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ON THE INTERSECTION OF RATIONAL TRANSVERSAL SUBTORI

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Abstract

We show that under a suitable transversality condition, the intersection of two rational subtori in an algebraic torus $(\mathbb{C}^*)^n$ is a finite group which can be determined using the torsion part of some associated lattice. We also give applications to the study of characteristic varieties of smooth complex algebraic varieties. As an example we discuss A. Suciu's line arrangement, the so-called deleted B_3-arrangement.

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1. Introduction

Let L be a free $\mathbb{Z}$-module of finite rank n, and let $A \subset L$ and $B \subset L$ be two primitive sublattices, that is, A and B are subgroups such that

$$\mathrm{Tors}(L/A) = \mathrm{Tors}(L/B) = 0.$$

Consider the associated $\mathbb{C}$-vector spaces

$$V = L \otimes_{\mathbb{Z}} \mathbb{C}, \quad V_A = A \otimes_{\mathbb{Z}} \mathbb{C} \quad \text{and} \quad V_B = B \otimes_{\mathbb{Z}} \mathbb{C}.$$

Let $\exp_L : V \to T = L \otimes_{\mathbb{Z}} \mathbb{C}^*$ be the associated exponential map given by

$$\exp_L = 1_L \otimes_{\mathbb{Z}} \exp$$

where $1_L : L \to L$ is the identity and $\exp : \mathbb{C} \to \mathbb{C}^*$ is defined by $t \mapsto \exp(2\pi i t)$. Then $\exp_L$ is a surjective group homomorphism with kernel $L = L \otimes_{\mathbb{Z}} \mathbb{Z} \subset V$. If a $\mathbb{Z}$-basis of L is chosen, then the identifications $L = \mathbb{Z}^n$, $V = \mathbb{C}^n$, $T = (\mathbb{C}^*)^n$ are obvious and $\exp_L : \mathbb{C}^n \to (\mathbb{C}^*)^n$ is given by

$$(t_1, \ldots, t_n) \mapsto (\exp(2\pi i t_1), \ldots, \exp(2\pi i t_n)).$$

The main result of this note is the following theorem.

THEOREM 1.1. *If $V_A \cap V_B = 0$, then there is a group isomorphism*

$$\mathrm{Tors}(L/(A+B)) \to \exp_L(V_A) \cap \exp_L(V_B).$$

In fact any algebraic subtorus $S \subset T$, that is, S is a closed algebraic subset and a subgroup in T, comes from a primitive lattice $A(S) \subset L$ (see Arapura's paper [1, Lemma 2.1 in Section II]). Hence, Theorem 1.1 applies to any pair of such algebraic subtori.

This theorem is proved in Section 2. In Section 3 we show how to use Theorem 1.1 to describe the intersections of the irreducible components of the characteristic varieties of smooth complex algebraic varieties. A specific example coming from hyperplane arrangement theory concludes the paper.

2. The proof

Let $n = \mathrm{rank}\, L$, $a = \mathrm{rank}\, A$ and $b = \mathrm{rank}\, B$. Consider the quotient group $L' = L/A$, which is again a lattice, of rank $n - a$. The composition $B \to L \to L'$ of the inclusion $B \to L$ and the projection $L \to L'$ gives rise to an injective morphism $\iota : B \to L'$ identifying B to the sublattice $B' = \iota(B) \subset L'$.

Then there is a basis $e'_1, \ldots, e'_{n-a}$ of the lattice L' such that B' is the subgroup spanned by $d_1 e'_1, \ldots, d_b e'_b$ for some positive integers d_j. Moreover, there is an integer m with $1 \le m \le b+1$ such that

$$1 = d_1 = \cdots = d_{m-1} < d_m \le \cdots \le d_b \quad \text{and} \quad d_m \mid d_{m+1} \mid \cdots \mid d_b. \tag{2.1}$$

It follows that

$$\mathrm{Tors}(L/(A+B)) = \mathrm{Tors}\left(\frac{L/A}{(A+B)/A}\right) = \mathrm{Tors}(L'/B') \tag{2.2}$$

and hence

$$\mathrm{Tors}(L/(A+B)) = \mathbb{Z}/d_m\mathbb{Z} \oplus \mathbb{Z}/d_{m+1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_b\mathbb{Z}. \tag{2.3}$$

Let $e_1, \ldots, e_{n-a}$ be any lifts of the vectors e'_j to L and let $f_1, \ldots, f_a$ be a $\mathbb{Z}$-basis of A. Then $\mathcal{B} = \{e_1, \ldots, e_{n-a}, f_1, \ldots, f_a\}$ is a $\mathbb{Z}$-basis of L.

For $j = 1, \ldots, b$, let $g_j \in B$ be vectors such that their classes g'_j in L' satisfy $g'_j = d_j e'_j$. It follows that $g_j = d_j e_j + a_j$ for some vectors $a_j \in A$. Now write

$$a_j = \sum_{i=1,a} \alpha_{ji} f_i \tag{2.4}$$

for some $\alpha_{ji} \in \mathbb{Z}$. By replacing e_j by $e_j + r_j$ for suitable vectors $r_j \in A$, we may and do assume throughout the remainder of this paper that

$$0 \leq \alpha_{ji} < d_j \tag{2.5}$$

for all $i = 1, \ldots, a$ and $j = 1, \ldots, b$. In particular, $a_j = 0$ for $j = 1, \ldots, m-1$.

LEMMA 2.1. *The vectors $g_1, \ldots, g_b$ form a $\mathbb{Z}$-basis of the lattice B.*

PROOF. Note that the vectors $g_1, \ldots, g_b$ are all contained in B and their images under ι span the lattice B'. □

Assume now that $\exp_L(v_A) = \exp_L(v_B)$ for some vectors

$$v_A = p_1 f_1 + \cdots + p_a f_a \in V_A$$

and

$$v_B = q_1 g_1 + \cdots + q_b g_b \in V_B$$

where $p_i, q_j \in \mathbb{C}$. It follows that $v_A - v_B \in \ker \exp_L = L$. More precisely, we obtain

$$q_j d_j \in \mathbb{Z} \quad \text{for } j = 1, \ldots, b$$

and

$$z_i := p_i - \sum_{j=1,b} q_j \alpha_{ji} \in \mathbb{Z} \quad \text{for } i = 1, \ldots, a.$$

It follows that $q_j = k_j/d_j$. We may and do assume that $0 \leq k_j < d_j$, since the value of $\exp_L(v_B)$ is not changed when the coefficients q_j are modified by integers. Note that with this choice one has $k_j = 0$ for $j = 1, \ldots, m-1$. In this way we obtain a surjective group homomorphism

$$\theta : \mathbb{Z}/d_m\mathbb{Z} \oplus \mathbb{Z}/d_{m+1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_b\mathbb{Z} \to \exp_L(V_A) \cap \exp_L(V_B) \tag{2.6}$$

given by

$$\hat{k} = (\hat{k}_m, \ldots, \hat{k}_b) \mapsto \exp_L\left(\frac{k_m}{d_m} g_m + \cdots + \frac{k_b}{d_b} g_b\right).$$

This morphism θ is indeed defined correctly since for any choice of the q_j as above we may use the defining equation of z_i, set $z_i = 0$ and determine the values for p_i, that is, find a vector v_A such that $\exp_L(v_A) = \exp_L(v_B)$.

To show that θ is injective, we have to show that $\ker \theta = 0$. Since B is primitive we can take the set

$$\{g_1, \ldots, g_b, h_1, \ldots, h_{n-b}\}$$

as a $\mathbb{Z}$-basis of L, where $h'_1, \ldots, h'_{n-b}$ is a $\mathbb{Z}$-basis for the lattice L/B. Let $\hat{k} \in \ker \theta$. Then

$$\theta(\hat{k}) = \exp_L \left(\frac{k_m}{d_m} g_m + \cdots + \frac{k_b}{d_b} g_b \right) = 1,$$

which implies that $((k_i)/d_i) \in \mathbb{Z}$, for all $m \leq i \leq b$. Therefore,

$$\hat{k} = (\hat{k}_m, \ldots, \hat{k}_b) = (\hat{0}, \ldots, \hat{0})$$

that is, $\ker \theta = 0$.

3. On the intersection of irreducible components of characteristic varieties

3.1. Local systems, characteristic and resonance varieties Let M be a quasi-projective smooth complex algebraic variety. The rank-one local systems on M are parameterized by the algebraic group

$$\mathbb{T}(M) = \mathrm{Hom}(H_1(M), \mathbb{C}^*). \tag{3.1}$$

The connected component $\mathbb{T}^0(M)$ of the unit element $1 \in \mathbb{T}(M)$ is an algebraic torus, that is, it is isomorphic to $(\mathbb{C}^*)^n$, where $n \in \mathbb{N}$ is the first Betti number of M, that is, $n = b_1(M)$. It is clear that $\mathbb{T}^0(M) = \mathbb{T}(M)$ if and only if the integral homology group $H_1(M)$ is torsion free. For $\rho \in \mathbb{T}(M)$, we denote the corresponding local system on M by $\mathcal{L}_\rho$.

The computation of the twisted cohomology groups $H^j(M, \mathcal{L}_\rho)$ is one of the major problems in many areas of topology. To study these cohomology groups, one idea is to study the *characteristic varieties* defined by

$$\mathcal{V}^j_m(M) = \{\rho \in \mathbb{T}(M) \mid \dim H^j(M, \mathcal{L}_\rho) \geq m\}. \tag{3.2}$$

To simplify the notation, we set $\mathcal{V}_m(M) = \mathcal{V}^1_m(M)$. It is known that the following theorem holds, see Beauville [2] and Simpson [12] in the proper case and Arapura [1] in the quasi-projective case.

THEOREM 3.2. *The positive-dimensional irreducible components of $\mathcal{V}_m(M)$ are subtori in $\mathbb{T}(M)$ translated by elements of finite order. More precisely, for each positive-dimensional irreducible component W of $\mathcal{V}_m(M)$ we can write $W = \rho \cdot f^*(\mathbb{T}(S))$, where $f : M \to S$ is a surjective regular mapping to a curve S having a connected general fiber and $\rho \in \mathbb{T}(M)$ is a finite-order character.*

If $1 \in W$, then we can take $\rho = 1$ in the above equality. Let $T_1 W$ denote the tangent space to W at the identity 1 in such a case. The following theorem is taken from [6, Theorem 2, (b)].

THEOREM 3.3. *Let M be a quasi-projective smooth complex algebraic variety. Let W_1 and W_2 be two distinct irreducible components of the characteristic variety $\mathcal{V}_1(M)$ such that $1 \in W_1 \cap W_2$. Then $T_1 W_1 \cap T_1 W_2 = 0$.*

Note that any such tangent space $T_1W \subset H^1(M, \mathbb{C})$ is *rationally defined*, that is, there is a primitive lattice $L \subset H^1(M, \mathbb{Z})$ such that $T_1W = L \otimes_{\mathbb{Z}} \mathbb{C}$ under the identification

$$H^1(M, \mathbb{C}) = H^1(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Indeed, one can take L to be the primitive sublattice $f^*(H^1(S, \mathbb{Z}))$, in view of the functoriality of the exponential mapping

$$\exp : T_1\mathbb{T}(M) = H^1(M, \mathbb{C}) \to \mathbb{T}(M)$$

and of the following lemma.

LEMMA 3.4. *Let $f : M \to S$ be a surjective regular mapping to a curve S, having a connected general fiber. Then $f^*(H^1(S, \mathbb{Z}))$ is a primitive sublattice in $H^1(M, \mathbb{Z})$.*

PROOF. Under these conditions, it is well known that the morphism

$$f_* : H_1(M, \mathbb{Z}) \to H_1(S, \mathbb{Z})$$

is surjective. Let L_0 be a primitive sublattice in $H_1(M, \mathbb{Z})$ such that

$$H_1(M, \mathbb{Z}) = \ker f_* \oplus L_0.$$

Then $f^*(H^1(S, \mathbb{Z}))$ can be identified with the dual

$$L_0^\vee = \mathrm{Hom}(L_0, \mathbb{Z}) = \{u \in \mathrm{Hom}(H_1(M, \mathbb{Z}), \mathbb{Z});\ u\,|_{\ker f_*} = 0\}.$$

Moreover

$$H^1(M, \mathbb{Z}) = H_1(M, \mathbb{Z})^\vee = (\ker f_*)^\vee \oplus L_0^\vee,$$

which completes the proof of the claim. □

Applying Theorem 1.1 to this setting, we obtain the following corollary.

COROLLARY 3.5. *Let W_1 and W_2 be two distinct irreducible components of the characteristic variety $\mathcal{V}_1(M)$ such that $1 \in W_1 \cap W_2$. Let L_1 and L_2 be the primitive sublattices in $H^1(M, \mathbb{Z})$ associated to W_1 and W_2, respectively, by the above construction. Then there is a group isomorphism*

$$\mathrm{Tors}(H^1(M, \mathbb{Z})/(L_1 + L_2)) = W_1 \cap W_2.$$

REMARK 3.6. Let W_1 and W_2 be two distinct irreducible components of the characteristic variety $\mathcal{V}_1(M)$, where at least one of them, say W_1, is translated, that is, $1 \notin W_1$, and that meet at a point ρ. Then we may write $W_1 = \rho \cdot W_1'$ and $W_2 = \rho \cdot W_2'$, where W_j' are subtori in $\mathbb{T}(M)$.

Assume that $\dim W_j > 1$ for $j = 1, 2$ (the claim is obvious when one of the two components is one dimensional) and that M is a hyperplane arrangement complement. Then $T_1W_1' \cap T_1W_2' = 0$, since W_1' and W_2' are again two distinct irreducible components of the characteristic variety $\mathcal{V}_1(M)$, see [5]. Moreover, the tangent spaces T_1W_j' are rationally defined and the above corollary yields a set bijection

$$\mathrm{Tors}(H^1(M, \mathbb{Z})/(L_1' + L_2')) = W_1 \cap W_2$$

where L_j' is the primitive sublattice associated to W_j' by the above construction.

Note that any character in such an intersection $W_1 \cap W_2$ has finite order. Indeed, let $W_1 = \rho_1 \cdot W_1'$ and $W_2 = \rho_2 \cdot W_2'$, where ρ_1 and ρ_2 are finite-order characters, that is, $\rho_1^{m_1} = 1$ and $\rho_2^{m_2} = 1$. Then $\rho \in W_1 \cap W_2$ implies that $\rho = \rho_1 w_1 = \rho_2 w_2$, where $w_j \in W_j'$. Let $m = \mathrm{lcm}(m_1, m_2)$. Then $\rho^m = \rho_1^m w_1^m$ and $\rho^m = \rho_2^m w_2^m$ which implies that

$$\rho^m = w_1^m = w_1^m \Rightarrow \rho^m \in W_1' \cap W_2'$$

so, by Corollary 3.5, ρ^m is of finite order. Thus, ρ is a finite-order character.

A completely different proof of the finiteness of the intersection $W_1 \cap W_2$ of two distinct irreducible components of the first characteristic variety was given in [7].

Let $H^*(M, \mathbb{C})$ be the cohomology algebra of the variety M with $\mathbb{C}$-coefficients. Right multiplication (cup-product) by an element $z \in H^1(M, \mathbb{C})$ yields a cochain complex $(H^*(M, \mathbb{C}), \mu_z)$. The *resonance varieties* of M are the jumping loci for the cohomology of this complex, namely

$$\mathcal{R}_m^j(M) = \{z \in H^1(M, \mathbb{C}) \mid \dim H^j(H^*(M, \mathbb{C}), \mu_z) \geq m\}. \tag{3.3}$$

To simplify the notation, we set $\mathcal{R}_m(M) = \mathcal{R}_m^1(M)$.

One of the main results relating the characteristic and resonance varieties is the following theorem.

THEOREM 3.7. *Let M be a hypersurface arrangement complement. The map* $\exp :$ $H^1(M, \mathbb{C}) \to \mathbb{T}^0(M)$ *induces, for any m, $j \geq 1$,*

$$(\mathcal{R}_m^j(M), 0) \simeq (\mathcal{V}_m^j(M), 1).$$

This equality of germs implies that the resonance variety $\mathcal{R}_m^j(M)$ is exactly the tangent cone at one of the characteristic varieties $\mathcal{V}_m^j(M)$, a fact established by Cohen and Suciu [4] and which can also be derived from [8]. (See also [3, Theorem 3.7].) It was claimed by Libgober that this property holds for any smooth quasi-projective variety [9] but now there are counter-examples to this claim [6].

REMARK 3.8. According to Arapura [1, Theorem 1.1 in Section V], under the assumption that $H^1(M, \mathbb{Q})$ has a pure Hodge structure, the positive-dimensional irreducible components of all characteristic varieties $\mathcal{V}_m^j(M)$ are (translated) subtori. Our Theorem 1.1 applies to this more general setting as well. The major difference with the case of first characteristic varieties is that distinct components do not necessarily meet only at the origin. In the following we give an example, for which we are grateful to Professor A. Suciu.

Consider the central hyperplane arrangement in $\mathbb{C}^4$ defined by the equation

$$xyzw(x+y+z)(y-z+w)=0.$$

Then the corresponding resonance variety $\mathcal{R}_1^2(M)$ has two three-dimensional components, say E_1 and E_2, defined respectively by the ideals

$$I_1=(x_1+x_2+x_3+x_6,\, x_4,\, x_5)$$

and

$$I_2=(x_1,\, x_2+x_3+x_4+x_5,\, x_6).$$

These two components intersect in the line $D=(x_1, x_2+x_3, x_4, x_5, x_6)$. This implies that the irreducible components $W_1=\exp(E_1)$ and $W_2=\exp(E_2)$ of the characteristic variety $\mathcal{V}_1^2(M)$ intersect along the one-dimensional subtorus $\exp(D)$.

The fact that M has a simply connected compactification implies the following.

COROLLARY 3.9. *The irreducible components of $\mathcal{R}_1(M)$ are precisely the maximal linear subspaces $E \subset H^1(M,\mathbb{C})$, isotropic with respect to the cup product on M*

$$\cup : H^1(M,\mathbb{C}) \times H^1(M,\mathbb{C}) \to H^2(M,\mathbb{C})$$

PROOF. Let E be a component of $\mathcal{R}_1(M)$. By the above theorem there is a component W in $\mathcal{V}_1(M)$ such that $1 \in W$ and $T_1W=E$. By Arapura's results in [1] we can write $W=f_E^*(\mathbb{T}(S))$, where $f_E : M \to S$ is a regular mapping to a curve S. (Such a mapping f_E is said to be associated with the subspace E.) Since in our case S is rational, $T_1W=f_E^*(H^1(S,\mathbb{C}))$ is isotropic with respect to the cup product, since the cup product on $H^1(S,\mathbb{C})$ is trivial. Maximality of E comes from the fact that E is a component of $\mathcal{R}_1(M)$. The restriction $\dim E \geq 2$ comes from [1]. □

3.2. An example: the deleted B_3-arrangement Let $\mathcal{A}$ be the deleted B_3-arrangement which is obtained from the B_3 reflection arrangement by deleting the plane $x+y-z=0$. A defining polynomial for $\mathcal{A}$ is

$$Q=xyz(x-y)(x-z)(y-z)(x-y-z)(x-y+z).$$

The decone $d\mathcal{A}$ is obtained by setting $z=1$. Let

$$L_1:\ell_1=x=0,\quad L_2:\ell_2=y=0,\quad L_3:\ell_3=x-y=0,\quad L_4:\ell_4=x-1=0,$$
$$L_5:\ell_5=y-1=0,\quad L_6:\ell_6=x-y-1=0,\quad L_7:\ell_7=x-y+1=0$$

be the lines of the associated affine arrangement in $\mathbb{A}^2$. Let $L_8:\ell_8=z=0$ be the line at infinity and let M be the complement of $d\mathcal{A}$ in $\mathbb{A}^2$. The resonance variety $R_1(d\mathcal{A})$

has 12 irreducible components of dimension two and three. These components E and their associated maps $f_E : M \to S$ are given below, see [13, 14]. Denote by $e_1, \ldots, e_7$ the $\mathbb{Z}$-basis of $H^1(M, \mathbb{Z})$ given by

$$e_j = \frac{1}{2\pi i}\frac{d\ell_j}{\ell_j},$$

see [11]. Then each of the components E is the $\mathbb{C}$-vector space spanned by a primitive lattice denoted by E^0, that is, $E = E^0 \otimes_{\mathbb{Z}} \mathbb{C}$, so it is enough in each case to indicate a $\mathbb{Z}$-basis of E^0.

(1) The local components There are seven local components, corresponding to six triple points and one quadruple point.

(i) For the triple $L_1 \cap L_2 \cap L_3$,

$$E_1^0 = \langle e_1 - e_3, e_2 - e_3\rangle \quad \text{and} \quad f_{E_1}(x, y) = \frac{x}{y},$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

(ii) For the triple $L_3 \cap L_4 \cap L_5$,

$$E_2^0 = \langle e_4 - e_5, e_4 - e_3\rangle \quad \text{and} \quad f_{E_2}(x, y) = \frac{x-1}{y-1},$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

(iii) For the triple $L_2 \cap L_4 \cap L_6$,

$$E_3^0 = \langle e_4 - e_2, e_6 - e_2\rangle \quad \text{and} \quad f_{E_3}(x, y) = \frac{x-1}{y},$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

(iv) For the triple $L_1 \cap L_5 \cap L_7$,

$$E_4^0 = \langle e_1 - e_7, e_5 - e_1\rangle \quad \text{and} \quad f_{E_4}(x, y) = \frac{x}{y-1},$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

(v) For the triple $L_1 \cap L_4 \cap L_8$,

$$E_5^0 = \langle e_1, e_4\rangle \quad \text{and} \quad f_{E_6}(x, y) = x,$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

(vi) For the triple $L_2 \cap L_5 \cap L_8$,

$$E_6^0 = \langle e_5, e_2\rangle \quad \text{and} \quad f_{E_7}(x, y) = y,$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

(vii) For the quadruple $L_3 \cap L_6 \cap L_7 \cap L_8$,

$$E_7^0 = \langle e_3, e_6, e_7\rangle \quad \text{and} \quad f_{E_5}(x, y) = x - y,$$

where $S = \mathbb{C} \setminus \{0, \pm 1\}$.

(2) The non-local components There are five non-local components, corresponding to braid subarrangements.

(viii) For $X = (L_1L_6|L_3L_4|L_2L_8)$,

$$E_8^0 = \langle e_1 - e_3 - e_4 + e_6, e_2 - e_3 - e_4\rangle \quad \text{and} \quad f_{E_8}(x, y) = \frac{x(x-y-1)}{(x-y)(x-1)},$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

(ix) For $Y = (L_4L_8|L_2L_3|L_5L_6)$,

$$E_9^0 = \langle -e_2 - e_3 + e_5 + e_6, e_2 + e_3 - e_4\rangle \quad \text{and} \quad f_{E_9}(x, y) = \frac{x-1}{y(x-y)},$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

(x) For $Z = (L_1L_5|L_2L_4|L_3L_8)$,

$$E_{10}^0 = \langle e_1 - e_2 - e_4 + e_5, e_2 - e_3 + e_4\rangle \quad \text{and} \quad f_{E_{10}}(x, y) = \frac{x(y-1)}{y(x-1)},$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

(xi) For $W = (L_1L_3|L_4L_7|L_5L_8)$,

$$E_{11}^0 = \langle e_1 + e_3 - e_5, e_5 - e_7 - e_4\rangle \quad \text{and} \quad f_{E_{11}}(x, y) = \frac{x(x-y)}{(x-y+1)(x-1)},$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

(xii) For $V = (L_1L_8|L_2L_7|L_3L_5)$,

$$E_{12}^0 = \langle e_1 - e_2 - e_7, e_3 + e_5 - e_2 - e_7\rangle \quad \text{and} \quad f_{E_{12}}(x, y) = \frac{x}{y(x-y+1)},$$

where $S = \mathbb{C} \setminus \{0, 1\}$.

One way to obtain these 12 irreducible components E_j is to compute the cup-product

$$H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \to H^2(M, \mathbb{C})$$

and then to use the computer program SINGULAR to list the irreducible components of the determinantal variety corresponding to $R_1(d\mathcal{A})$; for details see [10].

For each f_E in the list above we can use the method described in [5] to show that there is no translated component in $\mathcal{V}_1(d\mathcal{A})$ associated to such an f_E.

It was discovered by A. Suciu (again by using computer computations) that $\mathcal{V}_1(d\mathcal{A})$ has one one-dimensional translated component W associated with the mapping $f : M \to \mathbb{C}^*$ defined in affine coordinates as

$$f(x, y) = \frac{x(y-1)(x-y-1)^2}{(x-1)y(x-y+1)^2}$$

and with

$$\rho_W = (1, -1, -1, -1, 1, 1, 1) \in (\mathbb{C}^*)^7.$$

(See [13, 14].) In other words,

$$W = \rho_W \cdot \{(t, t^{-1}, 1, t^{-1}, t, t^2, t^{-2}) \mid t \in \mathbb{C}^*\}.$$

Let V_i be the component of $\mathcal{V}_1(d\mathcal{A})$ corresponding to each E_i for $i = 1, \ldots, 12$, that is, $V_i = \exp(E_i)$. Then it is known [13, 14] that

$$W \cap V_8 \cap V_9 \cap V_{10} = \rho_W$$

and

$$W \cap V_{10} \cap V_{11} \cap V_{12} = \rho'_W,$$

where

$$\rho'_W = (-1, 1, -1, 1, -1, 1, 1) \in (\mathbb{C}^*)^7.$$

Since these results were obtained as a result of computer computations, it is useful to provide a direct proof based on Corollary 3.5.

Let $A = E_8^0$ and $B = E_9^0$ be the primitive lattices in $H^1(M, \mathbb{Z})$ introduced above and apply the construction explained in Section 2 to them with $L = H^1(M, \mathbb{Z})$. Here $n = 7$, $a = b = 2$. The basis $e'_1, \ldots, e'_5$ can be chosen as given by the following equivalence classes

$$e'_1 = [e_1 - e_3 - e_4 + e_6], \quad e'_2 = [e_3], \quad e'_3 = [e_2], \quad e'_4 = [e_5], \quad e'_5 = [e_7].$$

Then $m = 1$ and $d_2 = 2$. Let $f_1 = e_2 + e_3 - e_5 - e_6$ and $f_2 = e_4 - e_2 - e_3$. Then

$$\mathcal{B} = \{e_1 - e_3 - e_4 + e_6, e_3, e_2, e_5, e_7, f_1, f_2\}$$

is a $\mathbb{Z}$-basis of L (the coefficient matrix is unimodular) and we can take

$$g_1 = e_1 - e_3 - e_4 + e_6 \quad \text{and} \quad g_2 = 2e_3 + f_2.$$

Therefore

$$\mathrm{Tors}(H^1(M, \mathbb{Z})/(E_8^0 + E_9^0)) = \mathbb{Z}_2.$$

Now, by the morphism $\theta : \mathbb{Z}_2 \to V_8 \cap V_9$ used in Section 2,

$$\hat{1} \mapsto \exp_L(\tfrac{1}{2}(g_2)) = \exp_L(\tfrac{1}{2}(-e_2 + e_3 + e_4)) = (1, -1, -1, -1, 1, 1, 1) = \rho_W.$$

By Corollary 3.5, it follows that

$$V_8 \cap V_9 = \theta(\mathbb{Z}_2) = \{1, \rho_W\}.$$

In exactly the same way one can show that $V_8 \cap V_{10} = V_9 \cap V_{10} = \{1, \rho_W\}$. Since clearly $\rho_W \in W$, it follows that the four components V_8, V_9, V_{10} and W meet exactly in one point.

Similarly one can show that $W \cap V_{10} \cap V_{11} \cap V_{12} = \rho'_W$ and that all of the other intersections of two irreducible components are trivial.

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