Royal Society Publishing
Informing the science of the future

The Relations Connecting the Angle-Sums and Volume of a Polytope in Space of n Dimensions
Author(s): D. M. Y. Sommerville
Source: *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, Vol. 115, No. 770 (Jun. 1, 1927), pp. 103-119
Published by: The Royal Society
Stable URL: http://www.jstor.org/stable/94871
Accessed: 31/12/2009 16:11

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/action/showPublisher?publisherCode=rsl.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

The Royal Society is collaborating with JSTOR to digitize, preserve and extend access to *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character.*

http://www.jstor.org

The Relations Connecting the Angle-Sums and Volume of a Polytope in Space of n Dimensions.

By D. M. Y. Sommerville, Victoria University College, Wellington, N.Z.

(Communicated by Major P. A. MacMahon, F.R.S.—Received November 29, 1926.)

§ 1. *Introduction.*

1.1. In two-dimensional spherical or elliptic geometry we have the familiar relation between the area of a triangle and its angle-sum,

$$\Delta = k^2 (\Sigma\alpha - \pi), \tag{1.11}$$

2π being the measure of the whole angle at a point, and k the space-constant, or, in spherical geometry, the radius of the sphere. It is well known that in three-dimensional spherical or elliptic geometry there is no corresponding relation involving the volume of a tetrahedron.* For elliptic or hyperbolic space of four dimensions it was proved by Dehn† that the volume of a simplex can be expressed linearly in terms of the sums of the dihedral angles (angles at a face), angles at an edge, and angles at a vertex, but for space of five dimensions the linear relations do not involve the volume. He indicates also, in a general way, the extensions of these results for spaces of any odd or even dimensions. He shows further that these results are connected with the form of the Euler‡ polyhedral theorem, which is expressed by a linear relation connecting the numbers of boundaries of different dimensions, and which for space of odd dimensions is not homogeneous, *e.g.*, $N_2 - N_1 + N_0 = 2$ in three dimensions, but for space of even dimensions is homogeneous, *e.g.*, $N_1 - N_0 = 0$ in two dimensions, $N_3 - N_2 + N_1 - N_0 = 0$ in four. The connection, as Dehn points out, was made use of by Legendre in a proof which he gave for the Euler formula in three dimensions.§

* H. W. Richmond has investigated an expression for the volume of a tetrahedron in elliptic space in terms of an integral, 'Q. J. Math.,' vol. 34, p. 175 (1903). Lobachevsky himself investigated similarly the volume of a tetrahedron in hyperbolic space, "Foundations of geometry," 1829, and "Application of imaginary geometry to some integrals," 1836.

† M. Dehn, "Die Eulersche Formel im Zusammenhang mit dem Inhalt in der nicht-euklidischen Geometrie," 'Math. Ann.,' vol. 61, p. 561 (1905).

‡ L. Euler, 'Mém. Pétersb.,' 1758.

§ A. M. Legendre, "Eléments de géométrie," Liv. vii, Prop. xxv. (1794). The proof is reproduced in Todhunter's "Spherical Trigonometry," Chap. xii, or Todhunter and Leathem, Chap. xvi.

1.2. Dehn extends this connection in detail for four and five dimensions, and states the following general results in space of n dimensions R_n for simplexes and for polytopes bounded entirely by simplexes :

(1) In R_n there are $\frac{1}{2}n+1$ or $\frac{1}{2}(n+1)$ relations (according as n is even or odd) between the numbers of boundaries of a polytope bounded by simplexes.

(2) There is a linear relation between the various angle-sums of a simplex (a *Zerlegungsinvariante*) which does or does not involve the volume according as n is even or odd.

In the present paper these relations are actually obtained, and it is found for any polytope bounded by simplexes that the two kinds of relations, those which connect the numbers of boundaries, and those which connect the angle-sums, are of precisely the same form.

§ 2. *Measure of angles.*

2.1. We must first consider the different types of angles and their measure.

In two dimensions there is just one type of angle to be considered, the angle between two directed straight lines or rays. Let O be the vertex; draw a circle with centre O, and let the two rays cut the circle in P, Q. The ratios of the arc PQ to the circumference, and the sector POQ to the area of the circle, are equal, and either may be taken as a measure of the angle. The whole angle at a point would then have the measure unity. In the *radian measure*, 2π is taken as the measure of the complete angle at a point; this is equivalent, in euclidean geometry, to taking as the measure of any angle the ratio of the arc to the radius.

2.2. In non-euclidean geometry the latter ratio is not constant for a given angle, but we may still take as the radian measure of the angle 2π times the ratio of the arc to the circumference, or 2π times the ratio of the sector to the area of the circle. In spherical geometry, if the radius is increased until the circle reduces to the point antipodal to O, the radian measure of the angle becomes 2π times the ratio of the whole area enclosed by the two rays to the whole area of the plane. In elliptic geometry, if the radius is increased until the circle becomes a straight line (the polar of O), the radian measure of the angle becomes 2π times the ratio of the whole area enclosed by the two rays to double the area of the plane. But if k is the space-constant the area of the whole plane is: in spherical geometry $4\pi k^2$, and in elliptic geometry $2\pi k^2$. In both spherical and elliptic geometry therefore the radian measure of an angle is the ratio of the area enclosed by the two rays to $2k^2$. (In spherical geometry the rays begin at O and end at the antipodal point O′; in elliptic geometry they begin and end at O.)

2.3. In three dimensions it is usual to distinguish two types of angles: the angle between two planes (dihedral angle), and the angle between three or more planes (solid angle). These, however, are to be considered as of the same type. Let O be the vertex, or, in the case of the dihedral angle, any point on the edge, and draw a sphere with centre O. Then we may take as a measure of the angle the ratio of the area which the planes cut out on the surface of the sphere to the whole surface of the sphere, or the corresponding ratio of volumes. The measure of the whole angle at a point or an edge would then be unity. In euclidean geometry it is customary to take as the measure of a solid angle the ratio of the area cut out on the surface of the sphere to the square of the radius. We may call this the *radian measure*; the measure of the complete solid angle at a point would then be 4π. We may therefore define the radian measure of a solid or dihedral angle as 4π times the ratio of the area cut out by the bounding planes on the surface of the sphere to the whole surface of the sphere, or 4π times the corresponding ratio of volumes. In spherical geometry the ratio of the volume enclosed between the planes and the sphere to the whole volume of the sphere becomes in the extreme case the ratio of the whole volume enclosed between the planes (on a specified side of each of them) to the whole volume of space. As the whole volume of spherical space of space-constant k is equal to the hypersurface of a hypersphere of radius k, $= 2\pi^2k^3$, we have: the radian measure of a solid or dihedral angle in spherical or elliptic geometry is the ratio of the volume enclosed between the planes to $\frac{1}{2}\pi k^3$.

2.4. Generally, in n dimensions there are angles bounded by $2, 3, \dots, n-1$, or more than $n-1$, hyperplanes. In euclidean geometry we define the radian measure of an angle as the ratio of the hypersurface cut out of a hypersphere whose centre is on the axis (vertex, edge, etc.) to the $(n-1)$th power of the radius. The hypersurface of a hypersphere of radius k in n dimensions is $k^{n-1}n\pi^{\frac{1}{2}n}/\Gamma\{\frac{1}{2}(n+2)\}$. Hence the radian measure of the complete angle at a point is $n\pi^{\frac{1}{2}n}/\Gamma\{\frac{1}{2}(n+2)\}$. In spherical or elliptic geometry similarly the radian measure of an angle is the ratio of the hypervolume enclosed by the bounding hyperplanes to

$$\frac{k^n(n+1)\pi^{\frac{1}{2}(n+1)}}{\Gamma\{\frac{1}{2}(n+3)\}}\cdot\frac{\Gamma\{\frac{1}{2}(n+2)\}}{n\pi^{\frac{1}{2}n}}, \text{ i.e., to } k^n\frac{n+1}{n}\frac{\sqrt{\pi}\,\Gamma\{\frac{1}{2}(n+2)\}}{\Gamma\{\frac{1}{2}(n+3)\}}, \tag{2.41}$$

and the radian measure of the complete angle is

$$\frac{n\pi^{\frac{1}{2}n}}{\Gamma\{\frac{1}{2}(n+2)\}}. \tag{2.42}$$

The total volume of spherical space is

$$\frac{k^n (n+1) \pi^{\frac{1}{2}(n+1)}}{\Gamma\{\frac{1}{2}(n+3)\}}. \tag{2.43}$$

2.5. In hyperbolic space also we may lay down corresponding measures of angles, although the total volumes are not now available. Thus we may define the radian measure of an angle as proportional to the volume or the surface cut out of the surrounding hypersphere, the numerical factor being so adjusted that the measure of the complete angle has the same value as in elliptic or euclidean space.

§ 3. *The relations connecting the angle-sums of a simplex in spherical space of n dimensions.*

3.1. A simplex in space of n dimensions is bounded by $n+1$ hyperplanes, and, generally, ${}_{n+1}C_{r+1}$ r-dimensional boundaries.

A single hyperplane divides space into two regions, which may be regarded as the positive and the negative side of the hyperplane, and denoted by $+$ and $-$. Two hyperplanes divide space into four regions, which may be distinguished by the signs $++$, $--$; $+-$, $-+$, and fall into two pairs of opposite or antipodal regions. Three hyperplanes give eight regions, and finally the $n+1$ hyperplanes divide space into 2^{n+1} regions, which may be distinguished by the $n+1$ signs $+$ and $-$ taken in a definite order. Each of these regions is the interior of a simplex, and we shall take the $(n+1)$ $+$ signs as denoting the interior of the particular simplex with which we are dealing.

Every set of n hyperplanes determines a pair of antipodal points. We shall denote the $n+1$ vertices of the simplex by $0, 1, 2, \ldots, n$, the antipodal points being denoted by $0', 1', 2', \ldots, n'$. The vertices of the 2^{n+1} simplexes are represented by the $n+1$ digits $0, 1, 2, \ldots, n$ with or without accents. But we shall find it convenient to suppress the accented figures. Thus the antipodal regions (interiors of simplexes) $(0'12\ldots n)$ and $(01'2'\ldots n')$ will be denoted by $(12\ldots n)$ and (0) respectively, showing their relation to the given simplex as standing on an $(n-1)$-dimensional boundary and a vertex respectively. $(012\ldots n)$ is the interior of the given simplex, and $(\ )$ is the interior of the antipodal simplex.

The number of regions of the type $(012\ldots r)$ is ${}_{n+1}C_{r+1}$, and there are the same number of antipodal regions $(r+1, r+2, \ldots, n)$. When n is even, antipodal regions are always of different type, but when n is odd the antipodal regions $\{01\ldots\frac{1}{2}(n-1)\}$ and $\{\frac{1}{2}(n+1), \ldots, n\}$, which both involve $\frac{1}{2}(n+1)$ figures, are of the same type. Thus, in three dimensions, there are 4 regions on the faces of a tetrahedron, and the antipodal regions are the 4 regions on the vertices, while the 6 regions on the edges form three pairs of antipodal regions. We shall denote any region of the same type as $(01\ldots r)$ by $[r+1]$.

3.2. We shall now consider the angular regions contained by any number of the bounding hyperplanes. The positive side of a hyperplane has already been implicitly defined as that region which contains the interior of the simplex, the region $(012\ldots n)$. The interior of any angular region is that which contains the interior of the simplex, or the region on the positive side of each of the bounding hyperplanes, and the antipodal region is that on the negative side of each of the bounding hyperplanes; we shall call the latter the *co-interior* of the angular space. We shall denote the content or volume of the interior of the angular space bounded by the two hyperplanes $123\ldots n$ and $023\ldots n$ by $\alpha_{23\ldots n}$, and that of the co-interior by $\alpha'_{23\ldots n}$; thus the number of suffixes is equal to the dimensions of the *axis* of the angular region. $\alpha_{23\ldots n}$ contains all those regions whose symbols include both 0 and 1, $\alpha'_{23\ldots n}$ all those whose symbols exclude both 0 and 1. Similarly $\alpha_{34\ldots n}$ denotes the content of the interior of the angular space bounded by the three hyperplanes $123\ldots n$, $023\ldots n$, $013\ldots n$, and contains all those regions whose symbols include 0, 1 and 2; and so on. $\alpha_{12\ldots n}$ will be taken to mean the whole region on the positive side of the hyperplane $12\ldots n$, $\alpha'_{12\ldots n}$ the negative side; *i.e.*, $\alpha_{12\ldots n}$ is equal to the half of space $= \frac{1}{2}S = \alpha'_{12\ldots n}$. $\alpha_{01\ldots n}$ can be taken to mean the whole of space $= S$. α will be taken to denote the volume V of the interior of the simplex, α' that of the antipodal simplex.

3.3. The number of angular spaces of the type $\alpha_{r+1,\ldots n}$ is ${}_{n+1}C_{r+1} = {}_{n+1}C_{n-r}$. This angle contains the following regions:

1 region $(012\ldots n)$,
${}_{n-r}C_1$ regions whose symbols are formed with n of the digits
$0, 1, \ldots, n$ always including $0, 1, 2, \ldots, r$,
${}_{n-r}C_2$ regions with $n-1$ digits, and so on,
1 region $(012\ldots r)$.

3.4. We next consider the sums of the angular regions of the same type. Let $\Sigma\,[r]$ be denoted by A_r, so that A_{n+1} denotes the volume of the simplex V, and A_0 the (equal) volume of the antipodal simplex. Let A'_r denote the sum of the regions antipodal to those which compose A_r. Then $A'_{r+1} = A_{n-r}$, both in total volume and in separate parts, and $A_{r+1} = A_{n-r}$ in volume.

Let $\Sigma\alpha_{01\ldots r}$, the summation extending to all angular regions with $r+1$ suffixes, be denoted by Σ_r, $\Sigma\alpha'_{01\ldots r}$ by Σ'_r. $\Sigma\alpha\,(=\alpha)$ may be denoted by

Σ_{-1} so that $\Sigma_{-1} = V$. $\Sigma\alpha_{01\ldots n} = \alpha_{01\ldots n} = \Sigma_n$ = the whole volume of space S. $\Sigma\alpha_{12\ldots n} = \Sigma_{n-1} = \frac{1}{2}(n+1)\,S = \frac{1}{2}(n+1)\,\Sigma_n$.

Now a given region $[s]$ $(s \geqq n-r)$ is contained in each of the angular regions of type $\alpha_{01\ldots r}$ whose suffixes contain all the $n+1-s$ numbers which are not included in the symbol of $[s]$, and the number of these angular regions is ${}_sC_{n-r}$; hence in the sum $\Sigma\alpha_{01\ldots r}$ each region $[s]$ occurs ${}_sC_{n-r}$ times. Hence

$$\begin{aligned}\Sigma_r &= {}_{n+1}C_{n-r}\,A_{n+1} + {}_nC_{n-r}\,A_n + \ldots + {}_{n-r}C_{n-r}\,A_{n-r} \\ &= {}_{n+1}C_{r+1}\,V + {}_nC_r\,A_1 + \ldots + {}_{n-r}C_0\,A_{r+1}. \qquad (3.41)\end{aligned}$$

Thus, putting $r = 0, 1, \ldots, n$, we have the following $n+1$ equations in $A_1, \ldots, A_n$.

$$\begin{aligned}\Sigma_0 &= {}_{n+1}C_1\,V + A_1.\\ \Sigma_1 &= {}_{n+1}C_2\,V + {}_nC_1\,A_1 + A_2,\\ &\ldots\ldots\ldots\ldots\\ \Sigma_{n-1} &= {}_{n+1}C_n\,V + {}_nC_{n-1}\,A_1 + {}_{n-1}C_{n-2}\,A_2 + \ldots + A_n,\\ \Sigma_n &= V + A_1 + A_2 + \ldots + A_n + V.\end{aligned}$$

Also the relations

$$A_{r+1} = A_{n-r}\,\{r = 0, 1, \ldots, \tfrac{1}{2}n - 1 \text{ or } \tfrac{1}{2}(n-1)\} \qquad (3.42)$$

supply $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$ further equations, according as n is even or odd. Hence by eliminating the A's we get, connecting the volume V and the angle-sums Σ_r, $\frac{1}{2}n+1$ or $\frac{1}{2}(n+1)$ relations, *i.e.*, $[\frac{1}{2}n]+1$, where $[\frac{1}{2}n]$ denotes as usual the integral part of $\frac{1}{2}n$.

3.5. We may form the eliminants by solving the equations (3.41) for the A's and substituting in (3.42).

From (3.41) we have

$$\begin{aligned}A_1 &= \Sigma_0 - {}_{n+1}C_1\,V,\\ A_2 &= \Sigma_1 - {}_nC_1\,\Sigma_0 + {}_{n+1}C_2\,V,\\ A_3 &= \Sigma_2 - {}_{n-1}C_1\,\Sigma_1 + {}_nC_2\,\Sigma_0 - {}_{n+1}C_3\,V.\end{aligned}$$

By induction we may show that

$$\begin{aligned}A_r = \Sigma_{r-1} - {}_{n-r+2}C_1\,\Sigma_{r-2} + \ldots + (-)^{r-s-1}\,{}_{n-s}C_{r-s-1}\,\Sigma_s + \ldots \\ + (-)^{r-1}\,{}_nC_{r-1}\,\Sigma_0 + (-)^r\,{}_{n+1}C_r\,V, \qquad (3.51)\end{aligned}$$

for assuming this true up to r, we have by (3.41)

$$A_{r+1} = \Sigma_r - {}_{n+1}C_{r+1}\,V - {}_nC_r\,A_1 - {}_{n-1}C_{r-1}\,A_2 - \ldots - {}_{n-r+1}C_1\,A_r.$$

Substituting the values of $A_1, \ldots, A_r$ in the right-hand side, the coefficient of Σ_s is

$$- {}_{n-s}C_{r-s} + {}_{n-s}C_{r-s} \cdot {}_{n-s}C_1 - \ldots + (-)^{p+1} {}_{n-s-p}C_{r-s-p} \cdot {}_{n-s}C_p + \ldots$$
$$+ (-)^{r-s} {}_{n-r+1}C_1 \cdot {}_{n-s}C_{r-s-1}.$$

But

$$_{n-s-p}C_{r-s-p} \cdot {}_{n-s}C_p = {}_{n-s}C_{r-s} \cdot {}_{r-s}C_p.$$

Hence the coefficient of Σ_s is

$$- {}_{n-s}C_{r-s} \{1 - {}_{r-s}C_1 + \ldots + (-)^{r-s-1} {}_{r-s}C_{r-s-1}\} = (-)^{r-s} {}_{n-s}C_{r-s},$$

which establishes (3.51).

Hence, finally, equating A_r and A_{n-r+1} we have

$$\Sigma_{r-1} - {}_{n-r+2}C_1 \Sigma_{r-2} + \ldots + (-)^{r-s-1} {}_{n-s}C_{r-s-1} \Sigma_s + \ldots$$
$$+ (-)^{r-1} {}_nC_{r-1} \Sigma_0 + (-)^r {}_{n+1}C_r V$$
$$= \Sigma_{n-r} - {}_{r+1}C_1 \Sigma_{n-r-1} + \ldots + (-)^{n-r-s} {}_{n-s}C_{n-r-s} \Sigma_s + \ldots$$
$$+ (-)^{n-r} {}_nC_{n-r} \Sigma_0 + (-)^{n-r+1} {}_{n+1}C_{n-r+1} V, \quad (3.52)$$

as the linear relations connecting the volume and angle-sums of a simplex. We notice that *if n is odd*, V *disappears from all the equations.*

The functions on the two sides of this equation occur frequently. We shall define the notation

$$_n\phi_s(\Sigma) \equiv {}_{n-s}C_0 \Sigma_{s-1} - {}_{n-s+1}C_1 \Sigma_{s-2} + \ldots + (-)^s {}_nC_s \Sigma_{-1}. \quad (3.53)$$

Then the equations (3.52) can be written

$$_{n+1}\phi_r(\Sigma) = {}_{n+1}\phi_{n-r+1}(\Sigma). \quad (3.54)$$

For $r = n + 1$ or 0,

$$\Sigma_n - \Sigma_{n-1} + \ldots + (-)^n \Sigma_0 + (-)^{n+1} V = V. \quad (3.55)$$

We have also the simple relation, already noted in 3.4,

$$\Sigma_{n-1} = \tfrac{1}{2}(n + 1) \Sigma_n, \quad (3.56)$$

which can also be derived by elimination. In fact, since $A_1 = A_n$, $A_2 = A_{n-1}$, etc., the last two equations of (3.41) give

$$2\Sigma_{n-1} = 2(n+1) V + (nA_1 + A_n) + \{(n-1) A_2 + 2A_{n-1}\} + \ldots$$
$$= (n+1)(2V + A_1 + A_2 + \ldots) = (n+1) \Sigma_n.$$

3.6. The foregoing investigation applies to spherical geometry. In elliptic geometry antipodal points coincide, the regions $(01\ldots r)$ and $(r+1, \ldots, n)$ are continuously connected and form one region, and the sums A_{r+1} and A_{n-r} are coincident instead of being merely of equal volume. But instead of Σ_n we have $2\Sigma_n$. The final equations are therefore the same except for this modification.

3.7. Now let S_r denote the sum of all the angles at r-dimensional edges expressed in radian measure, so that

$$\Sigma_r = S_r \, . \, k^n \frac{n+1}{n} \frac{\sqrt{\pi}\Gamma\{\frac{1}{2}(n+2)\}}{\Gamma\{\frac{1}{2}(n+3)\}}, \quad (r = 0, 1, \ldots, n-2) \qquad (3.71)$$

and

$$\Sigma_n = k^n \frac{(n+1)\pi^{\frac{1}{2}(n+1)}}{\Gamma\{\frac{1}{2}(n+3)\}}, \qquad (3.72)$$

$$\Sigma_{n-1} = \tfrac{1}{2}(n+1)\Sigma_n. \qquad (3.73)$$

Equation (3.55) then becomes

$$\{1+(-1)^n\}V$$
$$= k^n \frac{n+1}{n} \frac{\sqrt{\pi}\Gamma\{\frac{1}{2}(n+2)\}}{\Gamma\{\frac{1}{2}(n+3)\}} \left\{S_{n-2} - S_{n-3} + \ldots + (-)^n S_0 - \tfrac{1}{2}n(n-1)\frac{\pi^{\frac{1}{2}n}}{\Gamma\{\frac{1}{2}(n+2)\}}\right\}, \quad (3.74)$$

and the other equations of (3.52) can be transformed similarly. The equations thus expressed in radian measure are the same in spherical and elliptic geometry.

Excluding the equation $\Sigma_{n-1} = \frac{1}{2}(n+1)\Sigma_n$ we have therefore $[\frac{1}{2}n]$ linear equations connecting the volume V and the $n-1$ angle-sums, and in these V disappears when n is odd.

The following are the equations up to $n = 4$:—

$n = 2$ $\qquad V = k^2(S_0 - \pi)$,

$n = 3$ $\qquad S_0 - S_1 + 4\pi = 0$,

$n = 4$ (*) $\qquad V = \frac{2}{3}k^4(S_2 - S_1 + S_0 - 3\pi^2)$

$\qquad V = \frac{2}{15}k^4(2S_2 - 3S_1 + 5S_0 - 5\pi^2)$.

* From these equations we derive

$$-3V/k^4 = S_2 - 2S_0 - 4\pi^2,$$

and

$$0 = S_2 - \tfrac{2}{3}S_1 - \tfrac{10}{3}\pi^2.$$

The expressions on the right are, with different notation, Dehn's "Zerlegungsinvarianten" (*loc. cit.*, p. 572). With a disregard for homogeneity he takes different units for the angles at a face and the angles at a vertex or an edge, viz., for the complete dihedral angle at a face the value 2π and for the complete angle at a vertex or an edge the value unity. In our formulæ the complete angle in each case has the value $2\pi^2$, the surface content of a hypersphere of radius unity in euclidean space of four dimensions, *i.e.*, the total volume of spherical space of three dimensions and space-constant unity.

3.8. In euclidean geometry the space-constant $k \longrightarrow \infty$; the resulting equations are equivalent to putting $V = 0$, giving in all dimensions only linear identities involving the angle-sums. For hyperbolic geometry k is a pure imaginary; when n is odd V and k disappear, and the relations between the angle-sums are the same in all three geometries; when n is even k occurs to an even power; *e.g.*, for $n = 2$ in hyperbolic geometry, putting ik instead of k, we have

$$V = k^2 (\pi - S_0),$$

while for $n = 4$ the equations are the same as in elliptic geometry.

§ 4. *The relations connecting the numbers of boundaries of different dimensions of a polytope in space of n dimensions.*

4.1. Let the number of r-dimensional boundaries be denoted by N_r, and let the number of p-dimensional boundaries which are incident with (pass through or lie in) a particular q-dimensional boundary be denoted by N_{pq}. If we sum the numbers N_{pq} for all the q-dimensional boundaries the sum is equal to the number of p-dimensional boundaries each counted as often as there are q-dimensional boundaries incident with it, and this is the same as the number of q-dimensional boundaries each counted as often as there are p-dimensional boundaries incident with it. Hence

$$\Sigma N_{pq} = \Sigma N_{qp}. \qquad (4.11)$$

For a particular p-dimensional boundary Euler's formula for p dimensions gives

$$N_{p-1,p} - N_{p-2,p} + \dots + (-)^{p-1} N_{0,p} + (-1)^p = 1.$$

Summing for all the p-dimensional boundaries

$$\Sigma N_{p-1,p} - \Sigma N_{p-2,p} + \dots + (-)^{p-1} \Sigma N_{0,p} + (-)^p N_p = N_p. \qquad (4.12)$$

4.2. Take any vertex and draw a small hypersphere round it. This is cut by the boundaries at the vertex in a hyperspherical polytope with N_{r0} $(r-1)$-dimensional boundaries. Hence by Euler's formula

$$N_{n-1,0} - N_{n-2,0} + \dots + (-)^{n-3} N_{20} + (-)^{n-2} N_{10} + (-1)^{n-1} = 1,$$

and summing for all the vertices

$$\Sigma N_{n-1,0} - \Sigma N_{n-2,0} + \dots + (-)^{n-2} \Sigma N_{10} + (-)^{n-1} N_0 = N_0.$$

Take next an edge and any point O on the edge. Draw a hyperplane through O perpendicular to the edge, and a hypersphere of $n-1$ dimensions with centre O and lying in the hyperplane. This hypersphere is cut by the boundaries at the edge in a hyperspherical polytope of $n-2$ dimensions with N_{r1} $(r-2)$-dimensional boundaries. Hence by Euler's formula

$$N_{n-1,1} - N_{n-2,1} + \dots + (-)^{n-3} N_{21} + (-1)^{n-2} = 1.$$

Summing for all the edges

$$\Sigma N_{n-1,1} - \Sigma N_{n-2,1} + \dots + (-)^{n-3} \Sigma N_{21} + (-)^{n-2} N_1 = N_1.$$

Proceeding similarly with the p-dimensional boundaries we have

$$\Sigma N_{n-1,p} - \Sigma N_{n-2,p} + \dots + (-)^{n-p-2} \Sigma N_{p+1,p} + (-)^{n-p-1} N_p = N_p. \quad (4.21)$$

These relations are not all independent, and when n is even it can be shown that Euler's formula for the whole polytope is derived from them algebraically; for taking the term in N_p to the right, multiplying all the equations (4.12) by $+1$ and equations (4.21) by -1 and adding, we have on the right

$$2(N_{n-1} - N_{n-2} + \dots + N_1 - N_0),$$

and on the left we have the terms

$$(-)^{p-q-1} \Sigma N_{qp} - (-)^{n-p-q-1} \Sigma N_{pq} = 0.$$

Thus *Euler's theorem for even values of n follows from its truth for all smaller values of n and the equations* (4.11).

For odd values of n the equations do not involve the numbers N_{2r} at all as these disappear both from equations (4.12) and from equations (4.21). This is noted by Dehn for $n = 5$, and he establishes Euler's formula for $n = 5$ by dividing the polytope into simplexes.

4.3. *Polytope bounded entirely by simplexes.*—When the boundaries of the polytope are all simplexes we can derive relations connecting the numbers N_r alone, without the individual numbers N_{pq}. In this case we have

$$N_{pq} = {}_{q+1}C_{p+1} \qquad (p < q) \quad (4.31)$$

and

$$\Sigma N_{pq} = \Sigma N_{qp} = {}_{q+1}C_{p+1} N_q \qquad (p < q). \quad (4.32)$$

Equations (4.12) become identities, while equations (4.21) become

$${}_nC_{p+1} N_{n-1} - {}_{n-1}C_{p+1} N_{n-2} + \dots + (-)^{n-p-2} {}_{p+2}C_{p+1} N_{p+1} + (-)^{n-p-1} N_p = N_p \qquad (p = 0, 1, \dots, n-1), \quad (4.33)$$

For $p = n-1$, however, we get merely the identity $N_{n-1} = N_{n-1}$, and the equations for $p = n-2$ and $p = n-3$ both reduce to

$$nN_{n-1} = 2N_{n-2}. \quad (4.34)$$

If we assume also Euler's formula for the whole polytope, it may be considered as included in (4.33) for $p = -1$, with the understanding that $N_{-1} = 1$, thus

$$N_{n-1} - N_{n-2} + \dots + (-)^{n-1} N_0 + (-1)^n = 1. \qquad (4.35)$$

4.4. The equations (4.33) are in fact not all independent. They may be expressed in another form, which is sometimes more useful, by first eliminating N_{n-1} between the first two (Euler's equation (4.35) being counted as the first one), then N_{n-1} and N_{n-2} between the first three, and so on. Thus, multiplying the first two equations respectively by $-n$ and 1, and adding, we get

$$N_{n-2} - 2N_{n-3} + 3N_{n-4} - \dots + (-)^{n-2}(n-1) N_0 + (-)^{n-1} n = N_0 - n.$$

Multiplying the first three equations by ${}_nC_2$, $-{}_{n-1}C_1$, and 1 and adding, we get

$$N_{n-3} - {}_3C_2 N_{n-4} + {}_4C_2 N_{n-5} - \dots + (-)^{n-1} {}_{n-1}C_2 N_0 + (-)^n {}_nC_2$$
$$= N_1 - {}_{n-1}C_1 N_0 + {}_nC_2;$$

and, generally, multiplying the first $r+1$ equations by ${}_nC_{n-r}$, $-{}_{n-1}C_{n-r}$, ${}_{n-2}C_{n-r}$, …, $(-1)^r$ respectively, and adding, we get

$$N_{n-r-1} - {}_{r+1}C_1 N_{n-r-2} + \dots + (-)^{n-r-1} {}_{n-1}C_{n-r-1} N_0 + (-)^{n-r} {}_nC_{n-r}$$
$$= N_{r-1} - {}_{n-r+1}C_1 N_{r-2} + \dots + (-)^{r-1} {}_{n-1}C_{r-1} N_0 + (-)^r {}_nC_r, \qquad (4.41)$$

that is,

$${}_n\phi_r(N) = {}_n\phi_{n-r}(N). \qquad (4.42)$$

Thus *the equations which connect the number of boundaries of a polytope bounded entirely by simplexes are precisely the same as those which connect the volume and angular regions of a simplex, but in one dimension less.* The number of these equations is $[\frac{1}{2}(n+1)]$.

4.5. *Relations between the number of boundaries of a pyramid.*—We may interpolate here the special relations for a pyramid.

Consider a pyramid whose base is a polytope of $(n-1)$ dimensions with N'_r r-dimensional boundaries, and let N_r be the number of r-dimensional boundaries of the pyramid.

Then

$$N_0 = N'_0 + 1, \dots, N_r = N'_r + N'_{r-1}, \dots, N_{n-1} = 1 + N'_{n-2}.$$

Therefore

$$N'_0 = N_0 - 1,$$
$$N'_1 = N_1 - N'_0 = N_1 - N_0 + 1,$$
$$\dots\dots\dots$$
$$N'_r = N_r - N_{r-1} + N_{r-2} - \dots + (-)^r N_0 + (-1)^{r+1},$$
$$\dots\dots\dots$$
$$N'_{n-2} = N_{n-2} - N_{n-3} + \dots + (-)^{n-2} N_0 + (-1)^{n-1},$$
$$1 = N_{n-1} - N_{n-2} + \dots + (-)^{n-1} N_0 + (-1)^n. \qquad (4.51)$$

The last relation is the first Euler relation for the pyramid, viz., ${}_n\phi_n(\mathrm{N}) = 1$. This direct proof is of interest as it holds for odd or even dimensions and does not involve induction from lower dimensions or other extraneous assumptions.

We have also for the base of the pyramid

$$\begin{aligned} 1 &= \mathrm{N}'_{n-2} - \mathrm{N}'_{n-3} + \dots + (-)^{n-2}\mathrm{N}'_0 + (-1)^{n-1} \\ &= \mathrm{N}_{n-2} - 2\mathrm{N}_{n-3} + \dots + (-)^{n-2}(n-1)\,\mathrm{N}_0 + (-)^{n-1}\,n, \end{aligned}$$

i.e.,

$${}_n\phi_{n-1}(\mathrm{N}) = 1 \tag{4.52}$$

If the base is itself a pyramid of $(n-1)$ dimensions it follows that ${}_{n-1}\phi_{n-2}(\mathrm{N}') = 1$,

i.e., $\mathrm{N}'_{n-3} - 2\mathrm{N}'_{n-4} + \dots + (-)^{n-3}(n-2)\,\mathrm{N}'_0 + (-)^{n-2}(n-1) = 1.$

Hence

$$\begin{aligned} 1 &= \mathrm{N}_{n-3} - \mathrm{N}_{n-4} + \dots + (-)^{n-3}\mathrm{N}_0 + (-1)^{n-2} \\ &\qquad - 2\mathrm{N}_{n-4} + \dots + (-)^{n-3}\,2\mathrm{N}_0 + 2(-1)^{n-2} \dots \\ &= \mathrm{N}_{n-3} - 3\mathrm{N}_{n-4} + 6\mathrm{N}_{n-5} - \dots + (-)^{n-3}{}_{n-1}\mathrm{C}_{n-3}\,\mathrm{N}_0 + (-)^{n-2}{}_n\mathrm{C}_{n-2}, \end{aligned}$$

i.e.,

$${}_n\phi_{n-2}(\mathrm{N}) = 1. \tag{4.53}$$

We may call the pyramid in this case a pyramid of the second order. A pyramid of order r is one whose base is a pyramid of order $r-1$. In two dimensions a pyramid of first order is a triangle, in three dimensions a pyramid of second order is a tetrahedron, and generally in n dimensions a pyramid of order $(n-1)$ is a simplex.

It can be shown generally by induction that for a pyramid of order r we have the $r+1$ relations

$${}_n\phi_s(\mathrm{N}) = 1, \quad (s = n, n-1, \dots, n-r). \tag{4.54}$$

§ 5. *Relations between the volume and the angle-sums of a polytope bounded entirely by simplexes.*

5.1. Take any point in the interior of the polytope, and join it to all the vertices, thus dividing it centrally into simplexes. Let Σ_r denote the sum of the angular regions at the r-dimensional edges for the polytope, and Σ'_r the corresponding sum for a constituent simplex; V the whole volume and V′ that of a constituent simplex.

Then

$$\Sigma\Sigma'_n = N_{n-1}\Sigma_n,$$
$$\Sigma\Sigma'_{n-1} = \Sigma_{n-1} + N_{n-2}\Sigma_n,$$
$$\Sigma\Sigma'_{n-2} = \Sigma_{n-2} + N_{n-3}\Sigma_n,$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$\Sigma\Sigma'_1 = \Sigma_1 + N_0\Sigma_n,$$
$$\Sigma\Sigma'_0 = \Sigma_0 + \Sigma_n,$$
$$\Sigma V' = V.$$

5.2. For each simplex we have the relations

$$_{n+1}\phi_r(\Sigma') = {}_{n+1}\phi_{n-r+1}(\Sigma').$$

Summing these equations and substituting for $\Sigma\Sigma'_r$ we have for $r = 1, 2, \ldots [\frac{1}{2}(n-1)]$

$$_{n+1}\phi_r(\Sigma) + (N_{r-2} - {}_{n-r+2}C_1 N_{r-3} + \ldots + (-)^{r-1}{}_nC_{r-1})\Sigma_n$$
$$= {}_{n+1}\phi_{n-r+1}(\Sigma) + (N_{n-r-1} - {}_{r+1}C_1 N_{n-r-2} + \ldots + (-)^{n-r}{}_nC_{n-r})\Sigma_n,$$

that is,

$$_{n+1}\phi_r(\Sigma) - {}_{n+1}\phi_{n-r+1}(\Sigma) = \{{}_n\phi_{n-r}(N) - {}_n\phi_{r-1}(N)\}\Sigma_n$$
$$= \{{}_n\phi_r(N) - {}_n\phi_{r-1}(N)\}\Sigma_n \qquad \text{by (4.42)}$$
$$= {}_{n+1}\phi_r(N)\,.\,\Sigma_n = -\,{}_{n+1}\phi_{n-r+1}(N)\,.\,\Sigma_n, \qquad (5.21)$$

since ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.
But for $r = 0$

$$_{n+1}\phi_0(\Sigma) - {}_{n+1}\phi_{n+1}(\Sigma) = \{(N_{n-1} - 1) - N_{n-2} + \ldots + (-)^{n-1}N_0 + (-1)^n\}\Sigma_n$$
$$= 0, \qquad (5.22)$$

i.e., the first relation, viz.,

$$\Sigma_n - \Sigma_{n-1} + \Sigma_{n-2} - \ldots + (-)^n\Sigma_0 + (-)^{n+1}V = V,$$

is the same for all polytopes bounded entirely by simplexes.

We have also the relation,

$$2\Sigma_{n-1} = N_{n-1}\,.\,\Sigma_n. \qquad (5.23)$$

The relations between the volume and the angle-sums in radian measure for $n = 2, 3, 4$ are

$$n = 2: \quad V = k^2\{S_0 - (N_1 - 2)\pi\}, \qquad (5.24)$$
$$n = 3: \quad 0 = S_0 - S_1 + (N_2 - 2)2\pi, \qquad (5.25)$$
$$n = 4: \quad V = \tfrac{2}{3}k^4\{S_2 - S_1 + S_0 + (2 - N_3)\pi^2\}, \qquad (5.26)$$
$$V = \tfrac{2}{15}k^4\{2S_2 - 3S_1 + 5S_0 - (2N_0 + N_3 - 10)\pi^2\}.$$

§ 6. *Relations between the volume and the angle-sums for any Eulerian polytope.*

6.1. Take any r-dimensional boundary and divide it centrally. Then if α_r

is the volume of the angular region at that boundary as edge, we have, if Σ_s is the sum of the angular regions at an s-dimensional edge for the given polytope, and Σ'_s that for the transformed polytope,

$$\begin{aligned}
\Sigma'_s &= \Sigma_s \qquad (s = n, n-1, \dots, r+1)\\
\Sigma'_r &= \Sigma_r + (N_{r-1,r} - 1)\,\alpha_r,\\
\Sigma'_s &= \Sigma_s + N_{s-1,r}\alpha_r \qquad (s = r-1, r-2, \dots, 1)\\
\Sigma'_0 &= \Sigma_0 + \alpha_r,\\
V' &= V.
\end{aligned}$$

Then

$$\begin{aligned}
\Sigma'_n - \Sigma'_{n-1} + \Sigma'_{n-2} - \dots + (-)^{n-1}\Sigma'_1 + (-)^n\Sigma_0' + (-)^{n+1}V'\\
= \Sigma_n - \Sigma_{n-1} + \dots + (-)^{n-1}\Sigma_1 + (-)^n\Sigma_0 + (-)^{n+1}V\\
+ (-)^{n-r}\{(N_{r-1,r} - 1) - N_{r-2,r} + \dots + (-)^{r-1}N_{0,r} + (-1)^r\}\,\alpha_r\\
= \Sigma_n - \Sigma_{n-1} + \dots + (-)^n\Sigma_0 + (-)^{n+1}V,
\end{aligned}$$

i.e.,

$$_{n+1}\phi_{n+1}(\Sigma') = {}_{n+1}\phi_{n+1}(\Sigma).$$

Thus the function $_{n+1}\phi_{n+1}(\Sigma)$ is not altered if all the boundaries are divided centrally. If all the boundaries of all dimensions are divided centrally the polytope is transformed into one bounded entirely by simplexes, hence *the relation*

$$\Sigma_n - \Sigma_{n-1} + \Sigma_{n-2} - \dots + (-)^n\Sigma_0 + (-)^{n+1}V = V \tag{6.11}$$

is true for all Eulerian polytopes.

6.2. For the other relations it is found that the angles at the individual boundaries are involved. We shall work out as an example the case of $n = 4$.

Consider a polytope in space of 4 dimensions. Let N_r be the number of its r-dimensional boundaries, N_{pq} the number of p-dimensional boundaries of, or incident with, a particular q-dimensional boundary.

First choose any 2-dimensional boundary and divide it centrally. Let Σ_r be the sum of the angular regions for the polytope, Σ'_r those for the transformed polytope.

Then

$$\begin{aligned}
\Sigma'_4 &= \Sigma_4,\\
\Sigma'_3 &= \Sigma_3,\\
\Sigma'_2 &= \Sigma_2 + (N_{12} - 1)\,\alpha_2,\\
\Sigma'_1 &= \Sigma_1 + N_{02}\alpha_2,\\
\Sigma'_0 &= \Sigma_0 + \alpha_2,\\
V' &= V,
\end{aligned}$$

where α_2 is the angle at the 2-dimensional boundary. Also $N_{02} = N_{12}$.

Let the function

$$ {}_5\phi_4 (\Sigma) - {}_5\phi_1 (\Sigma) \equiv \Sigma_3 - 2\Sigma_2 + 3\Sigma_1 - 5\Sigma_0 + 10V \equiv \psi (\Sigma). $$

Then

$$ \begin{aligned} \psi (\Sigma') &= \Sigma'_3 - 2\Sigma'_2 + 3\Sigma'_1 - 5\Sigma'_0 + 10V' \\ &= \psi (\Sigma) + (-2N_{12} + 2 + 3N_{02} - 5)\, \alpha_2 \\ &= \psi (\Sigma) + (N_{12} - 3)\, \alpha_2. \end{aligned} $$

Hence if all the 2-dimensional boundaries are divided centrally

$$ \psi (\Sigma') = \psi (\Sigma) + \Sigma\, (N_{12} - 3)\, \alpha_2. \qquad (6.21) $$

Next choose any 3-dimensional boundary and divide it centrally. Let N'_{p3} be the number of its boundaries of p dimensions, Σ''_r the sum of the angular regions at an r-dimensional edge for the polytope thus further transformed.

Then

$$ \begin{aligned} \Sigma''_4 &= \Sigma'_4, \\ \Sigma''_3 &= \Sigma'_3 + (N'_{23} - 1)\, \alpha_3, \\ \Sigma''_2 &= \Sigma'_2 + N'_{13}\alpha_3, \\ \Sigma''_1 &= \Sigma'_1 + N'_{03}\alpha_3, \\ \Sigma''_0 &= \Sigma'_0 + \alpha_3, \\ V'' &= V', \end{aligned} $$

where α_3 is the angle at the 3-dimensional boundary, $= \frac{1}{2}\Sigma_4$.

Then $\psi (\Sigma'') = \psi (\Sigma') + (N'_{23} - 1 - 2N'_{13} + 3N'_{03} - 5)\, \frac{1}{2}\Sigma_4$, and when all the 3-dimensional boundaries are divided centrally

$$ \psi (\Sigma'') = \psi (\Sigma') + \Sigma\, (N'_{23} - 2N'_{13} + 3N'_{03} - 6)\, \tfrac{1}{2}\Sigma_4. \qquad (6.22) $$

But $N'_{03} = N_{03} + N_{23}$,

$N'_{13} = N_{13} + \Sigma N_{12}$ (the summation extending over the faces of the boundary)

$= N_{13} + 2N_{13} = 3N_{13}$,

$N'_{23} = \Sigma N_{12} = 2N_{13}$.

Hence

$$ \begin{aligned} N'_{23} - 2N'_{13} + 3N'_{03} - 6 &= 3N_{23} - 4N_{13} + 3N_{03} - 6 \\ &= -N_{13}, \end{aligned} $$

and, since the polytope is now bounded entirely by simplexes, by (5.21)

$$ {}_5\phi_4 (\Sigma'') - {}_5\phi_1 (\Sigma'') = -\, {}_5\phi_1 (N'') \,.\, \Sigma_4, $$

i.e.,

$$ \psi (\Sigma'') = (5 - N''_0)\, \Sigma_4. $$

But

$$N''_0 = N'_0 + N'_3,$$

$$N'_0 = N_0 + N_2, \qquad N'_3 = N_3.$$

Therefore

$$N''_0 = N_0 + N_2 + N_3.$$

Hence finally

$$\begin{aligned}\psi(\Sigma) &= \psi(\Sigma') - \Sigma(N_{12} - 3)\alpha_2 \\ &= (5 - N''_0)\Sigma_4 + \tfrac{1}{2}\Sigma_4 . \Sigma N_{13} - \Sigma(N_{12} - 3)\alpha_2 \\ &= (5 - N_0 - N_2 - N_3 + \tfrac{1}{2}\Sigma N_{13})\Sigma_4 - \Sigma(N_{12} - 3)\alpha_2 \qquad (6.23)\end{aligned}$$

6.3. For the regular polytopes, or more generally the homogeneous polytopes, these relations become simplified since N_{12} and N_{13} are the same for all the boundaries.

We have

$$N_{13} = N_{23} + N_{03} - 2,$$

$$\Sigma N_{23} = \Sigma N_{32} = N_2 N_{32} = 2N_2,$$

$$\Sigma N_{03} = \Sigma N_{30} = N_0 N_{30},$$

hence

$$\Sigma N_{13} = 2N_2 + N_{30} N_0 - 2N_3.$$

Also $\Sigma(N_{12} - 3)\alpha_2 = (N_{12} - 3)\Sigma_2$, and $\Sigma_3 = \tfrac{1}{2}N_3 . \Sigma_4$.

Hence we have

$$\{\tfrac{5}{2}N_3 + (1 - \tfrac{1}{2}N_{30})N_0 - 5\}\Sigma_4 + (N_{12} - 5)\Sigma_2 + 3\Sigma_1 - 5\Sigma_0 + 10V = 0, \qquad (6.31)$$

or in terms of the radian measures

$$2k^4\{[5N_3 + (2 - N_{30})N_0 - 10]\pi^2 + (N_{12} - 5)S_2 + 3S_1 - 5S_0\} + 15V = 0. \qquad (6.32)$$

We have also by (6.11)

$$2k^4\{(2 - N_3)\pi^2 + S_2 - S_1 + S_0\} = 3V. \qquad (6.33)$$

Eliminating V we get

$$N_{12}S_2 - 2S_1 = (N_{30} - 2)N_0\pi^2. \qquad (6.34)$$

6.4. In particular for the regular polytopes, if $\alpha_2, \alpha_1, \alpha_0$ are the radian measures of the angles at a face, an edge, and a vertex, $S_2 = N_2\alpha_2$, $S_1 = N_1\alpha_1$, $S_0 = N_0\alpha_0$. Equation (6.34) then becomes

$$N_{02}N_2\alpha_2 - 2N_1\alpha_1 = (N_{30} - 2)N_0\pi^2;$$

but $N_{02}N_2 = N_{20}N_0$ and $2N_1 = N_{01}N_1 = N_{10}N_0$,
hence

$$N_{20}\alpha_2 - N_{10}\alpha_1 = (N_{30} - 2)\pi^2. \qquad (6.41)$$

This result is easily verified directly by drawing a hypersphere round a vertex and applying equation (5.25), which gives

$$S'_1 - S'_2 + (N_{30} - 2)\pi^2 = 0,$$

S'_1 and S'_2 being the sums of the angles at the edges and faces at that vertex, and π^2 (for 4 dimensions) replacing 2π (for 3 dimensions), half the complete angle at a point. Then $S'_1 = N_{10}\alpha_1$ and $S'_2 = N_{20}\alpha_2$.

Since $N_{30} - N_{20} + N_{10} = 2$, equation (6.41) may also be written

$$N_{20}(\alpha_2 - \pi^2) = N_{10}(\alpha_1 - \pi^2). \tag{6.42}$$

If ${}_rN_{pq}$ denotes the number of p-dimensional boundaries through a q-dimensional boundary and lying in an r-dimensional boundary

$$N_{32}N_{20} = {}_3N_{20}N_{30} = {}_3N_{10}N_{30} = N_{31}N_{10},$$

and $N_{32} = 2$, hence $2N_{20} = N_{31}N_{10}$, and (6.42) may be further simplified to

$$N_{31}(\alpha_2 - \pi^2) = 2(\alpha_1 - \pi^2). \tag{6.43}$$

For the 5-, 8-, 24- and 120-cells $N_{31} = 3$, for the 16-cell $N_{31} = 4$, and for the 600-cell $N_{31} = 5$.

The numbers $N_{01} = N_{12} = k_1$, ${}_3N_{10} = {}_3N_{20} = k_2$, and $N_{21} = N_{31} = k_3$ are the *fundamental numbers* of the regular polytopes,* in terms of which all the numbers N_{pq} and the ratios of the numbers N_r can be expressed.

* See the author's paper "The regular divisions of space of n dimensions and their metrical constants," Palermo, 'Rend. Circ. mat.,' vol. 48, pp. 1–14 (1924).