Experimentally probing the shape of extra dimensions

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Theories with compact extra dimensions have become increasingly popular. In some theories the standard model particles are confined to our four-dimensional world and only gravity can propagate in the extra dimensions. In these models the size of the extra dimensions can be as large as hundreds of microns. The functional dependence of the gravitational force on distance is well known at both large and small distances compared to the size of the extra dimensions; for two extra dimensions, for example, it varies from $1/r^2$ to $1/r^4$ as the distance decreases. However, the dependence for intermediate distances has not been fully calculated. We determine this dependence as a function of both the size of the extra dimensions and the possible angle between the extra dimensional unit vectors and show that high precision measurements of the gravitational force would make possible the determination of the shape of the extra dimensions. © *2006 American Association of Physics Teachers.*
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I. INTRODUCTION

Since the realization 20 years ago that string theories were promising candidates for a unified theory of all interactions, there has been extensive interest in theories with more than three spatial dimensions. In string theory, consistency requires that there be nine spatial dimensions.[1] Because we only observe three such dimensions, it is generally assumed that each of the other six dimensions must be compact, curled into a very small size R. For example, a cylinder has one large and one compact direction. For a small creature on the cylinder, the world looks two dimensional, while for a large creature, the cylinder looks one dimensional. Thus we expect that at distance scales larger than R (corresponding to energy scales below $\hbar c/R$), the effects of these extra dimensions are unobservable. More precisely, if y is one of the compact dimensions, then y is identified with $y+2\pi R$. Thus, $f(y)=f(y+2\pi R)$ for any function. By Fourier's theorem, any such function can be Fourier-expanded. At distance scales larger than R (or energy scales smaller than $\hbar c/R$), only the zero mode, which is independent of the extra dimensions, is relevant.

The possibility of experimental detection of these extra dimensions has been considered negligible until recently, because it was believed that the scale R is far smaller than the scales that can be reached in colliders. As a particle's energy increases, its Compton wavelength shrinks, and at a high enough energy it will become smaller than the particle's Schwarzschild radius (at which point gravitational effects are important). Quantum gravitational effects will be critical at this energy scale, which is called the Planck energy. This energy scale is $\sqrt{\hbar c^5/G}\sim 10^{19}$ GeV (corresponding to a Planck length of 10^{-33} cm). We thus expect the compactification scale R to be of this order, which is far beyond the range of accelerators.

The reason that the Planck energy is so high is that at macroscopic length scales gravity is much weaker than the other interactions and thus enormous energies must be reached before it becomes comparably strong. A major problem in particle physics is trying to put into one single framework two widely different energy scales (the weak interaction scale of 100 GeV and the Planck scale). This problem is called the hierarchy problem and is related to the relative weakness of gravity.

In the past few years, new models have appeared[2] in which the size of the extra dimensions can be much larger, as large as a millimeter! In these models only gravitational interactions can propagate in the extra dimensions; other particles and interactions are stuck in our three-dimensional space. In this way, only gravitational interactions at distance scales less than a millimeter would be sensitive to these extra dimensions. Having nongravitational interactions stuck in our three-dimensional space is not as ad hoc as it might appear—string theories often have gravitons as closed strings and other particles as open strings, and the open ends can be stuck on a three-dimensional surface. Such large extra dimensions would have dramatic experimental implications. In higher dimensions, gravity would no longer be inverse square, but would be inverse-cube in four spatial dimensions, inverse-fourth in five, etc. This dependence follows from Gauss' law in many dimensions. Thus, we would experimentally see gravity change from being inverse-square at scales larger than R to inverse-cubed at scales smaller than R (if there is one extra dimension).

We have explicitly assumed that Gauss' law will describe the gravitational interaction in many spatial dimensions. This assumption follows naturally from string theories and is discussed in Ref. 2. Gauss' law relates the integral of the force over the surface of a volume to the amount of mass (or charge in the case of electromagnetism) inside that volume. By the divergence theorem the integral of the force over the surface can be converted into a divergence of the force over the volume. The divergence of a force is the Laplacian of the potential energy, which leads to Poisson's equation (giving $\nabla\cdot\boldsymbol{E}=4\pi\rho$ and $\nabla^2 V=-4\pi\rho$ for electromagnetic fields). Poisson's equation follows directly from the Einstein field equations in n dimensions, and thus Gauss' law follows from general relativity. See Ref. 2 for details.

The major attractive feature of models in which there are large extra dimensions is that they can substantially decrease the Planck scale to accessible energies. For example, suppose that there are six compact dimensions, all with a size R given by 20 fermis (corresponding to an energy scale of 10 MeV). As the energy scale increases past 10 MeV, gravity will

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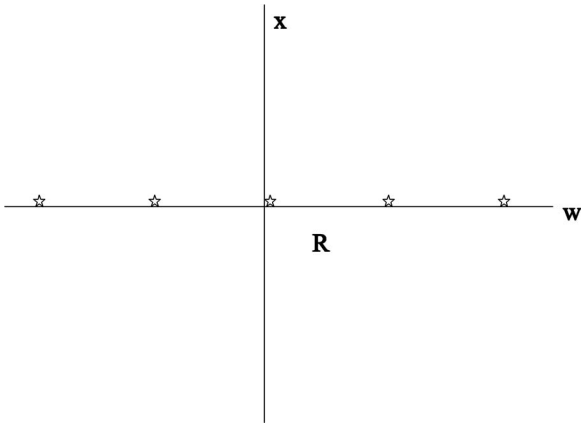

Fig. 1. A point mass in a space with one compact dimension w and one noncompact dimension x.

change from being $1/r^2$ to $1/r^8$. Its strength will then grow extremely rapidly at scales above 10 MeV and will become strong at approximately a few TeV, which is within reach of currently planned accelerators. Thus the Planck scale will only be a few TeV, and we can expect future colliders to directly probe quantum gravity and string effects, produce black holes, etc. The traditional hierarchy problem is then resolved.

We can reverse the argument and determine the value of R given that the requirement that the Planck scale be in the TeV range. This requirement will depend only on the number of extra dimensions. For a Planck scale of 1 TeV it is found[2] that $R \sim 10^{32/d} 10^{-17}$ cm, where d is the number of extra dimensions. For one extra dimension the value of R turns out to be millions of kilometers, which is clearly excluded. Thus, there cannot be just one extra dimension. For two extra dimensions R is a few hundred microns, which is currently being experimentally probed.[3,4] For three extra dimensions R is tens of nanometers. Although this distance is accessible in some proposed experiments,[5] reaching force strengths comparable to the gravitational force seems far out of reach. For more than three extra dimensions the value of R is too small to ever probe directly.

In this article we will focus on the case of two extra dimensions. With more than one compact dimension, the shape is not completely constrained, for example, with two extra dimensions, the ratio of the radii of the two dimensions can be other than unity and the unit vectors need not be orthogonal, leading to a shape angle. Thus, by precisely measuring the force of gravity at small distances, we can probe the shape as well as the size of the extra dimensions.

II. CALCULATIONAL METHODS AND GAUSS' LAW

Although we will focus on the case of two extra dimensions, we will first begin, as a toy model, with the case of a single extra dimension, compactified with periodicity R (R would be the circumference of a circle, not the radius). The spatial variables are x, y, z, and w. A point mass in the system will appear as in Fig. 1, where y and z coordinates have been suppressed, and the periodicity in the compact dimension is apparent. It should be pointed out that we are working in a "brane-world" scenario in which the known particles are confined to a four-dimensional brane and only gravity can propagate in the extra dimensions.

It is difficult to visualize four spatial dimensions. To simplify the visualization we consider a three-dimensional problem in which the masses in Fig. 1 are point charges and ask for the value of the electric field a distance r from the central point charge. For $r \ll R$, only the central point charge is relevant, and the electric field is given by drawing a spherical Gaussian surface around the charge, resulting in a $1/r^2$ force. As r increases, the other charges become important, and no simple Gaussian surface can be drawn; we must add the contributions explicitly, resulting in an infinite sum. As r becomes much bigger than R, the point charges look like a line charge. Now a cylindrical Gaussian surface can be used, resulting in a $1/r$ force law. An interesting exercise would be to write the sum explicitly, and show how it interpolates between the two different examples of Gauss' law (we will do this interpolation for the higher dimensional case). In standard electromagnetism texts the $1/r$ force law is derived by first considering a finite length line and then increasing the length to infinity. The method described here is an alternative derivation that provides some insight into the $1/r$ behavior.

In four spatial dimensions and $r \ll R$, the gravitational (or electromagnetic) force law is $1/r^3$, which is found by encircling the point mass with a four-sphere. As r increases, the full sum must be used. At very large distances the points look like a line, and a Gaussian four-cylinder must be used. Note that the surface area of a three cylinder (parallel to the axis) is given by the length of the cylinder times the circumference of the "two-sphere" (that is, circle) endcaps, that is, $2\pi rL$. Thus the surface area of a four-cylinder is the length times the area of the three-sphere endcaps, which is $4\pi r^2 L$. We now proceed to the calculation of the gravitational potential from the masses in Fig. 1 in four spatial dimensions.

For very small distance scales, $r \ll R$, the gravitational field depends only on the mass at $w=0$. Gauss' law in n-dimensions[6] is

$$\int F \cdot da = S_n G_n M_{\text{enc}}, \tag{1}$$

where S_n is the surface area of a unit n-sphere, M_{enc} is the mass enclosed in the Gaussian volume, and G_n is the n-dimensional Newton's constant. Note that Newton's constant can be defined through this equation—the constant S_n is only a convention. In three spatial dimensions the left-hand side of Eq. (1) gives $F 4\pi r^2$, and the right-hand side gives $4\pi G_3 M$. By equating these two terms, we find the familiar Newton's law for the gravitational field. In higher dimensions, the surface area of a unit sphere is

$$S_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}, \tag{2}$$

where Γ is the gamma function. For our purposes, it is sufficient to know that $\Gamma(n)=(n-1)\Gamma(n-1)$, $\Gamma(1)=1$, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. We see that the gravitational field will be given by $F=G_n M/r^{n-1}$. For one extra dimension, we see this reasoning gives $F=G_4 M/r^3$. Note that the dimensionality of G_4 and G_3 (the usual Newton's constant) are different.

At very large distances, $r \gg R$, the string of masses looks like a continuous line of uniform mass density, and we can use a cylindrical Gaussian surface to solve for the field.[2] Consider a four-dimensional cylinder of side length L, with

endcaps composed of three dimensional spheres of radius r. The mass enclosed by the cylinder is $M(L/R)$, $S_4=2\pi^2$, and the left-hand side of Gauss' law is $F4\pi r^2$. If we substitute these results into Eq. (1), the L drops out (as usual in cylindrical Gauss' law applications), and we find that

$$F = G_4 M \frac{2\pi^2}{4\pi} \frac{1}{r^2 R}. \tag{3}$$

Thus at large distances, we recover the usual inverse square law. If we compare Eq. (3) with the conventional Newton's law of gravity, we can find the expression for G_4 in terms of G_3 (the usual gravitational constant) and R. In n spatial dimensions, we can write

$$G_3 = \frac{S_n}{4\pi} \frac{G_n}{V_{n-3}}, \tag{4}$$

where V_{n-3} is the volume of the $(n-3)$-dimensional compactified space.

Thus, we find from Gauss' law that the gravitational field varies from an inverse-square law at large distances to an inverse $(n-1)$ law at small distances, where n is the number of spatial dimensions. This result is independent of the precise shape of the extra dimensions. To probe the shape we must look at intermediate distances. In this case, we must explicitly sum over the "mirror masses" and thus Gauss' law does not provide a useful way to calculate the force. For simplicity, we will calculate the gravitational potential at intermediate distances—the field can be determined by differentiation. Again, we first consider a single extra dimension as in Fig. 1. In four spatial dimensions the gravitational potential at a distance D from a point mass is $V_4=-G_4M/2D^2$ and so the general result for the potential a distance $r=\sqrt{x^2+y^2+z^2}$ from the mass M is

$$V(r) = -\frac{G_4 M}{2} \sum_{n=-\infty}^{\infty} \frac{1}{r^2 + n^2 R^2}. \tag{5}$$

In terms of the conventional Newton's constant G_3, Eq. (5) becomes

$$V(r) = -\frac{G_3 M}{\pi R} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + \Delta^2}, \tag{6}$$

where $\Delta \equiv r/R$. For large Δ, the sum is π/Δ, leading to the conventional gravitational potential.

We plot $rV(r)$ as a function of r in Fig. 2. As expected, the potential varies as $1/r$ for large r and as $1/r^2$ for small r. Because one extra dimension is already excluded, we now consider the case of two extra dimensions.

III. TWO EXTRA DIMENSIONS

With two extra dimensions the topology of the compactified space is no longer simple. Because the three dimensions of our universe are topologically flat, we will assume that the two compact dimensions are also flat. This assumption occurs naturally in string theory. Flat compact dimensions are not unnatural. For example, the surface of a cylinder is two-dimensional, and has one very large and one very small dimension. Yet it is topologically flat—the angles of a triangle on a cylinder always sum to 180°.

Technically, the space we are considering is the product of two circles, known as a 2-torus. It can have two circumferences, which we will denote as R_1 and R_2. In addition, there is no reason that the unit vectors in the two compact directions must be orthogonal.[7] Thus, the description of the image masses corresponding to Fig. 1 is given in Fig. 3. The three parameters that specify the shape of the extra dimensions are R_1, R_2, and θ.

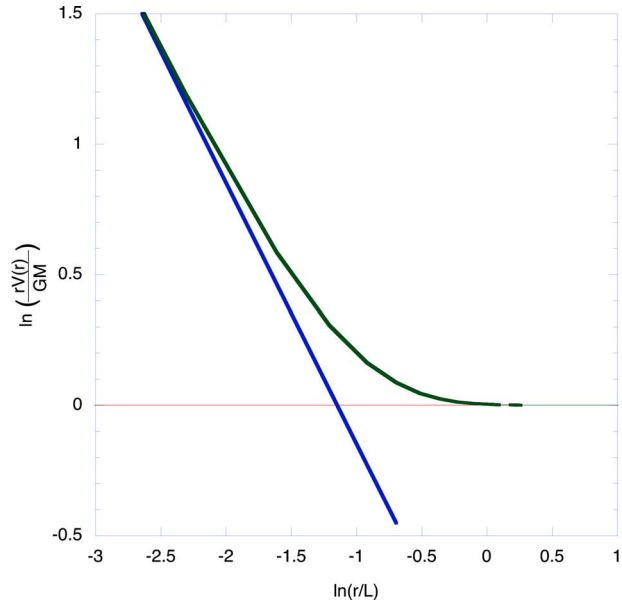

Fig. 2. The gravitational potential times r is plotted as a function of r for one extra dimension. Note that for large distances, the potential varies as $1/r$, and for short distances as $1/r^2$. The straight line has slope -1.

As before, we can look at a three-dimensional electromagnetic analogy with the masses in Fig. 3 replaced by point charges. At very short distances, only the central charge contributes, giving the $1/r^2$ force. At intermediate distances, we must calculate the full double sum. At very large distances, a Gaussian pillbox will yield a constant field. An interesting exercise would be to show how the double sum explicitly results in Coulomb's law at short distances and a constant field at large distances.

We will first consider the case in which the unit vectors are orthogonal. The potential is given by

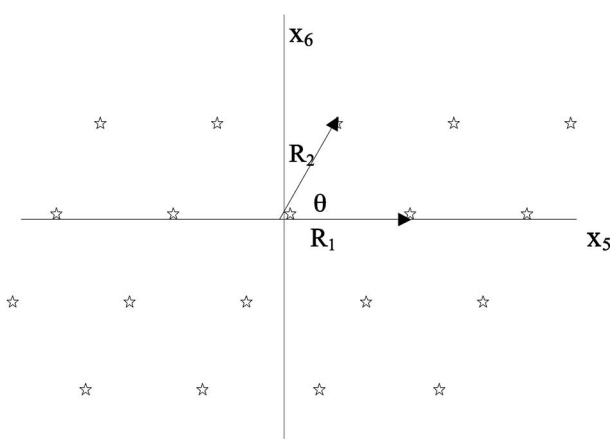

Fig. 3. A point mass located in a space of two compact dimensions of circumferences R_1 and R_2, with an angle θ between the unit vectors.

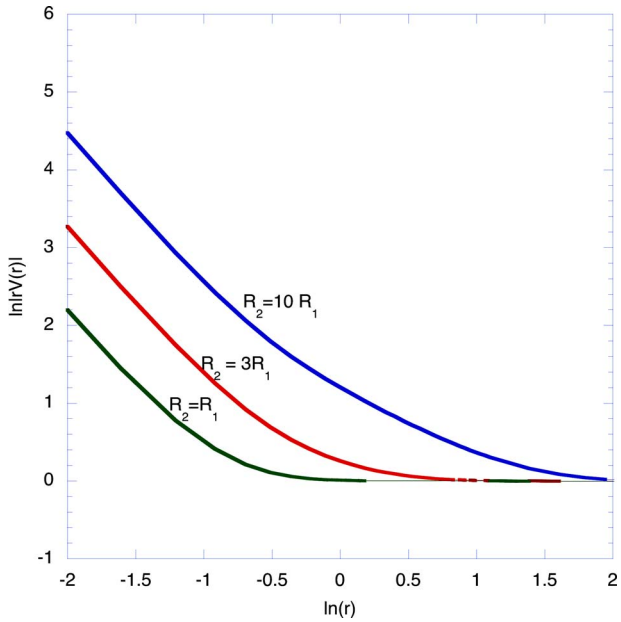

Fig. 4. The gravitational potential, times r as a function of distance for various values of R_2/R_1. We have fixed R_1 and thus the volume of the compact space will vary in each case.

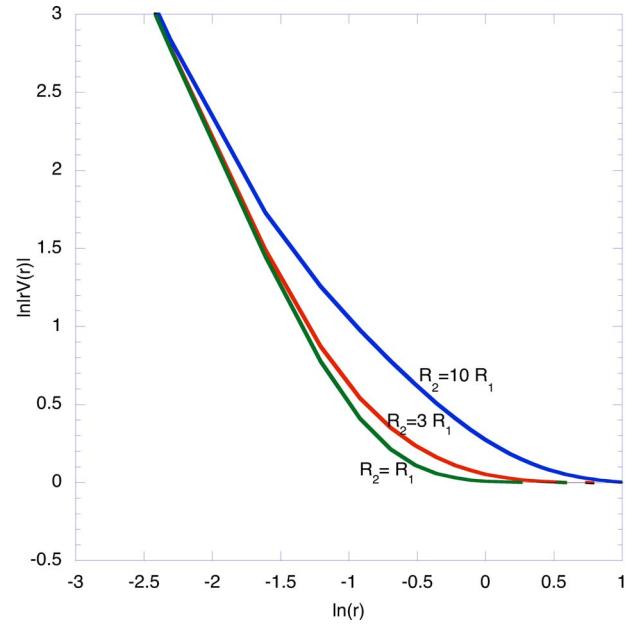

Fig. 5. The gravitational potential times r as a function of distance, where the value of R_1 has been rescaled so that the volume of the compact dimensions are unchanged as R_2 is varied.

$$V = -\frac{G_5 M}{3} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(r^2 + n^2 R_1^2 + m^2 R_2^2)^{3/2}}, \tag{7}$$

where without loss of generality we can choose $R_2 \geqslant R_1$; G_5 is Newton's constant in five spatial dimensions.

The result is plotted in Fig. 4 for $R_2 = R_1$, $R_2 = 3R_1$, and $R_2 = 10R_1$. Note that at large distances, the standard $1/r$ potential emerges as expected. For small distances, we can clearly see the $1/r^3$ dependence of the potential. It is important to note that a measurement of the strength of the gravitational potential at very small distances would enable us to distinguish between the three possibilities. We expect these possibilities to be distinguishable, because the strength of the potential depends on the five-dimensional Newton's constant, which depends on the volume of the extra dimensions [see Eq. (4)]. However, the volume of the extra dimensions does not determine the shape—we could not distinguish between $R_1 = R_2 = \sqrt{10} R_0$ and $R_2 = 10R_1 = 10R_0$, where R_0 is an unknown scale. Thus, to explore the shape of the extra dimensions, the full position dependence of the potential must be explored.

In Fig. 5 we have rescaled R_1 and R_2 so that the volume of the extra dimensions is the same in each of the three cases. We can now determine the precision necessary to distinguish the three cases. From Fig. 5 we can see that the value of the potential varies as R_2 varies from R_1 to $10R_1$ by a little over a factor of 2. A 30% measurement of the potential would enable us to distinguish between $R_2 = R_1$ and $R_2 = 3R_1$. Such precision seems experimentally feasible, once a deviation from the inverse-square law is found.

The full expression for the potential at intermediate distances has not previously been plotted. In many other papers (see Refs. 2, 8, and 9), the leading order deviation as r is decreased is given. This deviation is of the form $V = V_0[1 + \alpha \exp(-r/R)]$, and many experimenters make plots of the allowed region in the α-R plane. But a measurement of a nonzero value of α would only establish a deviation—the expression breaks down for slightly smaller values of r—and the full expression would then be necessary.

The other possibility is that the unit vectors in the extra dimensions are not orthogonal. Here, we assume $R_1 = R_2$ for simplicity and vary θ. If we again rescale the values of R so that the volumes are identical, we find the results in Fig. 6. We see that it is essentially impossible to distinguish between $\theta = \pi/2$ and $\theta = \pi/3$, but that a factor of 2 measurement of the potential would distinguish between these cases and $\theta = \pi/36$.

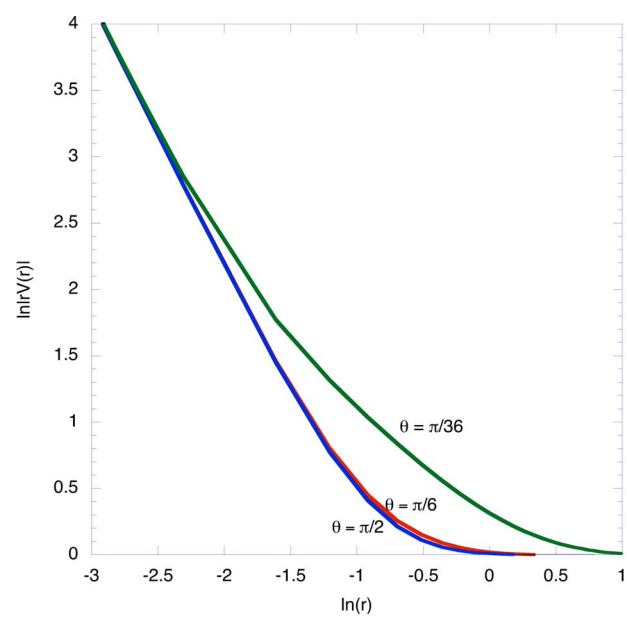

Fig. 6. The gravitational potential times r for $R_1 = R_2$ for various values of the shape angle. The results have been scaled so that the volumes are identical.

IV. DISCUSSION

Experiments probing the nature of gravity at short distances are now underway (see Ref. 3 for a review). Measurements at the nanometer-scale are possible. Should a deviation from the inverse-square law be discovered, there will be a huge effort to measure this deviation precisely. In large extra dimension models, the nature of the deviation is well known at very short distances, and the leading order term in the deviation is also well known. In this paper we have discussed the deviation in the intermediate region and have shown that experimentally constraining the shape as well as the volume of the extra dimensions is feasible. For two extra dimensions, in which the expected scale is in the tens to hundreds of microns region, a measurement of 30% accuracy in the potential would enable us to distinguish a difference of a factor of 3 in the size of the extra dimensions, and shape angles smaller than a few degrees would also be measurable.

There are several possible avenues for future study. We could consider three extra dimensions, which would be aesthetically appealing because the nine spatial dimensions of string theory could be grouped into three large, three intermediate, and three small dimensions. In this case there would be three different sizes as well as three different shape angles. As noted, it is unlikely that this case is measurable experimentally.

Another possibility is that the extra dimensions are not flat. Here we have assumed that they are flat and form a 2-torus, the simplest case. However, it could be that the extra dimensions are curved. For example, they could have the topology of the surface of a sphere.[9] In this case the simple description of Fig. 3 would not be accurate (because one can travel in any direction and return to one's starting point). An expansion in terms of Kaluza-Klein modes on a sphere would be appropriate in this case. This expansion was done in Ref. 9, where the leading order deviation from Newton's law is presented.

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