Visualizing Multi-Dimensional Functions in Economics *

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June, 1999

Abstract

This paper introduces a method of viewing multi-dimensional surfaces. Economists may find it useful when estimating complex econometric models or using sophisticated simulations, to gain some intuition on what their models "look like." Besides describing this method, this paper explores two estimation problems: a rational expectations model and the estimation of a translog production function.

Introduction

Economists use multi-dimensional functions most every day in their research. Obvious examples include statistical optimization by maximum likelihood and simulation of complex models. However, unlike their cousins with two dependent variables, multi-dimensional functions are difficult or nearly impossible to visualize. While formal analysis hardly requires visualization, a "feel" for the function certainly aids intuition and can help the researcher understand the model's quirks. It may also aid choosing the appropriate optimization algorithm. Conventional algorithms, such as conjugate gradient, variable metric (quasi-Newton), and the simplex method assume that the function is unimodal (and the first two assume that it is roughly quadratic). Of course, functions can be more complicated, and for these problems, a variety of methods have been proposed, including the genetic algorithm (Dorsey and Mayer, 1995) and simulated annealing (Goffe et al., 1994). However, they are more computationally expensive, so it is not necessarily wise to use them indiscriminately. To some degree this paper follows Goffe (1999), where three different methods were used to illustrate different functions and to aid in the selection of the correct optimization algorithm.

As described in Goffe (1999), the literature on the visualization of multi-dimension functions appears to be surprisingly limited. Some require sophisticated software, others require substantial

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user training, while others are not suitable for many dimensions. For instance, Feiner and Beshers (1990) develop an interesting method that unfortunately requires their complex software with stereo glasses. Bowyer et al. (1998) has the same complexity, but does not require stereo glasses. Inselberg and Dimsdale (1994) describe "parallel coordinates," but they are not suitable for arbitrary function surfaces (they can be useful for statistical data). Rucker (1985) is clearly limited to four dimensions.

The next section of this paper describes the method, and following that are two example sections. The first one contains simple functions to illustrate this method, and the second shows the surfaces of two econometric estimation problems (a rational expectations model and a translog production function). From these "pictures," one gains insight into these models.

Formulation

The method introduced in this paper is built upon polar coordinates in n dimensions using the formulation from Kendall (1961, pp. 15-16). Polar coordinates in n dimensions are

$$
\begin{aligned}
x_1 &= r \cdot cos(\theta_1) \cdot cos(\theta_2) \cdots cos(\theta_{n-2}) \cdot cos(\theta_{n-1}) \\
x_2 &= r \cdot cos(\theta_1) \cdot cos(\theta_2) \cdots cos(\theta_{n-2}) \cdot sin(\theta_{n-1}) \\
x_3 &= r \cdot cos(\theta_1) \cdot cos(\theta_2) \cdots sin(\theta_{n-2}) \\
\cdots &\quad \cdots \\
x_j &= r \cdot cos(\theta_1) \cdots cos(\theta_{n-j}) \cdots sin(\theta_{n-j+1}) \\
\cdots &\quad \cdots \\
x_n &= r \cdot sin(\theta_1),
\end{aligned}
\tag{1}
$$

with $0 \le r \le \infty, -\pi/2 \le \theta_j \le \pi/2, j = 1, \ldots, n - 2, 0 \le \theta_{n-1} \le 2\pi.$

This clearly simplifies to the the well-known set of polar coordinates in two dimensions:

$$
\begin{aligned}
x_1 &= r \cdot cos(\theta) \\
x_2 &= r \cdot sin(\theta),
\end{aligned}
$$

with $0 \le r \le \infty, 0 \le \theta \le 2\pi$, and spherical coordinates in three dimensions:

$$
\begin{aligned}
x_1 &= r \cdot cos(\theta_1) \cdot cos(\theta_2) \\
x_2 &= r \cdot cos(\theta_1) \cdot sin(\theta_2) \\
x_3 &= r \cdot sin(\theta_1),
\end{aligned}
$$

with $0 \le r \le \infty, -\pi/2 \le \theta_1 \le \pi/2, 0 \le \theta_2 \le 2\pi.$

Of course, this hardly aids the plotting of a multi-dimensional function—it simply changes the coordinate system. To plot a multidimensional surface as a function with two inputs, we will "collapse" the θ vector to a function of the scalar variable $t, 0 \le t \le 2\pi$, as follows:

$$
\begin{aligned}
\theta_{n-1} &= t \\
\theta_{n-2} &= \pi/2 \cdot sin(2t) \\
\theta_{n-3} &= \pi/2 \cdot sin(4t) \\
&\cdots \quad \cdots \\
\theta_1 &= \pi/2 \cdot sin(2^{(n-2)}t).
\end{aligned}
\tag{2}
$$

Note that the restrictions on the values of θ are achieved with this parameterization. Also note that with the exception of the value of θ_{n-1}, the values of θ are orthogonal to each other, so one can think of the θ's as roughly sampling θ-space in an efficient manner.

To plot a multidimensional function, one plots the function's value as a function of t and r. Thus, a function of several dimensions appears as a plot in polar coordinates with two inputs. The radius of the original function is maintained, and t subsumes the θ vector. As will be seen below, it does so in a particularly useful manner.

To illustrate this method for $n = 4$, the systems of equations (1) and (2) become

$$
\begin{aligned}
x_1 &= r \cdot cos(\theta_1) \cdot cos(\theta_2) \cdot cos(\theta_3) \\
x_2 &= r \cdot cos(\theta_1) \cdot cos(\theta_2) \cdot sin(\theta_3) \\
x_3 &= r \cdot cos(\theta_1) \cdot sin(\theta_2) \\
x_4 &= r \cdot sin(\theta_1),
\end{aligned}
$$

with

$$
\begin{aligned}
\theta_3 &= t \\
\theta_2 &= \pi/2 \cdot sin(2t) \\
\theta_1 &= \pi/2 \cdot sin(4t), \quad 0 \le t \le 2\pi.
\end{aligned}
$$

The function $f(x_1, x_2, x_3, x_4)$ is then plotted as a function of t and r. For larger n, the function is plotted the same way, but t contains a larger vector of θ. Thus, a number of variables can be plotted, but as described below, there are some practical limits.

Examples of several functions are shown below, but first it is useful to examine the Cartesian coordinate axes of the original function with this method. Rather tedious inspection shows that for arbitrary n, and for various values of t, this method "hits" each Cartesian coordinate axis. Thus, the Cartesian coordinate axes are "maintained" with this method. For $n = 3$, Table 1 shows the values of t and the resulting axes (x^j denotes the jth axis with the positive or negative end denoted by the appropriate sign). Figure 1 illustrates this with a unit sphere centered on the Cartesian coordinate origin. The curved dotted line on the sphere shows the values of (x_1, x_2, x_3) as a function of θ_1 and

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θ_2, which in turn is a function of t as it moves from 0 to 2π (the initial and ending point (x_1, x_2, x_3) is $(1, 0, 0)$). A careful look will show that the curved dotted line intersects each of the coordinate axes (an animation would clearly be useful here). Figure 2 shows the same sphere from overhead (along the x_3 axis) and Figure 3 show the sphere from the side. These figures give a sense of how the t "samples" various values of (x_1, x_2, x_3).

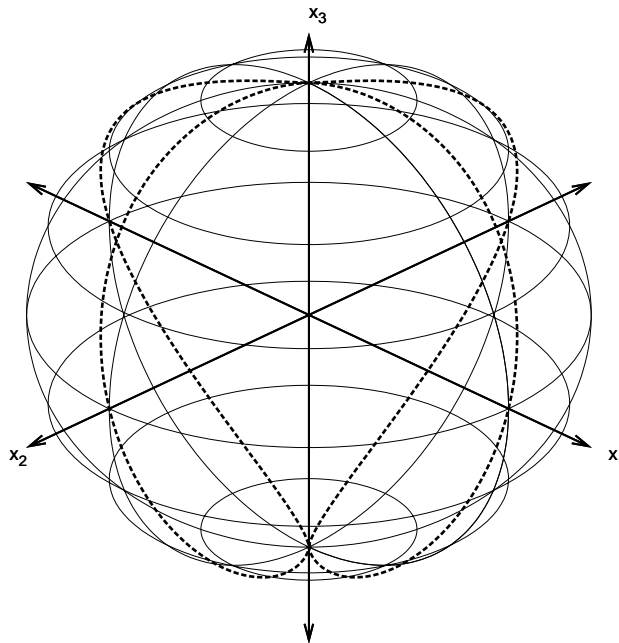| Table 1	
t	x axis
0	x^1
$\pi/4$	x^3
$\pi/2$	x^2
$3\pi/4$	x^{-3}
π	x^{-1}
$5\pi/4$	x^3
$3\pi/2$	x^{-2}
$7\pi/4$	x^{-3}

Figure 1: Unit Sphere with Coordinate Axes

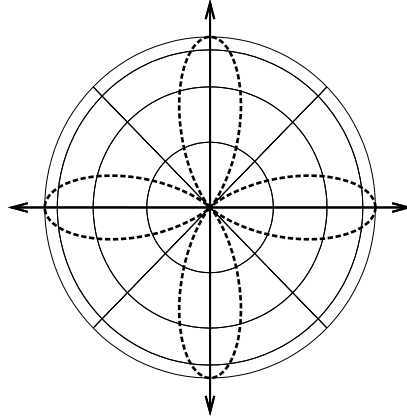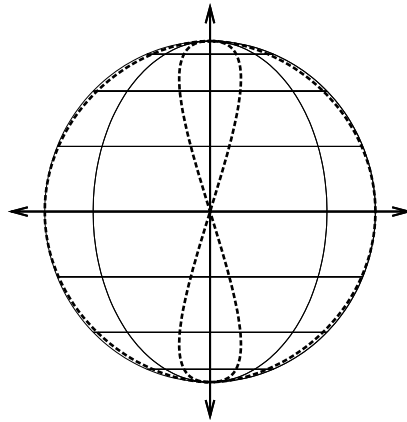Figure 2: Another View of the Unit Sphere

Figure 3: Yet Another View of the Unit Sphere

Non-Economic Examples

This section illustrates this method on several non-economic functions. As in the rest of this paper, only three variables are plotted. Four or perhaps five variables could be plotted, but the number of coordinate axes would be 16 and 32 respectively. Clearly, to plot a large number of variables with this method, one needs either a large monitor or large page. However, useful results can be obtained with even a relatively small number of variables.
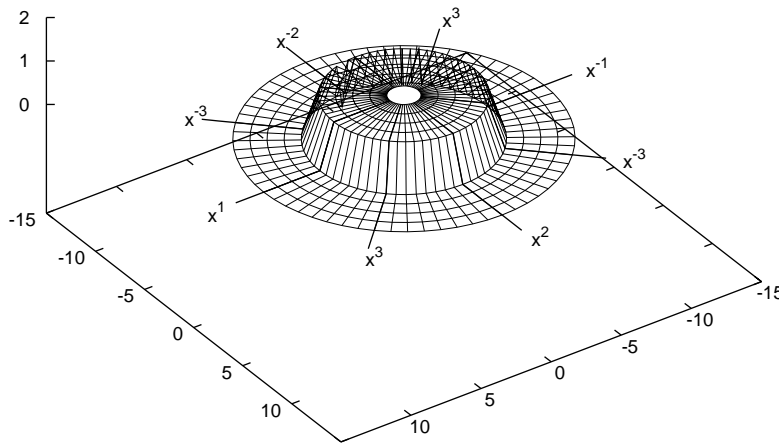
Figure 4 shows the function

$$f(x_1, x_2, x_3) = 1, \ \sqrt{(x_1{}^2 + x_2{}^2 + x_3{}^2)} \leq 5$$
$$f(x_1, x_2, x_3) = 0, \ \sqrt{(x_1{}^2 + x_2{}^2 + x_3{}^2)} > 5.$$

Using the notation of the previous section, the Cartesian coordinate axes are marked in the plot. The function is plotted for values of r from 1 to 10, and the symmetric nature of the function is very easily seen.

Figure 4: Symmetric Function

Another symmetric function is shown in Figure 5; it is

$$f(x_1, x_2, x_3) = (x_1{}^2 + x_2{}^2 + x_3{}^2).$$

Again, it appears exactly as one would expect.

Figure 6 shows a variation of Figure 5 that is not symmetric:

$$f(x_1, x_2, x_3) = (x_1{}^2 + 1.5 \cdot x_2{}^2 + x_3{}^2).$$

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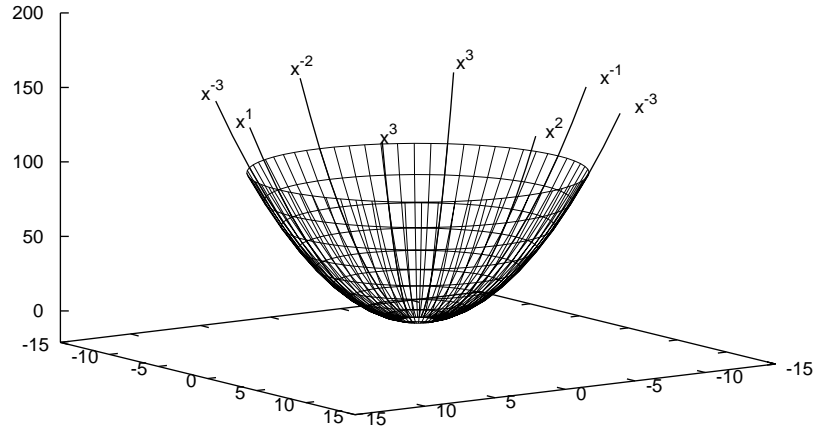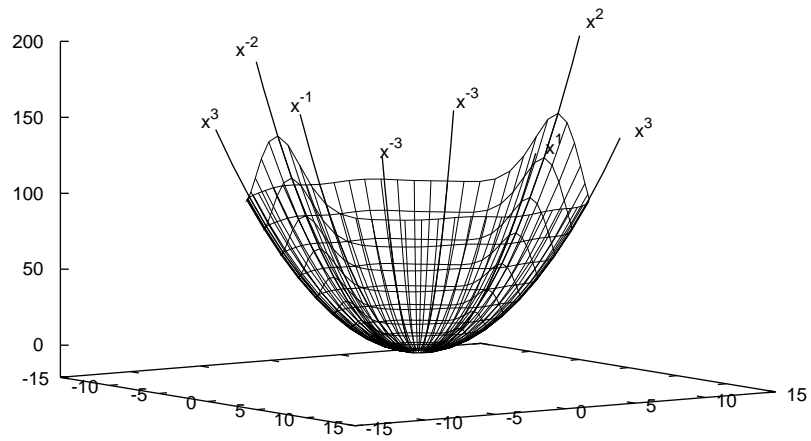Figure 5: Another Symmetric Function


Figure 6: Non-Symmetric Function

The impact on the x^2 and x^{-2} axes is very clear and entirely predictable.

Figure 7 shows another non-symmetric function:

$$\begin{aligned}
f(x_1, x_2, x_3) &= 1, \quad (x_1 - 3)^2 + (x_2 - 3)^2 + (x_3 - 3)^2 < 5 \\
f(x_1, x_2, x_3) &= 0, \quad (x_1 - 3)^2 + (x_2 - 3)^2 + (x_3 - 3)^2 \geq 5
\end{aligned}$$

Note how the positive regions appear in two places—there is some redundancy with this technique. Recall Figures 1, 2 and 3—the function is sometimes "sampled" at nearly identical points.

Figure 7: Another Non-Symmetric Function
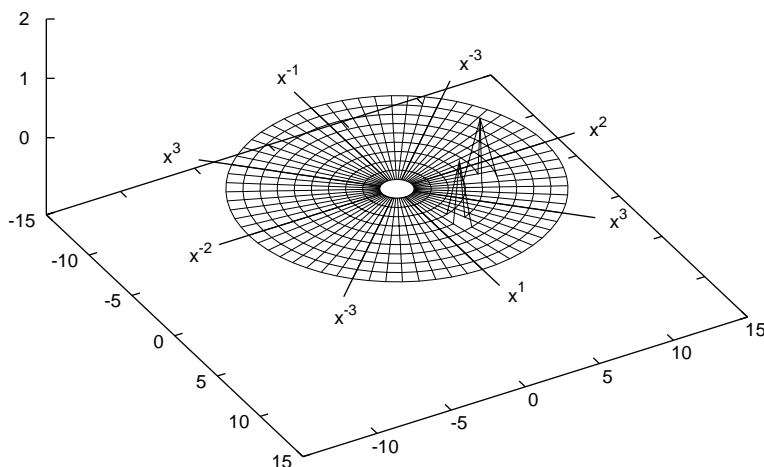
Economic Examples

Following Goffe (1999), the function from Hoffman and Schmidt (1981), with synthetic data from Veall (1990) is examined here. It is a rational expectations model estimated by maximum likelihood. It has 8 variables: γ, β_1, β_2, λ_1, λ_2, σ_0^2, σ_1^2, and σ_2^2. The model is described by

$$\begin{aligned}
Y_t &= \gamma E_{t-1} Y_t + \beta_1 X_{1t} + \beta_2 X_{2t} + \epsilon_{0t}, \ |\gamma| < 1, & (3) \\
X_{1t} &= \lambda_1 X_{1t-1} + \epsilon_{1t}, & (4) \\
X_{2t} &= \lambda_2 X_{2t-1} + \epsilon_{2t}, & (5) \\
\epsilon_{it} &\sim i.i.d. \ N(0, \sigma_i^2), & (6) \\
Y_t &= \beta_1 X_{1t} + \beta_2 X_{2t} + (\gamma/(1-\gamma))(\beta_1 \lambda_1 X_{1t-1} + \beta_2 \lambda_2 X_{2t-1}) + \epsilon_{0t}. & (7)
\end{aligned}$$

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Equation (7) is derived from the other equations, and (4), (5) and (7) are used to estimate the model.

Figure 8 shows the first three variables plotted. The center point is the minimum found via simulated annealing in Goffe (1996) (as described there, this function is relatively hard to optimize). The results are not terribly remarkable—they confirm that the center point does indeed seem to be the minimum (it has a value of 79.67), and that the function varies in these variables, which is hardly surprising.
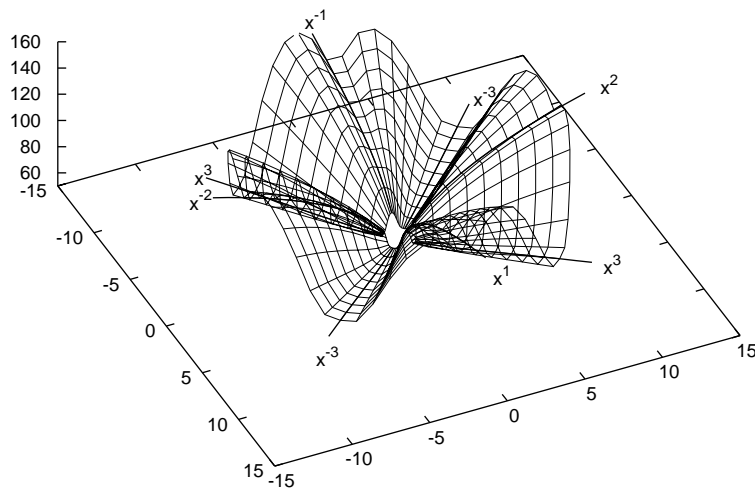
Figure 8: Rational Expectations Model

Figure 9 shows the function for the last three variables (all variances). As this plot makes immediately clear, this model cannot distinguish the values of these parameters. This was hinted at in runs of simulated annealing in Goffe (1996), but the ancient adage about pictures would seem to apply here.

To further illustrate this visualization method, a translog functional form was fitted to synthetic data generated from a CES production function. A standard CES production function, with inputs x_1 and x_2 and output q of the form

$$q = A[\alpha x_1^{-\rho} + (1-\alpha)x_2^{-\rho}]^{1/\rho}$$

was used, with A set to unity, α to .75, and ρ to 4. This value of ρ was chosen as it leads to an elasticity of substitution of .2, and as Wales (1977) reports, the translog has a difficult time with the CES when this elasticity strays from unity. The number of observations was set to 100, and both x_1 and x_2 trended upwards with random variation supplied by the pseudo-random number generator of Marsaglia and Zaman (1987). From the generated data, the translog functional form

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Figure 9: Another View of the Rational Expectations Model
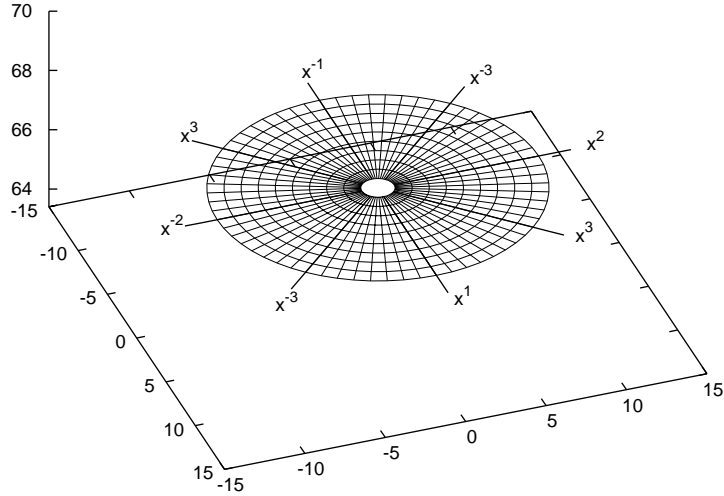
$$q = \beta_1 + \beta_2 \log x_1 + \beta_3 \log x_2 + \tfrac{1}{2}\beta_4 (\log x_1)^2 + \beta_5 (\log x_1 \cdot \log x_2) + \tfrac{1}{2}\beta_6 (\log x_2)^2$$

was fitted by minimizing the sum of squares using simulated annealing (granted, it could have been estimated with OLS, but hopefully this will provide insight for those translog models that cannot be estimated with OLS). The plot for the last three β's is shown in Figure 10 (the first three β's show a similar pattern). The extreme steepness of this function is clear from the labels on the z-axis; at a radius of 10 from the minimum (with a value of .19), the minimum value is 574, and the maximum value is 61,193. While not a serious estimation problem here (although simulated annealing did find it a bit of a challenge), it does suggest that for more complex translog models, the very steep valleys might hinder easy estimation.

Figure 10: Translog Production Function
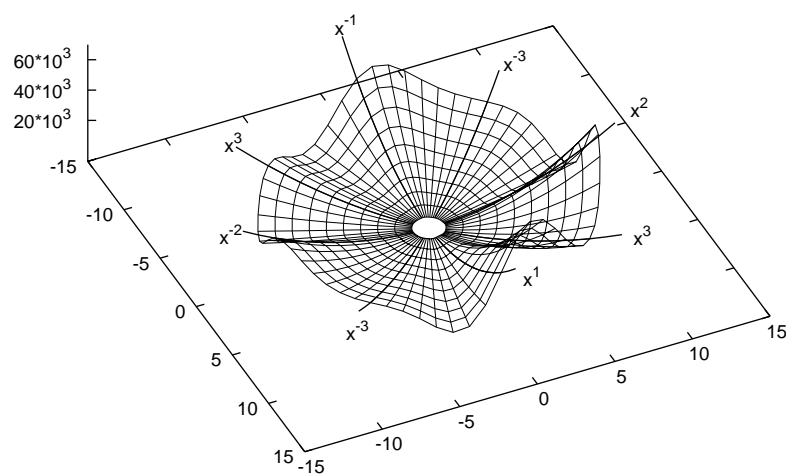
Conclusion

This paper introduced a method to visualize a function of multiple dimensions in a relatively intuitive manner. Economists may find it useful for dealing with difficult functions that arise in estimation and modeling. While one is limited to plotting four or perhaps five different variables, as the examples showed, this is often sufficient to gain valuable insights.

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