Perfect fluid warp drive solutions
with the cosmological constant

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Abstract

The Alcubierre metric describes a spacetime geometry that allows a massive particle inside a spacetime distortion, called warp bubble, to travel with superluminal global velocities. In this work we advance solutions of the Einstein equations with the cosmological constant for the Alcubierre warp drive metric having the perfect fluid as source. We also consider the particular dust case with the cosmological constant, which generalizes our previous dust solution (Santos-Pereira et al. 2020) and led to vacuum solutions connecting the warp drive with shock waves via the Burgers equation, as well as our perfect fluid solution without the cosmological constant (Santos-Pereira et al. 2021). All energy conditions are also analyzed. The results show that the shift vector in the direction of the warp bubble motion creates a coupling in the Einstein equations that requires off-diagonal terms in the energy-momentum source. Therefore, it seems that to achieve superluminal speeds by means of the Alcubierre warp drive spacetime geometry one may require a complex configuration and distribution of energy, matter and momentum as source in order to produce a warp drive bubble. In addition, warp speeds seem to require more complex forms of matter than dust for stable solutions and that negative matter may not be a strict requirement to achieve global superluminal speeds.

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1. INTRODUCTION

The *warp drive* is a mechanism based on General Relativity which in theory allows for massive particles to be propelled throughout the spacetime with global superluminal speeds [1, 2]. The theory describes this possibility by means of a localized spacetime distortion, called *warp bubble*, that would contain a local lightcone where the particle would follow special relativity, that is, move locally with speeds smaller than light. However, the metric is such that the warp bubble moves along a geodesic that creates an expansion of spacetime behind it and a contraction in front of it in such a way that an observer outside the warp bubble sees it moving with superluminal speed.

In the original paper by M. Alcubierre the warp drive metric was established without solving the Einstein equations [1]. The Einstein tensor components were calculated and it was noticed that negative energy density would be required to create the warp bubble, thus violating the weak and dominant energy conditions.

Ford and Roman [3] calculated via quantum inequalities the amount of negative energy required for the warp drive to be possible, concluding that it would be a massive amount, impossible to achieve. Pfenning and Ford [4] also concluded that it would be necessary an enormous amount of energy for the warp drive to be possible. They obtained a quantity ten orders of magnitude greater than the mass-energy of the entire visible Universe, also with negative density.

Krasnikov [5] discussed the possibility of a massive particle moving in space faster than a photon, arguing that this is not possible due to limitations on globally hyperbolic spacetimes properties with feasible physical assumptions. He is the creator of a specific spacetime topology with devices that would allow massive particles to travel between two points in space with superluminal velocities without the need for tachyons. Everet and Roman [6] coined the name for this spacetime topology as the *Krasnikov tube*. They generalized the metric designed by Krasnikov by proposing a tube in the direction of the particle's path, connecting the start and end point. Inside this tube, the spacetime is flat and the lightcones are opened to allow the one direction superluminal travel. The Krasnikov tube also requires huge amounts of negative energy density. Since the tube is designed to not possess closed timelike curves, it would be theoretically possible to construct a two way non-overlapping system that could work as a time machine. The energy-momentum tensor (EMT) for the Krasnikov metric is positive in some regions. Both the metric and the obtained EMT were thoroughly analyzed in Refs. [7, 8].

Van de Broeck [9] made a relevant contribution to warp drive theory by demonstrating that a small modification of the original Alcubierre geometry would reduce, to a few solar masses, the total negative energy necessary for the creation of the warp bubble distortion of spacetime. This result have led van de Broeck to suppose that other geometrical modifications of the Alcubierre's geometry for the warp drive could also reduce the amount of energy necessary to create a warp drive bubble in the same way.

Natario [10] stated that the spacetime contraction and expansion of the warp bubble is a peculiar consequence of the warp drive metric. Hence, he designed a spacetime where no contraction or expansion occurs for the warp drive bubble. Lobo and Visser [11, 12] discussed that the center of the warp bubble proposed by Alcubierre needs to be massless [see also

Refs. 13, 14]. They proposed a linearized model for both Alcubierre and Natario proposals and demonstrated that for small speeds, the energy stored in the warp fields must be a significant fraction of the mass of the spaceship inside the warp bubble. Quarra [15] discussed null geodesics moving faster-than-light according to far away observers when inside a region-delimited gravitational wave field. Lee and Cleaver [16, 17] analyzed how external radiation might affect the Alcubierre warp bubble to turn it unsustainable. They also claimed that a warp field interferometer could not detect spacetime distortions. Mattingly et al. [18, 19] studied the curvature invariants characteristic of Natario and Alcubierre warp drives, whereas Mattingly [20] discussed further curvature invariants for warped spacetimes.

Bobrick and Martire [21] claimed that any warp drive spacetime consists of a shell of regular or exotic material moving inertially with a certain speed, also reaching at a class of subluminal spherically symmetric warp drives. Lentz [22] and Fell and Heisenberg [23] advanced superluminal capable soliton solutions with positive energy warp drives. Santiago et al. [24, 25] argued that only comoving timelike Eulerian observers satisfy the weak energy condition, whereas this not the case for all timelike observers. Furthermore, they claimed that all physically reasonable warp drives will violate the null and weak energy conditions, therefore disputing the claim advanced by Refs. [21–23] that it would be theoretically possible to set up positive energy warp drives that satisfy the weak energy condition.

Motivated by the fact that neither the original paper by Alcubierre, nor the subsequent ones cited above, did actually solve the Einstein equations using the warp drive metric, we proceed to investigate possible solutions for a dust particle energy momentum tensor [26]. Our results showed that solutions of the Einstein equations for the Alcubierre warp drive metric having dust as source connect them in a particular case the warp drive geometry to the well-known Burgers equation, which describes the dynamics of the waves moving through an inviscid fluid. Hence, shock waves appear to be vacuum solutions of the Einstein equations endowing the warp drive metric [26].

In our second paper [27] we investigated solutions for the warp drive metric having the perfect fluid and a special case of anisotropic fluid with heat flux, but both with zero cosmological constant in the Einstein equations. The resulting solutions indicate that positive matter densities are possibly capable of generating superluminal speeds. In our third paper [28] a charged dust was used as source EMT for the Alcubierre metric and the Einstein equations which included the cosmological constant. We obtained solutions connecting the electric energy density with the cosmological constant and, again, some solutions were found having positive matter density and satisfying the energy conditions.

Motivated by the results we obtained in Ref. [28] we have pondered that even though the Alcubierre warp drive metric is a vacuum geometry, the warp bubble would be created by geometry alone, or if a vacuum energy would make it possible through other material sources of energy and momentum. Hence, in this paper we went back to the perfect fluid source but included the cosmological constant in the Einstein equations as an additional flexibility and geometrical properties for the solutions.

We calculated the Einstein equations and analyzed the null divergence of the energy momentum tensor together with the validity requirements for the energy conditions inequalities to be satisfied. We found that the perfect fluid with the cosmological constant as source for the Alcubierre warp drive results in four sets of differential equations, two of them are very

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similar and raise the possibility for the shift vector to be a complex function in one case, depending on the (t, y) coordinates, and in another case depending on the (t, z) coordinates. The other two sets of solutions are identical to each other and similar to the solution we found in Ref. [26], except that now there is a cosmological constant coupled with the Burgers equation and, again, the warp drive is connected to shock waves solutions. Considering that the zero pressure reduces the perfect fluid to the dust EMT, the solution for this case is identical to the one we found in Ref. [26], namely, the vacuum solution of the Einstein equations connecting the warp drive to shock waves.

The plan of the paper is as follows. Section 2 presents a brief review of the basic equations and concepts of the warp drive theory, and in Section 3 we solve the Einstein equations and calculated the covariant divergence for the EMT. Section 4 discusses the energy conditions inequalities and their validity for the warp drive with $\Lambda \neq 0$ and the perfect fluid as a source. In section 5 we analyze incoherent matter as a source assuming that this is as a special case considering the perfect fluid with null pressure. In section 6 we depict our conclusions and final remarks.

2.   WARP DRIVE BASIC CONCEPTS

2.1.   Warp drive metric

The warp drive metric is a generic metric in a foliated spacetime given by the following expression,

$$ds^2 = -\left(\alpha^2 - \beta_i\beta^i\right) dt^2 + 2\beta_i \, dx^i \, dt + \gamma_{ij} \, dx^i \, dx^j \ , \tag{2.1}$$

where $d\tau$ is the proper time lapse, α is the lapse function that controls the amount of time elapsed between two hypersurfaces of constant time coordinate, β^i is the spacelike shift vector and γ_{ij} is the spatial metric for the hypersurfaces. The lapse function α and the shift vector β_i are functions of the spacetime coordinates to be determined, γ_{ij} is a positive-definite metric on each one of the spacelike hypersurfaces and these features make this spacetime globally hyperbolic. Throughout this paper Greek indices will range from 0 to 3, whereas the Latin ones indicate the spacelike hypersurfaces and will range from 1 to 3.

We have the following choices for Eq. (2.1) [1],

$$\alpha = 1, \tag{2.2}$$

$$\beta_1 = -v_s(t)f\big[r_s(t)\big], \tag{2.3}$$

$$\beta_2 = \beta_3 = 0, \tag{2.4}$$

$$\gamma_{ij} = \delta_{ij}. \tag{2.5}$$

Hence, the *warp drive metric* is given by,

$$ds^2 = -\big[1 - v_s(t)^2 f(r_s)^2\big] dt^2 - v_s(t)f(r_s) \, dx \, dt + dx^2 + dy^2 + dz^2 \ , \tag{2.6}$$

where $v_s(t)$ is the velocity of the center of the bubble moving along the curve $x_s(t)$, given by,

$$v_s(t) = \frac{dx_s(t)}{dt} \ . \tag{2.7}$$

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The function $f(r_s)$ is the warp drive *regulating form function*. It describes the shape of the warp bubble, which is given by the expression,

$$f(r_s) = \frac{\tanh\left[\sigma(r_s + R)\right] - \tanh\left[\sigma(r_s - R)\right]}{2\tanh(\sigma R)} \ , \tag{2.8}$$

where σ and R are constants to be determined. The function $r_s(t)$ defines the distance from the center of the bubble $[x_s(t), 0, 0]$ to a generic point (x, y, z) on the surface of the bubble, given by the following equation,

$$r_s(t) = \sqrt{[x - x_s(t)]^2 + y^2 + z^2} \ . \tag{2.9}$$

From Eq. (2.9) one can see that the warp bubble is perturbed in a one-dimensional manner because of the term $x - x_s(t)$.

2.2. Einstein tensor components

The components of the Einstein tensor with a cosmological constant for the warp drive metric in Eq. (2.1) are given by the expressions below,

$$G_{00} = \Lambda(1 - \beta^2) - \frac{1}{4}(1 + 3\beta^2)\left[\left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2\right] - \beta\left(\frac{\partial^2\beta}{\partial y^2} + \frac{\partial^2\beta}{\partial z^2}\right), \tag{2.10}$$

$$G_{01} = \Lambda\beta + \frac{3}{4}\beta\left[\left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2\right] + \frac{1}{2}\left(\frac{\partial^2\beta}{\partial y^2} + \frac{\partial^2\beta}{\partial z^2}\right), \tag{2.11}$$

$$G_{02} = -\frac{1}{2}\frac{\partial^2\beta}{\partial x\partial y} - \frac{\beta}{2}\left(2\frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2\beta}{\partial x\partial y} + \frac{\partial^2\beta}{\partial t\partial y}\right), \tag{2.12}$$

$$G_{03} = -\frac{1}{2}\frac{\partial^2\beta}{\partial x\partial z} - \frac{\beta}{2}\left(2\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2\beta}{\partial x\partial z} + \frac{\partial^2\beta}{\partial t\partial z}\right), \tag{2.13}$$

$$G_{11} = \Lambda - \frac{3}{4}\left[\left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2\right], \tag{2.14}$$

$$G_{12} = \frac{1}{2}\left(2\frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2\beta}{\partial x\partial y} + \frac{\partial^2\beta}{\partial t\partial y}\right), \tag{2.15}$$

$$G_{13} = \frac{1}{2}\left(2\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial x} + \beta\frac{\partial^2\beta}{\partial x\partial z} + \frac{\partial^2\beta}{\partial t\partial z}\right), \tag{2.16}$$

$$G_{23} = \frac{1}{2}\frac{\partial\beta}{\partial z}\frac{\partial\beta}{\partial y}, \tag{2.17}$$

$$G_{22} = -\Lambda - \frac{1}{4}\left[\frac{\partial^2\beta}{\partial t\partial x} + \beta\frac{\partial^2\beta}{\partial x^2} + \left(\frac{\partial\beta}{\partial x}\right)^2\right] - \frac{1}{4}\left[\left(\frac{\partial\beta}{\partial y}\right)^2 - \left(\frac{\partial\beta}{\partial z}\right)^2\right], \tag{2.18}$$

$$G_{33} = -\Lambda - \frac{1}{4}\left[\frac{\partial^2\beta}{\partial t\partial x} + \beta\frac{\partial^2\beta}{\partial x^2} + \left(\frac{\partial\beta}{\partial x}\right)^2\right] + \frac{1}{4}\left[\left(\frac{\partial\beta}{\partial y}\right)^2 - \left(\frac{\partial\beta}{\partial z}\right)^2\right], \tag{2.19}$$

where $\beta = -\beta_1 = v_s(t)f(r_s)$, as in Eq. (2.3). Also noticed that we incorporated the cosmological constant into the Einstein tensor

$$G_{\mu\nu} \to G_{\mu\nu} - \Lambda g_{\mu\nu} \tag{2.20}$$

2.3. Energy conditions revisited

The components for the Eulerian (normal) observers' 4-velocities are given by,

$$u^\alpha = (1, -\beta, 0, 0), \quad u_\alpha = (-1, 0, 0, 0). \tag{2.21}$$

Using these results into the Einstein equations,

$$T_{\alpha\beta}u^\alpha u^\beta = \frac{1}{8\pi}G_{\alpha\beta}u^\alpha u^\beta, \tag{2.22}$$

results in an expression concerning the energy conditions. From Eqs. (2.21) and considering that the only non-zero terms of Eq. (2.22) are G_{00}, G_{01} and G_{11}, we obtain the following expression,

$$T_{\alpha\beta}u^\alpha u^\beta = \frac{1}{8\pi}\left(G_{00} - 2\beta G_{01} + \beta^2 G_{11}\right). \tag{2.23}$$

Substituting Eqs. (2.10), (2.11) and (2.14) into Eq. (2.23) the result may be written as,

$$T_{\alpha\beta}u^\alpha u^\beta = \Lambda - \frac{v_s^2}{32\pi}\left[\left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right]. \tag{2.24}$$

The bubble radius is given by using Eq. (2.9). So, Eq. (2.24) is given by,

$$T_{\alpha\beta}u^\alpha u^\beta = \Lambda - \frac{v_s^2}{16\pi}\frac{y^2 + z^2}{r_s^2}\left(\frac{\partial f}{\partial r_s}\right)^2. \tag{2.25}$$

This result is similar to the one found by Alcubierre [1], with the difference that Ref. [1] did not consider the cosmological constant. Considering the results in Ref. [1] we realized that both the weak and dominant energy conditions would be violated [26] if the bubble was formed. However, these same energy conditions would be satisfied in the case of a vacuum solution, which discloses the new result that the warp drive metric is a vacuum solution for the Einstein equations. Besides, the Burgers equation is connected to this geometry where shock waves are partial solutions. Here, with the inclusion of the cosmological constant it may be possible that the weak and dominant energy conditions could be satisfied if Λ is positive and large enough in Eq. (2.25).

3. MATTER CONTENT ENERGY-MOMENTUM TENSORS

3.1. Perfect fluid energy momentum tensor

For Eulerian observers 4-velocity $u^\alpha = (1, -\beta, 0, 0)$ and $u_\alpha = (-1, 0, 0, 0)$ the perfect fluid EMT for those observers is given by the expression below,

$$T_{\alpha\beta} = (\mu + p)\, u_\alpha u_\beta + p g_{\mu\nu} \,, \tag{3.1}$$

where μ is a scalar function that represents the matter density, p is the fluid pressure, and $g_{\mu\nu}$ is the metric tensor. It must be noted that the dust EMT is a particular case for the perfect fluid with null pressure.

From the Einstein tensor components, Eqs. (2.10) to (2.19), and the perfect fluid EMT it is possible to write all the components of Einstein equations. After some algebraic work we found the following set of equations,

$$\frac{4}{3}\Lambda = 8\pi \left[T_{00} + 2\beta T_{01} + \left(\beta^2 - \frac{1}{3} \right) T_{11} \right] = 8\pi \left(\mu - \frac{1}{3}p \right) \,, \tag{3.2}$$

$$\left(\frac{\partial\beta}{\partial y} \right)^2 + \left(\frac{\partial\beta}{\partial z} \right)^2 - 4\Lambda = -32\pi \left(T_{00} + 2\beta T_{01} + \beta^2 T_{11} \right) = -32\pi\mu \,, \tag{3.3}$$

$$\frac{\partial^2\beta}{\partial y^2} + \frac{\partial^2\beta}{\partial z^2} = 16\pi(T_{01} + \beta T_{11}) = 0 \,, \tag{3.4}$$

$$\left(\frac{\partial\beta}{\partial y} \right)^2 + \left(\frac{\partial\beta}{\partial z} \right)^2 - \frac{4}{3}\Lambda = -\frac{32}{3}\pi T_{11} = -\frac{32}{3}\pi p \,, \tag{3.5}$$

$$-\frac{\partial}{\partial x} \left(\frac{\partial\beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2) \right) - 2\Lambda = 8\pi(T_{33} + T_{22}) = 16\pi p \,, \tag{3.6}$$

$$\frac{\partial^2\beta}{\partial x \partial y} = -16\pi(T_{02} + \beta T_{12}) = 0 \,, \tag{3.7}$$

$$\frac{\partial^2\beta}{\partial x \partial z} = -16\pi(T_{03} + \beta T_{13}) = 0 \,, \tag{3.8}$$

$$\frac{\partial\beta}{\partial y}\frac{\partial\beta}{\partial z} = 16\pi T_{23} = 0 \,, \tag{3.9}$$

3.2. Solving the Einstein equations with Λ for the perfect fluid

From Eq. (3.9) it follows that either $\partial\beta/\partial z = 0$, or $\partial\beta/\partial y = 0$, or both vanish. From Eqs. (3.7) and (3.8) it is easy to see that $\partial\beta/\partial x$ can also be zero. Those cases reveals four possibilities, which we will discuss in detail as follows.

Case 1: $\left[\dfrac{\partial\beta}{\partial z}=0\right]$

Case 1a: $\left[\dfrac{\partial\beta}{\partial z}=0 \ \text{ and } \ \dfrac{\partial\beta}{\partial x}=0\right]$ For this case Eqs. (3.2) to (3.9) simplify to,

$$\Lambda = 6\pi\left(\mu - \frac{p}{3}\right),\tag{3.10}$$

$$\left(\frac{\partial\beta}{\partial y}\right)^2 = 4(\Lambda - 8\pi\mu),\tag{3.11}$$

$$\left(\frac{\partial\beta}{\partial y}\right)^2 = \frac{4}{3}\pi(\Lambda - 8\pi p).\tag{3.12}$$

The set of these last equations implies that the shift vector β is not uniquely defined. It is a complex valued function that depends only on (t,y) spacetime coordinates. In the case of the dust EMT as $p = 0$, the warp drive metric is no longer a vacuum solution as it was found in [26], because the existence of the cosmological constant as another parameter originated a solution that does not consider shock waves via the Burgers equation.

Case 1b: $\left[\dfrac{\partial\beta}{\partial z}=0 \ \text{ and } \ \dfrac{\partial\beta}{\partial y}=0\right]$ For this case one has to solve the following equations,

$$\Lambda = 8\pi\mu = 8\pi p = 0,\tag{3.13}$$

$$-\frac{\partial}{\partial x}\left(\frac{\partial\beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2)\right) = 0.\tag{3.14}$$

Eq. (3.14) is the Burgers equation that connects the warp drive to shock waves, as discussed in Ref. [26]. The cosmological constant, fluid pressure and matter density are all equal to zero and the warp drive metric (2.6) is a vacuum solution for the Einstein equations.

Case 2: $\left[\dfrac{\partial\beta}{\partial y}=0\right]$

Case 2a: $\left[\dfrac{\partial\beta}{\partial y}=0 \ \text{and} \ \dfrac{\partial\beta}{\partial x}=0\right]$ For this configuration, the set of Eqs. (3.2) to (3.9) simplify to

$$\Lambda = 6\pi\left(\mu - \frac{p}{3}\right),\tag{3.15}$$

$$\left(\frac{\partial\beta}{\partial z}\right)^2 = 4(\Lambda - 8\pi\mu),\tag{3.16}$$

$$\left(\frac{\partial \beta}{\partial z}\right)^2 = \frac{4}{3}\left(\Lambda - 8\pi p\right). \tag{3.17}$$

The above set of equations are very similar to Case 1a, where the shift vector is a complex valued function and it is not uniquely defined, but in this case β depends on the (t, z) coordinates.

Case 2b: $\left[\dfrac{\partial \beta}{\partial z} = 0 \ \text{ and } \ \dfrac{\partial \beta}{\partial y} = 0\right]$ For this case, one has to solve the following equations

$$\Lambda = 8\pi\mu = 8\pi p = 0\,, \tag{3.18}$$

$$-\frac{\partial}{\partial x}\left(\frac{\partial \beta}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(\beta^2)\right) = 0\,, \tag{3.19}$$

which is the same case as Case 1b.

3.3. Divergence for the perfect fluid EMT

Calculating the divergence for the perfect fluid EMT, one arrives at the following equations,

$$T^{0\nu}{}_{;\nu} = -(\mu + p)\frac{\partial \beta}{\partial x} - \beta\frac{\partial(p + \mu)}{\partial x} - \frac{\partial \mu}{\partial t}\,, \tag{3.20}$$

$$T^{1\nu}{}_{;\nu} = \frac{\partial p}{\partial x}\,, \tag{3.21}$$

$$T^{2\nu}{}_{;\nu} = \frac{\partial p}{\partial y}\,, \tag{3.22}$$

$$T^{3\nu}{}_{;\nu} = \frac{\partial p}{\partial z}\,. \tag{3.23}$$

Besides, imposing the null divergence condition, Eqs. (3.20) to (3.23) implies that the pressure p does not depend on the spatial coordinates. Considering cases 1a and 2a there is another partial differential equation to solve,

$$\beta\frac{\partial \mu}{\partial x} + \frac{\partial \mu}{\partial t} = 0\,, \tag{3.24}$$

and for Cases 1b and 2b Eq. (3.20) is trivially satisfied since $\mu = p = 0$.

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4. ENERGY CONDITIONS

4.1. Weak Energy Conditions

For this case the EMT at each point of the spacetime must obey the inequality

$$T_{\alpha\sigma} \, u^\alpha u^\sigma \geq 0 \tag{4.1}$$

for any timelike vector $\mathbf{u}$ $(u_\alpha u^\alpha < 0)$ and any null zero vector $\mathbf{k}$ $(k_\alpha k^\alpha = 0)$. For an observer with unit tangent vector $\mathbf{v}$ at a certain point of the spacetime, the local energy density measured by any observer is non-negative [29]. For the perfect fluid EMT the expression $T_{\alpha\sigma} \, u^\alpha u^\sigma$ is

$$T_{\alpha\sigma} \, u^\alpha u^\sigma = \mu \ , \tag{4.2}$$

and the weak energy condition from Eq. (4.1) is satisfied if the matter density μ is positive. This is also the case for the dust EMT.

4.2. Dominant Energy Conditions

For every timelike vector u_α, the following inequality must be satisfied,

$$T^{\alpha\beta} \, u_\alpha u_\beta \geq 0, \quad \text{and} \quad F^\alpha F_\alpha \leq 0 \ , \tag{4.3}$$

where $F^\alpha = T^{\alpha\beta} u_\beta$ is a non-spacelike vector, and the following condition must also be satisfied

$$T^{00} \geq |T^{\alpha\beta}|, \text{ for each } \alpha, \beta \ . \tag{4.4}$$

Evaluating the first condition for the perfect fluid EMT we have that,

$$T^{\alpha\beta} \, u_\alpha u_\beta = \mu. \tag{4.5}$$

The other condition $F^\alpha F_\alpha$ is given by the result

$$F^\alpha F_\alpha = -\mu^2 \leq 0 \ . \tag{4.6}$$

Hence, the dominant energy condition is satisfied for $\mu > 0$, as can be seen in Eq. (4.5). Besides, Eq. (4.6) is always satisfied no matter the sign of the matter density. This condition also holds true if one considers the dust EMT as a particular case for the perfect fluid with null pressure.

4.3. Strong Energy Conditions

For the strong energy condition the expression

$$\left(T_{\alpha\beta} - \frac{1}{2} T \, g_{\alpha\beta} \right) u^\alpha u^\beta \geq 0 \tag{4.7}$$

is true for any timelike vector u. Computing the strong energy condition in Eq. (4.7) yields,

$$\left(T_{\alpha\beta} - \frac{1}{2}T\,g_{\alpha\beta}\right)u^\alpha u^\beta = \frac{1}{2}(3p + \mu) \,, \tag{4.8}$$

and the strong energy condition stated in Eq. (4.7) is satisfied if $3p + \mu \geq 0$. The same is valid for the dust EMT, considering $p = 0$ for the perfect fluid, if $\mu \geq 0$.

4.4. Null Energy Conditions

The null energy conditions are satisfied in the limit of null observers. For the null vector $\mathbf{k}$ the following conditions must be satisfied,

$$T_{\alpha\sigma}\,k^\alpha k^\sigma \geq 0, \qquad \text{for any null vector } k^\alpha \,. \tag{4.9}$$

Assuming that the following null vector k^α is given by,

$$k^\alpha = (a, b, 0, 0) \,, \tag{4.10}$$

we have that the relation between the components a and b are obtained by solving $k_\alpha k^\alpha = 0$. The two solutions given by,

$$a = \frac{b}{\beta + 1} \qquad \text{and} \qquad a = \frac{b}{\beta - 1} \,. \tag{4.11}$$

Then, the null energy condition reads,

$$T_{\alpha\sigma}\,k^\alpha k^\sigma = \left(\frac{b}{\beta \pm 1}\right)^2 (\mu + p) \,, \tag{4.12}$$

and the null energy condition may be satisfied if the following conditions are written as

$$\mu + p \geq 0 \,. \tag{4.13}$$

Eq. (4.13) is also true for the dust EMT if one considers it as a particular case for the perfect fluid with zero pressure, then the null energy condition is satisfied for the dust if the matter density is positive.

5. DUST AS A PARTICULAR CASE FROM THE PERFECT FLUID

Table I summarizes the results found for the energy conditions for the perfect fluid with the cosmological constant that are widely known [29]. Considering the dust EMT as a particular case for the perfect fluid by imposing the pressure p to be zero, the energy conditions for the warp drive metric and the dust EMT would be trivially satisfied, since for this case, the solution of the Einstein equations is a vacuum solution [26].

TABLE I: Summary results for the perfect fluid energy conditions

Energy condition	Results
Weak	$\mu \geq 0$
Strong	$\mu \geq 0$
Dominant	$\mu + 3p \geq 0$
Null	$\mu + p \geq 0$

Table II summarizes the solutions of the Einstein equations for the perfect fluid EMT with the cosmological constant and the warp drive metric. As can be seen there are two types of solutions and each is divided in two sub cases. Solutions 1b and 2b are identical and require that $\Lambda = p = \mu = 0$, where the two solutions are the ones already found in Ref. [26] for the dust of non interacting particles EMT. This led to a vacuum solution of the Einstein equations and the connection between shock waves and the warp drive via the Burgers equation.

Solutions 1a and 2a in table II have structures very similar to the ones of the same type of equations, but for the solution 1a the shift vector is a function of both the time and the y-spatial coordinates, i.e., $\beta = \beta(y, t)$. For solution 2a it is a function of both the time and the z-spatial coordinate, i.e., $\beta = \beta(z, t)$.

If we consider the dust solution as a particular case of perfect fluid with the imposition that the pressure is zero, we have that the four sets of partial differential equations in Table II become a solution for the warp drive metric and the dust EMT with the cosmological constant. In the case of dust EMT there is no longer a set of equations 1a and 2a to be solved, only 1b and 2b, that are identical to the ones appearing in Ref. [26]. Even with a cosmological constant the dust EMT seems to be not a stable source of matter, energy and momentum for the warp drive.

6.   CONCLUSIONS AND FINAL REMARKS

In this work we investigated how the presence of a cosmological constant would affect the solutions of the Einstein equations endowed with the Alcubierre warp drive metric and the perfect fluid EMT as the source. Firstly, we solved the Einstein equations and obtained two solutions, Cases 1b and 2b, that are similar to the solutions we found for the dust particle without the cosmological constant [26], and two other solutions, Case 1a with $\beta = \beta(t, y)$ and Case 2a with $\beta = \beta(t, z)$, having the following equation of state relating the cosmological constant Λ, the matter density μ and the fluid pressure p: $\Lambda = 2\pi(3\mu - p)$.

The presence of the cosmological constant allows the shift vector to be a real valued function as can be seen from Eqs. (3.11) and (3.12) for Case 1a in Table II, and Eqs. (3.16) and (3.17) for Case 2a, namely, $\Lambda - 8\pi\mu \geq 0$ and $\Lambda - 8\pi p \geq 0$.

If we do not consider the cosmological constant, then the shift vector would become a complex valued function for Cases 1a and 2a in Table II. The energy conditions are all satisfied for the perfect fluid if the conditions in Table I are satisfied. Solutions 1b and 2b shown in Table II connect the Burgers equation to both the warp drive and the perfect fluid

Case	Condition	Results
1) $\dfrac{\partial \beta}{\partial z} = 0$	1a) $\dfrac{\partial \beta}{\partial x} = 0$	$\Lambda = 6\pi \left(\mu - \frac{p}{3}\right)$ $\beta = \beta(y, t)$ $\dfrac{\partial \beta}{\partial y} = \pm\sqrt{4(\Lambda - 8\pi\mu)}$ $\dfrac{\partial \beta}{\partial y} = \pm\sqrt{\frac{4}{3}(\Lambda - 8\pi p)}$ $\beta \dfrac{\partial \mu}{\partial x} + \dfrac{\partial \mu}{\partial t} = 0$ (null divergence)
	1b) $\dfrac{\partial \beta}{\partial y} = 0$	$\Lambda = 8\pi\mu = 8\pi p = 0$ $\beta = \beta(x, t)$ $\dfrac{\partial \beta}{\partial t} + \dfrac{1}{2}\dfrac{\partial}{\partial x}(\beta^2) = h(t)$ Null divergence is trivially satisfied This is the solution found in Ref. [26]
2) $\dfrac{\partial \beta}{\partial y} = 0$	2a) $\dfrac{\partial \beta}{\partial x} = 0$	$\Lambda = 6\pi \left(\mu - \frac{p}{3}\right)$ $\beta = \beta(y, t)$ $\dfrac{\partial \beta}{\partial z} = \pm\sqrt{4(\Lambda - 8\pi\mu)}$ $\dfrac{\partial \beta}{\partial z} = \pm\sqrt{\frac{4}{3}(\Lambda - 8\pi p)}$ $\beta \dfrac{\partial \mu}{\partial x} + \dfrac{\partial \mu}{\partial t} = 0$ (null divergence)
	2b) $\dfrac{\partial \beta}{\partial z} = 0$	$\Lambda = 8\pi\mu = 8\pi p = 0$ $\beta = \beta(x, t)$ $\dfrac{\partial \beta}{\partial t} + \dfrac{1}{2}\dfrac{\partial}{\partial x}(\beta^2) = h(t)$ Null divergence is trivially satisfied This is the solution found in Ref. [26]

TABLE II: Summary of all solutions of the Einstein equation with the cosmological constant and the Alcubierre warp drive metric having the perfect fluid EMT as mass-energy source. This table is also valid for the dust particle if considered as a particular case for the perfect fluid with null pressure.

solution. In Ref. [26] we found this intrinsic relationship between the warp drive and shock waves by solving Einstein equations for the warp drive metric and the dust particle EMT, but we concluded that there is an impossibility of coupling the dust as a source in this case. So, the presence of shock waves would imply that the Alcubierre metric shown in Eq. (2.6) is a vacuum solution for the warp drive. In Ref. [1] the Einstein equations were not solved,

since the metric was merely guessed with a form function (see Eq. 2.8) that rules the warp bubble shape.

The results found here led us to a kind of prescription where the warp drive requires more complex forms of matter than dust in order to obtain stable solutions. In addition, considering this work and the previous ones of this series of papers [26–28] it becomes increasingly clear and that negative matter density may not be a strict requirement to obtain warp speeds. The shift vector in the direction of the warp bubble movement creates a coupling in the Einstein equations that requires off-diagonal terms in the EMT source. In the light of these results we may conjecture that the key for engineering a superluminal propelling system for interstellar travel could be understood as a complex distribution of energy, matter and momentum sources that could stabilize the warp drive geometry, allowing then superluminal travel.

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