \tau_0). \tag{7.23}$$

So if $K(\tau_0) < 0$, then there will be some finite proper time increment, less than $3/|K_0|$, at which $K^{-1} \to 0^-$ implying $K \to -\infty$.

Integrating downwards, from $\tau = \tau_0$ to some $\tau < \tau_0$ we have

$$K_0^{-1} \geq K(\tau)^{-1} + \frac{1}{3}|\tau - \tau_0|. \tag{7.24}$$

That is,

$$K^{-1}(\tau) \leq K_0^{-1} - \frac{1}{3}|\tau_0|. \tag{7.25}$$

So if $K(\tau_0) > 0$, then there will be some finite proper time decrement, less than $3/|K_0|$, at which $K^{-1} \to 0^+$ implying $K \to +\infty$.

Either way, this is in contradiction to the fact that the Eulerian observers define a zero-vorticity congruence of timelike geodesics that by construction *cannot* have any focussing points. So either $K \equiv 0$ identically, or whenever $K(\tau_0) \neq 0$ somewhere for any arbitrary Eulerian observer, the SEC fails somewhere in the interval

$$\tau \in \left(\tau_0 - \frac{3}{|K(\tau_0)|},\ \tau_0 + \frac{3}{|K(\tau_0)|}\right). \tag{7.26}$$

(Of course we could relax the global condition in the ADM-like split in the warp drive. However, this would imply either the formation of singularities in a finite time in the region of non-zero extrinsic curvature—meaning the warp bubble would encounter it; or we would have to save the situation by opting for CTCs [6], in many ways an even worse situation for physics [83–88].)

Overall, this now gives us quite good control on *where* and *when* the SEC violations take place. The special case where $K \equiv 0$ identically reduces to the Natário warp drive, for which we had already argued that the SEC fails (as long as the extrinsic curvature K_{ij} is not identically zero everywhere in the spacetime).

NEC: For the NEC we had already argued that the NEC requires $\rho + \bar{p} > 0$, and we have the explicit calculation (4.12):

$$\rho + \bar{p} = \frac{1}{24\pi}\left(-2\mathcal{L}_n K + K^2 - 3\,\mathrm{tr}\big(K^2\big)\right). \tag{7.27}$$

So the NEC would require

$$2\mathcal{L}_n K - K^2 + 3\,\mathrm{tr}\big(K^2\big) \leq 0. \tag{7.28}$$

But, in terms of the trace-free part of extrinsic curvature, we have

$$\mathrm{tr}\big(K^2\big) = \mathrm{tr}\big([K^{\mathrm{tf}}]^2\big) + \frac{1}{3}K^2. \tag{7.29}$$

So the NEC would require

$$2\mathcal{L}_n K + 3\,\mathrm{tr}\big([K^{\mathrm{tf}}]^2\big) \leq 0. \tag{7.30}$$

That is,

$$\mathcal{L}_n K \leq -\frac{3}{2}\,\mathrm{tr}\big([K^{\mathrm{tf}}]^2\big). \tag{7.31}$$

That is, using $\mathcal{L}_n K = \mathrm{d}K/\mathrm{d}\tau$, and assuming the NEC:

$$\frac{\mathrm{d}K}{\mathrm{d}\tau} \leq -\frac{3}{2}\,\mathrm{tr}\big([K^{\mathrm{tf}}]^2\big). \tag{7.32}$$

Now add some extra physics: We want to be able to turn our warp drive on and off. So before some time t_i we have flat space, and after some time t_f we have flat space. That is, $K_{ij} = 0$ for $t \notin (t_i, t_f)$. (There is a minor approximation being made here: that the mass of your spacecraft is negligible.)

Now pick any arbitrary Eulerian observer — by construction that observer will cross both of the t_i and t_f hyperplanes, and along that geodesic we can calculate

$$\int_i^f \mathcal{L}_n K\;\mathrm{d}\tau = \int_i^f \frac{\mathrm{d}K}{\mathrm{d}\tau}\;\mathrm{d}\tau = K|_i^f = 0. \tag{7.33}$$

But, invoking the NEC, this now implies

$$0 \leq -\frac{3}{2}\int_i^f \mathrm{tr}\big([K^{\mathrm{tf}}]^2\big)\,\mathrm{d}\tau. \tag{7.34}$$

Equivalently,

$$\int_i^f \mathrm{tr}\big([K^{\mathrm{tf}}]^2\big)\,\mathrm{d}\tau \leq 0. \tag{7.35}$$

But, since $\mathrm{tr}\big([K^{\mathrm{tf}}]^2\big)$ is positive semidefinite, this implies that everywhere along the Eulerian trajectory in question we must have

$$K^{\mathrm{tf}}_{ij} = 0. \tag{7.36}$$

That is, the NEC, plus the ability to switch the warp drive on and off, implies that everywhere along every Eulerian observer, (and so everywhere in the spacetime), one must have

$$K_{ij} = \frac{1}{3} K\ \delta_{ij}. \tag{7.37}$$

This is a very strong constraint. In fact, the constraint is so strong as to preclude any useful warp drive behaviour. Let us re-cast the constraint as

$$v_{(i,j)} = \frac{1}{3} v_{k,k}\ \delta_{ij}. \tag{7.38}$$

Taking the divergence,

$$v_{(i,j),j} = \frac{1}{3} v_{k,kj} \delta_{ij}. \tag{7.39}$$

Thence,

$$\nabla^2 v_i = -\frac{1}{3} v_{j,ji}. \tag{7.40}$$

Now apply the boundary condition that on each time slice we have $v^i \to 0$ at spatial infinity, and use the standard flat-space Green function to write:

$$v_i(x) = -\frac{1}{3} \int \frac{1}{|\vec{x} - \vec{y}|}\ v_{j,ji}(y)\ \mathrm{d}^3 y. \tag{7.41}$$

Integrate by parts, (by construction we know we have suitable falloff conditions at spatial infinity), then

$$v_i(x) = \partial_i \left(\frac{1}{3} \int \frac{1}{|\vec{x} - \vec{y}|}\ v_{j,j}(y)\ \mathrm{d}^3 y \right). \tag{7.42}$$

So enforcing the NEC implies that generic warp drives must be of Lentz/Fell-Heisenberg zero-vorticity type: $v_i = \partial_i \Phi$. But this means that our condition is now that the NEC implies

$$\Phi_{,ij} = \frac{1}{3} \Phi_{,kk}\ \delta_{ij}. \tag{7.43}$$

But then, differentiating:

$$\Phi_{,ijj} = \frac{1}{3}\Phi_{,kki}. \tag{7.44}$$

Thence,

$$\Phi_{,ijj} = 0, \tag{7.45}$$

implying that

$$\Phi_{,jj} = (\text{constant}). \tag{7.46}$$

We are reduced to solving $\boldsymbol{\nabla}^2\Phi = (\text{constant})$, subject to $\boldsymbol{\nabla}\Phi \to 0$ at spatial infinity. This has the unique solution $\Phi = (\text{another constant})$, implying $v_i \equiv 0$, so one is in Minkowski space. Overall we see that the NEC + (the ability to switch the warp drive on and off), implies that one is in flat Minkowski space. That is, the warp drive is trivial. We conclude that any physically reasonable warp drive must violate the NEC.

8 Discussion and conclusions

We have now demonstrated that all members of the general class of warp drives defined by Natário [5] violate the NEC. This significantly extends previously known results. This argument applies, in particular, to all three of the Alcubierre [4], Natário zero-expansion [5], and Lentz/Fell–Heisenberg zero-vorticity [1, 3] warp drives, and also (for slightly different reasons) to the model warp drives considered by Bobrick–Martire [2]. Because the NEC is violated then so are the WEC, SEC, and DEC.

Consequently, insofar as one wishes to continue to entertain the possibility of warp drives as a real physical phenomenon, one has no choice but to face the violation of the energy conditions head on. Several possibilities arise: (i) modify the theory of gravity, (ii) modify the definition of warp drive, (iii) modify the energy conditions, (iv) appeal to macroscopic quantum physics, (v) allow for singularities or CTCs (time travel). None of these options are particularly palatable. All of these options have serious draw-backs. Thus it is our melancholy duty to report that none of the recent claims of positive-mass physical warp drives survive careful inspection of the proffered arguments.

A "Spherically symmetric" warp drives

"Spherically symmetric" warp drives date back to the original article by Alcubierre [4]. Specifically, let us reconsider the explicit line element (2.12) and rephrase it as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \Big(\mathrm{d}z - v_*(t)\; f\left(r_s(x,y,z,t)\right)\mathrm{d}t\Big)^2, \tag{A.1}$$

where we set

$$r_s(x,y,z,t) = \sqrt{x^2 + y^2 + \left(z - z_*(t)\right)^2}\,, \qquad z_*(t) = \int v_*(t)\,\mathrm{d}t, \tag{A.2}$$

and enforce both $f(0) = 1$ and $f(\infty) = 0$. This represents a spherically symmetric warp bubble, centred on the moving point $(0, 0, z_*(t))$, with a shape function $f(r_s)$ that depends only on the Euclidean distance from the moving centre. We emphasize that while the warp bubble is spherically symmetric the spacetime is not — there is certainly a preferred axis due to the direction of motion of the warp bubble. (Hence the warp bubble is spherically symmetric, but the spacetime is only "spherically symmetric".) At large spatial distances $\mathrm{d}s^2 \to -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, the usual representation of Minkowski space, while at the centre of the warp drive

$$\mathrm{d}s^2 \to -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \left(\mathrm{d}z - v_*(t)\mathrm{d}t\right)^2. \tag{A.3}$$

Consider now the coordinate transformation

$$\tilde{t} = t, \qquad \tilde{x} = x, \qquad \tilde{y} = y, \qquad \tilde{z} = z - z_*(t), \tag{A.4}$$

which is designed to bring the warp bubble "to rest". Then

$$\mathrm{d}\tilde{t} = \mathrm{d}t, \qquad \mathrm{d}\tilde{x} = \mathrm{d}x, \qquad \mathrm{d}\tilde{y} = \mathrm{d}y, \qquad \mathrm{d}\tilde{z} = \mathrm{d}z - v_*(t)\,\mathrm{d}t, \tag{A.5}$$

and the line element becomes

$$\mathrm{d}s^2 = -\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{x}^2 + \mathrm{d}\tilde{y}^2 + \left(\mathrm{d}\tilde{z} + v_*(t)[1 - f(r_s)]\,\mathrm{d}\tilde{t}\right)^2, \tag{A.6}$$

where now $r_s = \sqrt{\tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2}$ is time independent. At the centre of the warp bubble we now have $\mathrm{d}s^2 \to -\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{x}^2 + \mathrm{d}\tilde{y}^2 + \mathrm{d}\tilde{z}^2$, the usual representation of Minkowski space, while at spatial infinity it is now the outside universe that is streaming by with velocity $-v_*(t)$.

At spatial infinity the line element becomes

$$\mathrm{d}s^2 \to -\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{x}^2 + \mathrm{d}\tilde{y}^2 + \left(\mathrm{d}\tilde{z} + v_*(t)\,\mathrm{d}\tilde{t}\right)^2. \tag{A.7}$$

This is again Minkowski space, but now in "moving coordinates". Note the "conservation" of relative motion — either spatial infinity is standing still and the bubble is moving, or the bubble is standing still and spatial infinity is moving [4, 5].

If one wishes to make the walls of the warp bubble thin, (but still of finite thickness), this is a *choice*, not a *necessity*, then one might pick

$$f(r_s) = \begin{cases} 1 & r_s \leq r_{\text{inner}}, \\ \text{smooth} & r_s \in [r_{\text{inner}}, r_{\text{outer}}], \\ 0 & r_s \geq r_{\text{outer}}. \end{cases} \tag{A.8}$$

If one makes this *choice*, then one needs to enforce $r_{\text{outer}} > r_{\text{inner}}$ to keep the metric continuous. A discontinuity in the metric leads to delta functions in the Christoffel symbols, squares of delta functions in the Riemann tensor, and cubes of delta functions in the Bianchi identities. Even when working with the Israel–Lanczos–Sen thin-shell formalism (the junction condition formalism) one needs to keep the metric piecewise differentiable, C^{1-}, since then the Christoffel symbols at worst contain step functions and the Riemann tensor and Bianchi identities at worst contain delta functions.

The reason for being so explicit is that in reference [2] the authors present a deeply flawed discussion of spherically symmetric warp drives. In their implementation of spherically symmetric warp drives the warp bubble is certainly at rest, but in addition their choice of asymptotic boundary conditions also forces the warp bubble to be at rest with respect to spatial infinity. This defeats the whole purpose of a warp drive.

Furthermore, these authors take the interior of the warp bubble to be a portion of flat Minkowski space, the wall of the warp bubble to be a non-negative density barotropic and isotropic fluid, and the exterior region to be a portion of Schwarzschild spacetime. They then attempt to apply the Tolman–Oppenheimer–Volkov (TOV) equation to the bubble wall. But under the assumptions they are imposing (zero pressure at the inner edge of the bubble wall, zero pressure at the outer edge of the bubble wall, non-negative energy density within the bubble wall, and a barotropic equation of state) the unique solution to the TOV is the trivial solution $p(r) = 0 = \rho(r)$, and the mass of the exterior region must be zero. Thus their specific model (as presented in [2]) just reduces, globally, to flat Minkowski space — it is not a warp drive.

These authors [2] also assert that both Alcubierre [4] and Natário [5] must *necessarily* "truncate" their warp fields, forcing them to be exactly Minkowski space, at some finite distance from the centre of the warp bubble. This is simply not an accurate reflection of what is actually done in those references [4, 5]. Specifically, in Alcubierre's original article [4, equation (6)] the concrete realization of the shape function $f(r_s)$ he chooses, see equation (A.8), uses tanh functions and so does not have compact support. In this terminology, Alcubierre sets $r_{\text{inner}} = 0$ and $r_{\text{outer}} = \infty$, which is certainly not a truncation. Unfortunately this terminology of "truncation" has been uncritically adopted by subsequent authors [3].

The authors of [2] also assert in passing, see their section (5.2), that the Alcubierre warp drive [4] does not satisfy the continuity equations. They also assert that the velocity of the Alcubierre warp bubble cannot be time dependent, claiming a violation of the conservation of momentum. These statements are both false, and seem to arise (at best) from adopting a naive Newtonian viewpoint. What Alcubierre has actually done is to "reverse engineer" the warp drive — once one writes down a suitable metric, (at least C^{1-}, piecewise differentiable), one simply calculates the Einstein tensor to find what the stress-energy is that would be required to support that spacetime geometry. The continuity equation is automatically enforced via the Bianchi identities, and there is no difficulty in making the velocity of the Alcubierre warp bubble time-dependent. (Reference [3] seems to repeat this error in their discussion. Reference [1] is more careful in this regard, setting the 3-velocity of their model warp bubble constant for simplicity, but without making any claim as to necessity.)

Overall, while the idea of a "spherically symmetric" warp drive is certainly well-defined and useful, the specific implementation of this notion in reference [2] is not a successful one, and is not useful.

B More general warp drives

There are two somewhat more general warp drive spacetimes that do not fall into Natário's general classification [5], and are incompatible with the Alcubierre, Natário, and Lentz/Fell–Heisenberg warp drives. These more general warp drives are obtained by either relaxing the condition that the lapse be unity, (so $N(x, y, z, t) \neq 1$), or relaxing the condition that the spatial slices be flat, (so $g_{ij} \neq \delta_{ij}$). See reference [24] which discusses the case $N \neq 1$, and reference [12] which permits the spatial slices to be conformally flat $g_{ij} = e^{2\theta(x,y,z,t)}\, \delta_{ij}$ rather than Riemann flat.

Unhelpfully, in appendices A.1 and A.2 of reference [2] those authors incorrectly claim that both of these more general spacetimes can, by a coordinate transformation, be brought into Alcubierre form. The simplest way to see that both these claims are wrong is to set the flow vector to zero. The claims made in appendices A.1 and A.2 of reference [2] then reduce to the claims that the line elements

$$\mathrm{d}s^2 = -N(x,y,z,t)^2 \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \tag{B.1}$$

and

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2\theta(x,y,z,t)}\{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\}, \tag{B.2}$$

are actually Riemann flat. This is manifestly incorrect. What has gone wrong? The transformations these authors invoke are just not coordinate transformations. For a transformation $\mathrm{d}x^a \to \mathrm{d}\bar{x}^a = J^a{}_b\, \mathrm{d}x^b$ to actually be a coordinate transformation the critical requirement is that $J^a{}_{[b,c]} = 0$. (Additionally, one would want $\det(J^a{}_b) \neq 0$, except on sets of measure zero.) While the transformations made in appendices A.1 and A.2 of reference [2] satisfy the determinant condition, they fail the more basic $J^a{}_{[b,c]} = 0$ condition; they are simply not coordinate transformations.

For our purposes then it means that the non-unit-lapse and van den Broeck warp drive spacetimes cannot simply be dismissed out of hand. There are other problematic issues with both of these models, but they are physically different from the Natário class [5], (unit lapse, spatially flat 3-geometry), and must be directly addressed using different techniques.

C Some defective warp drives

Some of the recently proposed warp drive spacetimes are defective for other reasons.

For instance, in the first explicit example in reference [3] the velocity potential Φ is certainly C^0 but is only piecewise differentiable, C^{1-}. That is, the flow $\vec{v} = \boldsymbol{\nabla}\Phi$ is discontinuous, which implies that the metric is discontinuous. This then leads to delta functions in the Christoffel symbols, squares of delta functions in the Riemann tensor, and cubes of delta functions in the Bianchi identities. This is mathematically and physically not viable.

In their second explicit example, [3, equation (9)], the velocity potential Φ is C^∞, except at the centre of the spacetime, but suffers other problems. If one takes $r = \sqrt{x^2+y^2+z^2}$, then even at its most general their velocity potential is of the form

$$\Phi(x,y,z) = F(r,\sigma(x,y,z)) + v_*\, z. \tag{C.1}$$

Then

$$\vec{v} = \boldsymbol{\nabla}\Phi(x,y,z) = \partial_r F(r,\sigma(x,y,z))\,\hat{r} + \partial_\sigma F(r,\sigma(x,y,z))\,\boldsymbol{\nabla}\sigma + v_*\,\hat{z}. \tag{C.2}$$

But then, in their specific example, they set $\sigma(x,y,z) \to 1$ and $v_* \to 0$. That is

$$\Phi(x,y,z) \to F(r,1), \tag{C.3}$$

which is a function of r only. But then

$$\vec{v} = \boldsymbol{\nabla}\Phi \to \frac{\partial F(r,1)}{\partial r}\;\hat{r}. \tag{C.4}$$

So their flow vector is always pointing radially outwards/inwards. This is simply not viable for describing a warp drive spacetime.

Oddly enough, consider $\Phi(x,y,z) = 2\sqrt{2mr} = 2\sqrt{2m}\,\sqrt[4]{x^2+y^2+z^2}$, so that

$$\vec{v} = \boldsymbol{\nabla}\Phi(x,y,z) = \frac{\sqrt{2m}\,(x,y,z)}{(x^2+y^2+z^2)^{3/4}} = \sqrt{\frac{2m}{\sqrt{x^2+y^2+z^2}}}\;\hat{r}. \tag{C.5}$$

This is the Schwarzschild spacetime (in Painlevé–Gullstrand form, converted to Cartesian coordinates). This is not a warp drive spacetime.

Similarly, consider $\Phi(x,y,z) = \frac{(x^2+y^2+z^2)}{2\ell}$ so that

$$\vec{v} = \boldsymbol{\nabla}\Phi = \frac{(x,y,z)}{\ell} = \frac{\sqrt{x^2+y^2+z^2}}{\ell}\;\hat{r}. \tag{C.6}$$

This is the de Sitter spacetime (in Painlevé–Gullstrand form, converted to Cartesian coordinates, with $\Lambda = 3/\ell^2$). This is not a warp drive spacetime.

But the central issue is this: There is simply no way that a flow of the form

$$\vec{v} = f(r)\,\hat{r} + v_*\,\hat{z} \tag{C.7}$$

can represent a warp drive spacetime.

In reference [1] the claim is made that the author can self-consistently find a warp drive configuration that is sourced by a perfect fluid plasma (satisfying the WEC and so also the NEC) plus electromagnetic Maxwell stress-energy (satisfying all of NEC, WEC, SEC, and DEC). So if this claim were to be true, if these models truly were solutions of the Einstein equations with the specified source, one would have a warp drive spacetime satisfying the NEC, which we have just shown to be impossible. What has gone wrong? The point here is that the author of [1] has not actually solved the Einstein equations, he has only solved part of the Einstein equations — for the density, flux, and trace of the stress. This is not enough to obtain a valid solution of the Einstein equations — the author would also need to consider the remaining trace-free part $T_{ij} - \frac{1}{3}(T_{kl}\delta^{kl})g_{ij}$ of the stress tensor.

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