Photon propagation in a stationary warp drive space-time

Claes R Cramer*
School of Mathematical Sciences,
Queen Mary & Westfield College,
Mile End Road
London E1 4NS, UK

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Abstract

We simplify the warp drive space-time so that it becomes stationary and the distorsion becomes one-dimensional and static. We use this simplified warp drive space-time as a background for a photon field. We shall especially use the Drummond&Hathrell action in order to investigate the velocity effects on photons in this background. Finally, we discuss the limitations of this model.

*e-mail: C.R.Cramer@maths.qmw.ac.uk

1 Introduction

It is known that General Relativity allows for superluminal velocities and that the most common way of achieving this effect is to use wormholes. It is also known that this necessarily involves non-trivial topologies and use of exotic matter, see e.g Morris&Thorne for a nice introduction to the subject [1]. Another way to send information faster than the speed of light, but without introducing non-trivial topologies, would be to use M. Alcubierre's *warp drive* [2]. His model for superluminal travel uses a distorsion of the space-time such that an expansion is created behind the particle and a contraction in front of it. This model is also plagued with the need for exotic matter. However, it is not clear to what extent this is a real drawback since it is known that space-time curvature can induce negative stress-energy-momentum, at least at the quantum level, see e.g. Birrell&Davis [3].

The models mentioned above are both classical. In the non-classical realm has it been shown that if vacuum polarization induced interactions between a photon and a gravitational field is taken into consideration then there exist a possibility for superluminal velocities , e.g. Drummond and Hathrell have shown that in a Schwarzschild field photons in their one-loop state may propagate faster than unity [4].

In this paper we study photon propagation at the quantum level in a simplified warp drive space-time. This will be done by taking into consideration that vacuum polarization induces interactions between the gravitational field and the photon. In section 2 we briefly review the warp-drive model and simplify it so that the space-time becomes stationary and the space-time distorsion static. In section 3 we use the Drummond&Hathrell action in order to study photon propagation in this simplified warp drive spacetime. In section 4 we discuss our results.

2 The warp drive

In a recent letter to the editor M. Alcubierre discussed a model for hyperfast travel within General Relativity which does not make use of a non-trivival topology. The model is based on the following idea: create a local distorsion of the space-time such that it induces an expansion behind the particle subject to hyperfast travel and a contraction ahead of it. This implies that the particle in the distorted space-time will travel faster between two points, say A and B, than a particle in Minkowski space-time. In fact one can make the time it takes for the particle to travel between A and B arbitrarily small [2].

The warp drive space-time is taken to be globally hyperbolic in order to avoid anomalies such as closed non-spacelike curves. The line element is given by

$$ds = dt^2 - (dx + \dot{x}_s V dt)^2 - dy^2 - dz^2 \tag{1}$$

where $\dot{x}_s$ is the particle velocity and V a function given by

$$V(r_s) = \frac{\tanh\left(\sigma(r_s + R)\right) - \tanh\left(\sigma(r_s - R)\right)}{2\tanh\left(\sigma R\right)} \tag{2}$$

and where R and σ are arbitrary parameters and

$$r_s = \sqrt{(x - x_s(t))^2 + y^2 + z^2} \tag{3}$$

Notice that for large values of the parameter σ V very rapidly approaches a top-hat function, i.e. V is 1 when $r_s \in [-R, R]$ and 0 otherwise.

A quick look at the line element above immediately reveals that the space-time is not stationary, since the particle velocity and the top-hat function are not time independent. Now, we shall let $\dot{x}_s$ equal unity and remove the time dependency in the top-hat function so that it becomes static, since we in this paper intend to study photon propagation in a stationary space-time with a static distorsion. Furthermore, we shall let the top-hat be dependent on only one of the space-time coordinates, say x. The line element then becomes

$$ds = dt^2 - (dx + V dt)^2 - dy^2 - dz^2 \tag{4}$$

where $V = V(x)$ now is given by

$$V(x) = \frac{\tanh\left(\sigma(x + R)\right) - \tanh\left(\sigma(x - R)\right)}{2\tanh\left(\sigma R\right)}. \tag{5}$$

3 The photon field in the warp drive space-time

If space-time is curved as in the warp drive case, and if we take into consideration one-loop vacuum polarization effects then we cannot be sure that the photon travels with the speed of light in a manner independent of its polarization state. This follows since the transition of the photon into a virtual electron positron pair gives the photon a size related to the Compton wavelength of the electron. This means that the photon can be influenced by the gravitational field and hence the motion of the photon can be polarization dependent.

The equation of motion for the photon field in a gravitational field is

$$\frac{\delta S}{\delta A_\mu} = 0 \tag{6}$$

where S is the effective action given by

$$S = \frac{1}{m^2}\int d^4x\sqrt{-g}(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + f). \tag{7}$$

f is the *Drummond-Hathrell* functional which incorporates the effects of virtual electron loops [4]

$$f = f(R, R_{\mu\nu}, R_{\mu\nu\sigma\tau}) = aRF_{\mu\nu}F^{\mu\nu} + bR_{\mu\nu}F^{\mu\sigma}F^{\nu}{}_{\sigma} + cR_{\mu\nu\sigma\tau}F^{\mu\nu}F^{\sigma\tau} \tag{8}$$

where

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\frac{\alpha}{720\pi}\begin{pmatrix} 5 \\ -26 \\ 2 \end{pmatrix}. \tag{9}$$

Using equation (6) and (7) one easily finds the following equation of motion for the photon field

$$D_\mu F^{\mu\nu} + \frac{1}{m^2}D_\mu(4aRF^{\mu\nu} + 2b(R^{\mu}{}_{\sigma}F^{\sigma\nu} - R^{\nu}{}_{\sigma}F^{\sigma\mu}) + 4cR^{\mu\nu}{}_{\sigma\tau}F^{\sigma\tau}) = 0. \tag{10}$$

The equation above can be simplified by neglecting a number of terms of subleading order [5]

$$D_\mu F^{\mu\nu} + \frac{1}{m^2}(2bR^{\mu}{}_{\sigma}D_\mu F^{\sigma\nu} + 4cR^{\mu\nu}{}_{\sigma\tau}D_\mu F^{\sigma\tau}) = 0. \tag{11}$$

In order to derive the propagation equation for the photon we use the assumption, used in writing equation (11), that the photon wavelength is small compared to the curvature scale. We shall also assume that the photon wavelength is small compared to the typical length over which the amplitude, polarization and wavelength vary. This implies that we can write the field tensor $F_{\mu\nu}$ as

$$F_{\mu\nu} = f_{\mu\nu} e^{i\theta} \tag{12}$$

where $f_{\mu\nu}$ is a slowly varying amplitude and θ a rapidly varying phase. This is the geometric optics approximation, see e.g. Wheeler et al. [6]. If we now let the wave vector k^μ be defined by

$$k_\mu := D_\mu \theta \tag{13}$$

and use the geometrical optics approximation we obtain from equation (11)

$$k_\mu f^{\mu\nu} + \frac{1}{m^2}(2bR^\mu{}_\sigma k_\mu f^{\sigma\nu} + 4cR^{\mu\nu}{}_{\sigma\tau} k_\mu f^{\sigma\tau}) = 0. \tag{14}$$

Now, multiply the above equation with k^λ and antisymmetrize

$$-k_\mu k^{[\lambda} f^{\nu]\mu} + \frac{1}{m^2}(-2bR^\mu{}_\sigma k_\mu k^{[\lambda} f^{\nu]\sigma} - 4ck^{[\lambda} R^{\nu]\mu}{}_{\sigma\tau} k_\mu f^{\sigma\tau}) = 0. \tag{15}$$

Combining this equation with the Bianchi identity

$$k_\lambda f_{\mu\nu} + k_\mu f_{\nu\lambda} + k_\nu f_{\lambda\mu} = 0 \tag{16}$$

gives

$$(k^2 + \frac{2}{m^2} bR^\mu{}_\sigma k_\mu k^\sigma) f^{\lambda\nu} - \frac{8c}{m^2} k^{[\lambda} R^{\nu]\mu}{}_{\sigma\tau} k_\mu f^{\sigma\tau} = 0. \tag{17}$$

This is a homogeneous equation for $f^{\lambda\nu}$ which constrains it to have no more than three independent components. Furthermore, in order to have non-trivial solutions the coefficient determinant has to vanish. It is this determinant condition which will enable us to find the propagation equation for the photons in the warp drive space-time.

The line element for the simplified stationary warp drive was given in section 2 as

$$ds^2 = dt^2 - (dx + V dt)^2 - dy^2 - dz^2. \tag{18}$$

From the line element above we easily find that the contravariant components of the metric are given by

$$g^{\mu\nu} = \begin{pmatrix} 1 & -V & 0 & 0 \\ -V & V^2 - 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{19}$$

If we introduce a Newman-Penrose nulltetrad $(l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$ we can write the metric as

$$g^{\mu\nu} = 2l^{(\mu} n^{\nu)} - 2m^{(\mu} \bar{m}^{\nu)}. \tag{20}$$

In order to construct the null tetrad for the warp drive space-time we make the following ansatz

$$l^\mu = (A, B, 0, 0) \tag{21}$$

$$n^\mu = (C, D, 0, 0) \tag{22}$$

$$m^\mu = (0, 0, E, F) \tag{23}$$

where A, B, C, D, E and F are determined by equation (20) and the contravariant components for the space-time. Thus after some simple algebra we find that the null tetrad is

$$l^\mu = (\frac{-1}{V+1}, 1, 0, 0) \tag{24}$$

$$n^\mu = (-\frac{V+1}{2}, \frac{V^2-1}{2}, 0, 0) \tag{25}$$

$$m^\mu = \frac{1}{\sqrt{2}}(0, 0, 1, i). \tag{26}$$

If we now introduce the restricted Minkowski tetrad

$$e_{\hat{0}}{}^{\mu} := \frac{1}{\sqrt{2}}(l^\mu + n^\mu) \tag{27}$$

$$e_{\hat{1}}{}^{\mu} := \frac{1}{\sqrt{2}}(m^\mu + \bar{m}^\mu) \tag{28}$$

$$e_{\hat{2}}{}^{\mu} := \frac{i}{\sqrt{2}}(m^\mu - \bar{m}^\mu) \tag{29}$$

$$e_{\hat{3}}{}^{\mu} := \frac{1}{\sqrt{2}}(l^\mu - n^\mu), \tag{30}$$

the bivectors

$$U^{\hat{0}\hat{1}}_{\mu\nu} := e^{\hat{0}}{}_{\mu} e^{\hat{1}}{}_{\nu} - e^{\hat{0}}{}_{\nu} e^{\hat{1}}{}_{\mu} \tag{31}$$
$$V^{\hat{0}\hat{2}}_{\mu\nu} := e^{\hat{0}}{}_{\mu} e^{\hat{2}}{}_{\nu} - e^{\hat{0}}{}_{\nu} e^{\hat{2}}{}_{\mu} \tag{32}$$
$$W^{\hat{1}\hat{2}}_{\mu\nu} := e^{\hat{1}}{}_{\mu} e^{\hat{2}}{}_{\nu} - e^{\hat{1}}{}_{\nu} e^{\hat{2}}{}_{\mu}, \tag{33}$$

the symmetric tensors

$$H^{\hat{0}\hat{0}}_{\mu\nu} := e^{\hat{0}}{}_{\mu} e^{\hat{0}}{}_{\nu} \tag{34}$$
$$G^{\hat{0}\hat{1}}_{\mu\nu} := -2 e^{\hat{0}}{}_{(\mu} e^{\hat{1}}{}_{\nu)} \tag{35}$$
$$I^{\hat{1}\hat{1}}_{\mu\nu} := e^{\hat{1}}{}_{\mu} e^{\hat{1}}{}_{\nu} \tag{36}$$

and use the non-vanishing components of the Riemann- and the Ricci tensor

$$p := R_{0101} = V \frac{d^2 V}{dx^2} + \left(\frac{dV}{dx} \right)^2 \tag{37}$$

$$q := R_{00} = (V^2 - 1)p \tag{38}$$
$$r := R_{01} = Vp \tag{39}$$
$$s := R_{11} = p \tag{40}$$

then we may write the Riemann and the Ricci tensor as

$$R_{\mu\nu\sigma\tau} = p U^{\hat{0}\hat{1}}_{\mu\nu} U^{\hat{0}\hat{1}}_{\sigma\tau} \tag{41}$$
$$R_{\mu\nu} = q H^{\hat{0}\hat{0}}_{\mu\nu} + r G^{\hat{0}\hat{1}}_{\mu\nu} + s I^{\hat{1}\hat{1}}_{\mu\nu}. \tag{42}$$

Hence, we may write equation (17) as

$$[k^2 + \frac{2b}{m^2}(q H^{\hat{0}\hat{0}}_{\mu\tau} + r G^{\hat{0}\hat{1}}_{\mu\tau} + s I^{\hat{1}\hat{1}}_{\mu\tau}) k^\mu k^\tau)] f^{\lambda\nu} - \frac{8cp}{m^2} l^{[\lambda} k^{\nu]} g = 0 \tag{43}$$

where g and l^ν are defined by

$$g := U^{\hat{0}\hat{1}}_{\mu\nu} f^{\mu\nu} \tag{44}$$
$$l^\nu := k_\mu U^{\mu\nu}_{\hat{0}\hat{1}} \tag{45}$$

Let us now introduce the physical components of $f^{\mu\nu}$ in the case of a plane wave propagating along the x-axis

$$v := V^{\hat{0}\hat{2}}_{\mu\nu} f^{\mu\nu} \tag{46}$$

$$w := W^{\hat{1}\hat{2}}_{\mu\nu} f^{\mu\nu}. \tag{47}$$

It then follows from equation (43) that the determinant condition is

$$k^2 + \frac{2b}{m^2}(qH^{\hat{0}\hat{0}}_{\mu\tau} + rG^{\hat{0}\hat{1}}_{\mu\tau} + sI^{\hat{1}\hat{1}}_{\mu\tau})k^\mu k^\tau + \frac{8l^2pc}{m^2} = 0 \tag{48}$$

$$k^2 + \frac{2b}{m^2}(qH^{\hat{0}\hat{0}}_{\mu\tau} + rG^{\hat{0}\hat{1}}_{\mu\tau} + sI^{\hat{1}\hat{1}}_{\mu\tau})k^\mu k^\tau = 0 \tag{49}$$

$$k^2 + \frac{2b}{m^2}(qH^{\hat{0}\hat{0}}_{\mu\tau} + rG^{\hat{0}\hat{1}}_{\mu\tau} + sI^{\hat{1}\hat{1}}_{\mu\tau})k^\mu k^\tau = 0 \tag{50}$$

where the first condition is the unphysical polarization.

In order to investigate the propagation velocity for physical photons we use the second condition. We then get

$$(1 + \frac{2b}{m^2}q)\dot{x}^2 + \frac{4br}{m^2}\dot{x} + \frac{2sb}{m^2} - 1 = 0 \tag{51}$$

where

$$\dot{x} := \frac{k_0}{k_1} \tag{52}$$

is the photon velocity.

Consider the case when V is close to one. Then it is clear from equation (38) that we can neglect the term $2bq/m^2$. A further simplification can be made by neglecting terms of order m^{-4}. Then it follows to $o(2)$ that

$$\dot{x} = 1 - \frac{b}{m^2}(2r + s) = 1 - \frac{b}{m^2}\left[(2V + 1)p\right] \tag{53}$$

where we have used equation (39) and (40). It is relatively easy to see that the expression inside the bracket is negative semi-definite for large values of the parameter σ and for V close to one. Thus we can conclude that the photon increases its velocity in the region where the space-time is distorted and V is close to one. If V is close to zero then it clear that $1 + 2bq/m^2$ can be taken to equal one, since the first and the second derivative of V are not going to be of significant magnitude. Thus this means the equation above still can be taken to represent the photon velocity and from it we then immediately conclude that the photon velocity is going to be less than unity since the expression inside the bracket is going to be larger than zero. This means that we can conclude that a photon traveling along the x-axis in a stationary warp drive space-time with a static distorsion will do so faster than the speed of light in some regions and slower than the speed of light in other regions.

4 Discussion

In the orginal warp drive there was no need to raise the question about a causality breach, since the space-time was taken to be globally hyperbolic and because locally a photon would always be massless, i.e the lightcone would remain unchanged. In the present case we cannot use global hyperbolicity as an argument to claim that causality is not breached. This follows since the local effective lightcone is larger than the ordinary local lightcone, see equation (17). What we can do is to argue that there does not necessarily have to occur any causality breach, since the action (7) involves terms which violates the strong equivalence principle. This follows since in special relativity the necessary conditions for a causality breach are the Lorentz group and spacelike motion, see e.g. Rindler for a simple example of this [7]. In general relativity the fiber is not of course identical with the Lorentz group and the closest we can get to the Lorentz group is the equivalence of the physical laws in every local inertial frame, but this is the condition which is breached since we allow strong equivalence principle violating terms in the action, see Shore for a similar discussion [5]. Thus it may seem that there does not occur any causality breach, but if we were to be sure that no closed curves exist at all then we would be forced to prove that the class of globally hyperbolic topologies do not allow any closed curves for photons described by the Drummond&Hathrell action.

Let us now consider some restrictions and drawbacks of this model. One restriction of this model is that the photon wavelength has to be much greater than the Compton wavelength [4]. This implies that this model can only be valid if the photons are of low energy. Furthermore, the next limitation comes through the approximation in the equation of motion for the photon field and the geometrical optics approximation which demands that the photon wavelength be much smaller than the curvature scale. If we were to consider a more general model we would have to add a term like

$$D_\mu F^{\mu\nu} D_\sigma F^\sigma_{\ \nu} \tag{54}$$

to the effective action. We would also be forced to consider derivatives of the Riemann, Ricci tensor and the curvature scalar in the equation of motion.

Let us also notice that since this model uses a one-dimensional static distorsion we can immediately conclude that it hardly can be physical. This can also be concluded by calulating the energy momentum tensor which must be regarded as unphysical, since it has pressure but no energy density. If we were to present a more physical model we would be forced to extend the distorsion at least in one more space dimension and we should also take into consideration back-reaction effects from the photon on the gravitational field.

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