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Transversal intersection formula for compacta

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Abstract

The main purpose of this paper is to present a unified treatment of the formula for dimension of the transversal intersection of compacta in Euclidean spaces. A new contribution is the proof of inequality $\dim(X \cap Y) \geqslant \dim(X \times Y) - n$ for transversally intersecting compacta $X, Y \subset \mathbb{R}^n$, based on a correct interpretation of the classical Čogošvili theorem. Also included is a short summary of a new direction of dimension theory, called extension theory, which is needed for the proof. © 1998 Elsevier Science B.V.

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1. Introduction

A remarkable progress in dimension theory observed in the last decade was stimulated by the discovery of a new geometrical phenomenon, namely that the formula

$$\dim(P \cap Q) = \dim P + \dim Q - n$$

for dimension of the transversal (i.e., general position) intersection of two polyhedra in $\mathbb{R}^n$ does not generalize to the class of compacta. As a consequence, a new branch

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of dimension theory, the so-called *extensional dimension*, arose in an attempt to find a correct version of the formula above for this, more general class of spaces. The expected formula for compacta $X, Y \subset \mathbb{R}^n$ which intersect transversally looks quite natural:

$$\dim(X \cap Y) = \dim(X \times Y) - n.$$

The main part of the formula, the inequality $\dim(X \cap Y) \leqslant \dim(X \times Y) - n$ has been proved by now for all compacta [11], except for codimension 2. The proof strongly relies on the theory of extensional dimension.

In this article we survey the history of this formula and we describe the structure of its proof which has been dispersed in several articles. We have also included a new result, namely a proof of the inequality

$$\dim(X \cap Y) \geqslant \dim(X \times Y) - n.$$

As it follows from the intersection formula, the difference between compacta and polyhedra concerning their intersections is due to the well-known "nonlogarithmic" behaviour of the dimension theory on the class of compacta [2,28].

In particular, by a theorem of Bockstein, for every n-dimensional compactum X with $\dim(X \times X) < 2\dim X$, the dimension of its transversal self-intersections in $\mathbb{R}^{2n}$ would have to be -1. This implies the nonstability of its self-intersections and consequently the density of the subspace of all embeddings $\mathcal{E}(X, \mathbb{R}^{2n})$ in the space $C(X, \mathbb{R}^{2n})$ of all maps of X into $\mathbb{R}^{2n}$, by the standard Pontryagin–Nöbeling argument.

The history of the transversal intersection formula for compacta begins with the following question due to Ancel [25]: *Does there exist an n-dimensional compactum X such that every mapping $f: X \to \mathbb{R}^{2n}$ is approximable by embeddings?*

McCullough and Rubin first announced a negative answer in [25]. Soon thereafter Krasinkiewicz found a gap in their argument and also constructed an example of "disjoint membranes" (later simplified by Lorentz, cf. [23]), thereby disproving the key lemma of [25]. Subsequently, Karno and Krasinkiewicz [21] and, independently, McCullough and Rubin [26], using the new concept of "disjoint membranes", constructed an example which provided a positive answer to the Ancel question.

Independent efforts by Krasinkiewicz [22], Spież [30,31], and these authors [14,15,17] resulted in the following theorem on the *approximations by embeddings*, which represents a generalization of the Pontryagin–Nöbeling theorem, combined with its converse:

Theorem 1.1. *For every integer $n \geqslant 0$ and every compactum X the following assertions are equivalent:*
(1) $\dim(X \times X) < n$; *and*
(2) *the subspace $\mathcal{E}(X, \mathbb{R}^n)$ of all embeddings is dense in the space $C(X, \mathbb{R}^n)$ of all continuous mappings of X into $\mathbb{R}^n$.*

The verification of the Pontryagin–Nöbeling direction (i.e., that (1) $\Rightarrow$ (2)) represents the more difficult part of the proof of Theorem 1.1. It relies on algebra, the Alexander duality and some special tricks in dimension 4. This part was independently proved by Spież [30,31] and these authors [14,15,17]. The converse implication was proved independently by Krasinkiewicz [22] and these authors [14] and turned out to be shorter and more geometric.

An interesting history is connected with an earlier idea of the proof of the (2) $\Rightarrow$ (1) part of Theorem 1.1, proposed earlier in [17], which was based on a classical theorem of Čogošvili [3] from the late 1930s. Since the logarithmic law holds for the dimension of the intersection of a compactum with a polyhedron, the intersections of compacta and polyhedra obey the polyhedral intersection formula. Already in 1928 Aleksandrov proved the following theorem:

Theorem 1.2 (P.S. Aleksandrov). *A compactum $X \subset \mathbb{R}^n$ has dimension $\dim X \leqslant k$ if and only if it is removable from every $(n-k-1)$-dimensional rectilinear polyhedron in $\mathbb{R}^n$.*

Recall that X removable from Y means the existence of arbitrary small ε-translations of X, where $\varepsilon > 0$, whose images miss Y. This theorem was studied already in 1938, by Čogošvili [3], in the "if" direction. He claimed that already the removability from every $(n-k-1)$-dimensional plane in $\mathbb{R}^n$ suffices to imply the inequality $\dim X \leqslant k$. For several years this was considered a classical theorem of dimension theory.

In [17] the easy part of Theorem 1.1 was deduced from this Čogošvili theorem. However, in 1989, Dranishnikov and Daverman found a gap in Čogošvili's proof. About the same time Engelking announced [20] that Pol had also found the same error in [3].

Attempts to prove or disprove Čogošvili's assertion led to several interesting papers—by Ancel and Dobrowolski [1], Dobrowolski et al. [4], Levin and Sternfeld [24], and Sternfeld [35], which combine the techniques arising from the work on Hilbert's 13th problem [34] with the theory of Bing atomic continua [24]. Finally, in 1995, a counterexample to the Čogošvili theorem was constructed in [12]:

Theorem 1.3 (A.N. Dranishnikov). *There exists a 2-dimensional compactum in $\mathbb{R}^4$ which is removable from every 2-dimensional plane.*

In Section 4 we present a weak, but correct version of Čogošvili's theorem which differs from the original only by one word—instead of removability one puts stable removability. This weak version is nevertheless sufficient to guarantee the correctness of the proof in [17]. The proof of the easier part of the transversal intersection formula (i.e., $\dim(X \cap Y) \geqslant \dim(X \times Y) - n$), presented in Section 4 is based on a generalization of this corrected Čogošvili theorem (see also [34]).

After the complete solution of Ancel's problem, provided by Theorem 1.1, the interest shifted to proving that the answer to the following *mapping intersection problem* is affirmative (recall that mappings f, g are said to be *intersecting unstably* if they have arbitrary close approximations with disjoint images).

Problem 1.4. Does every pair of maps $f : X \to \mathbb{R}^n$ and $g : Y \to \mathbb{R}^n$ of compacta X and Y into $\mathbb{R}^n$, such that $\dim(X \times Y) < n$, have an unstable intersection?

A positive solution of this problem for the case of compacta of complementary dimension (i.e., $\dim X + \dim Y = n$) was given independently in [14] and [32]. Next, the *metastable case*, i.e., when $2\dim X + \dim Y \leqslant 2n - 2$, was solved. The proof of this case, given by Spież, Segal and Toruńczyk [29,33], relies on Weber's paper [36] on isotopy of polyhedra and is very geometric. In particular, a theorem on disjoining of compacta was proved in [33] via ambient isotopies, which did not overlap by the subsequent developments. Another proof was obtained by these authors at about the same time [15].

An independent, different solution of the metastable case of Problem 1.4 was presented by Dranishnikov [6,7,9]. We refer the reader to our earlier paper [15] where the state of affairs after the solution of the metastable case is presented in greater detail.

The third part of history of Problem 1.4 begins with the Negligibility criterion proved in [6–8]. A subset $Y \subset \mathbb{R}^n$ is said to be *negligible* with respect to a compactum X if every mapping of X into $\mathbb{R}^n$ is removable from Y (i.e., has arbitrary close approximations whose images miss Y). Afterwards, the first solution of the easier part of Problem 1.4 appeared in [7,8]. The most difficult ingredient of this solution can be summarized as follows:

Theorem 1.5 (A.N. Dranishnikov). *Every tame codimension three compactum $Y \subset \mathbb{R}^n$, is negligible with respect to every compactum X such that* $\dim(X \times Y) < n$.

The next important step was made in [16], where the mapping intersection problem was reduced to the *problem of realization of dimension types*. The mapping intersection problem was split into two parts: the *approximation problem* and the *subsets intersection problem*. The second one, which differs from general mapping intersection problem only by one extra assumption (namely, that X and Y are assumed to lie in $\mathbb{R}^n$), was completely solved in [16].

Moreover, this paper introduced the notions of the *dimension type* and the *dimension complement*, and formulated *approximation* and *embedding problems* for cohomological dimension. Finally, the mapping intersection problem was solved in [10] via the realization of dimension types.

The hard part of transversal intersection formula, the inequality $\dim(X \cap Y) \leqslant \dim(X \times Y) - n$, was proved soon thereafter in [11]. To formulate it let us generalize the concepts of removability and negligibility.

Definition 1.6. We say that a mapping $f : X \to \mathbb{R}^n$ of a compactum is *k-removable* from a subset $Y \subset \mathbb{R}^n$, $k \leqslant n$, if it has an arbitrary close approximation $f' : X \to \mathbb{R}^n$ which satisfies the inequality:

$$\dim\big(f'(X) \cap Y\big) < k.$$

A subset $X \subset \mathbb{R}^n$ is said to be *k-removable* from another subset $Y \subset \mathbb{R}^n$ if the inclusion $X \subset \mathbb{R}^n$ is k-removable from Y. A subset $Y \subset \mathbb{R}^n$ is said to be *k-negligible* with respect to a compactum X if every mapping of X into $\mathbb{R}^n$ is k-removable from Y.

Clearly, for $k \leqslant 0$, these notions coincide with *removability* and *negligibility* as defined before.

Theorem 1.7 (A.N. Dranishnikov). *Every tame compact subset $Y \subset \mathbb{R}^n$ of dimension* $\dim Y < n - 2$ *is k-negligible with respect to any compactum X such that*

$$\dim(X \times Y) < n + k.$$

In the present paper we prove the following converse to this theorem:

Theorem 1.8. *If a compact subset $Y \subset \mathbb{R}^n$ is k-negligible with respect to a compactum X then*

$$\dim(X \times Y) < n + k.$$

Both results are then combined to yield the *transversal intersection formula*:

$$\dim(X \cap Y) = \dim(X \times Y) - n$$

for the dimension of the intersection of compacta in *general position*.

Let us define more precisely the meaning of a *transversal intersection* of compacta. To speak of general position one has to deal with classes of mappings. Let $\mathcal{F}$ and $\mathcal{G}$ be two classes of mappings of compacta X and Y to the Euclidean space $\mathbb{R}^n$, respectively. In this paper, *for almost all* means that the complement is of the *first Baire category*.

Definition 1.9. Mappings $f \in \mathcal{F}$ and $g \in \mathcal{G}$ are said to be *intersecting transversally* with respect to the classes $\mathcal{F}$ and $\mathcal{G}$ if

$$\dim\left(f'(X) \cap g'(Y)\right) = \dim\left(f(X) \cap g(Y)\right) \geqslant 0,$$

for almost all mappings $f' \in \mathcal{F}$ and $g' \in \mathcal{G}$, which are sufficiently close to f and g, respectively.

So according to this definition only *stably* intersecting mappings can intersect transversally. The default classes are $\mathcal{F} = \mathcal{C}(X, \mathbb{R}^n)$ and $\mathcal{G} = \mathcal{C}(Y, \mathbb{R}^n)$, i.e., the classes of all continuous mappings of X and Y into $\mathbb{R}^n$. So *transversality* with no mention of the classes means transversality with respect to these two classes.

We call subsets $X, Y \subset \mathbb{R}^n$ *transversally intersecting*, and denote this by $X \pitchfork Y$, if their inclusions into $\mathbb{R}^n$ are transversally intersecting mappings. The dimension of the intersection of two polyhedra is the same for every transversal intersections due to their *dimensional homogeneity* (i.e., all nonempty open sets are of the same dimension).

We are now ready to present the *transversal intersection* formula:

Theorem 1.10. *Let X and Y be compacta in $\mathbb{R}^n$ with dimensionally homogeneous product $X \times Y$ and such that $X \pitchfork Y$ and* $\dim Y < n - 2$. *Then*

$$\dim(X \cap Y) = \dim(X \times Y) - n.$$

In the case $\dim(X \times Y) < n$, Theorem 1.10 implies the nonexistence of transversal intersections. Since transversal pairs of mappings are dense (see Section 4) we obtain in this case unstability of intersections for every pair of mappings. So, in particular, the transversal intersection formula of Theorem 1.10 yields the solution of the mapping intersection problem for subsets.

Another result concerns the notion of *relative* transversality.

Definition 1.11. A compactum $X \subset \mathbb{R}^n$ is said to be *relatively* transversal to a compactum $Y \subset \mathbb{R}^n$, and this is denoted by $X \pitchfork Y \operatorname{rel} Y$, if the inclusions $X, Y \subset \mathbb{R}^n$ intersect $\mathcal{F}, \mathcal{G}$-transversally, for $\mathcal{F} = \mathcal{C}(X, \mathbb{R}^n)$ and $\mathcal{G} = \{Y \subset \mathbb{R}^n\}$.

The following result is the corresponding *relative* transversal intersection formula:

Theorem 1.12. *Let X and Y be compacta in $\mathbb{R}^n$ with dimensionally homogeneous product $X \times Y$, such that $X \pitchfork Y \operatorname{rel} Y$, Y is tame in $\mathbb{R}^n$, and* $\dim Y < n - 2$. *Then*

$$\dim(X \cap Y) = \dim(X \times Y) - n.$$

The main unresolved problem in this area is to prove the Negligibility criterion for codimension 2. This would yield the transversal intersection formula without any dimensional restrictions. Another interesting problem is to prove the *isotopical* transversal intersection formula, where by *isotopical transversality* we mean transversality with respect to the class of all autohomeomorphisms of $\mathbb{R}^n$. In this case even the codimension 3 case is unsolved. However, the metastable case follows by Spież and Toruńczyk [33].

We conclude this introduction with some comments on the organization of the paper. Section 2 represents a short exposition of the extensional dimension theory. The main results are formulated without proofs and some simple consequences are derived which are needed in the sequel. Section 3 represents an updated version of [16]. In particular, it contains a geometric proof of the realization problem for dimension types in codimension 2. The main result of this section is Approximation Theorem 3.14. It is proved only in codimension 3 and with additional restriction of the so-called simply connected dimensional type. Finally, Section 4 contains a correct version of Čogošvili's theorem and the proof of our main results, i.e., the transversal intersection formula of Theorem 1.10.

We use this opportunity to point out an error in the proof of the Reduction theorem in [16] which, in particular, claims the equivalence of the realization and approximation problems for all compacta of codimension 3, without any restrictions such as, e.g., the simple connectedness of dimension type. This additional restriction arises in connection with the current status of the Negligibility criterion, which remains unproved in codimension 2. Moreover, in the proof of Reduction theorem in [16] it is applied not only to the compactum itself but also to its dimensional complement. So the dimensional complement must also be of codimension 3, implying the simple connectedness of dimensional type of X.

2. Extensional dimension theory

By $\mathcal{C}$ we shall denote the class of all finite-dimensional compact metric spaces and by $\mathcal{S}$ the class of all countable CW-complexes. We recall that a CW-complex L is said to be *simple* if the action of the fundamental group $\pi_1(L)$ on higher-dimensional homotopy groups of L is trivial. In particular, it implies that $\pi_1(L)$ is abelian. Mostly we are going to work with simple or even simply connected L.

Kuratowski notation $X\tau L$ means that for every partial continuous map $\phi: A \to L$ given on a closed subset $A \subset X$ there is an extension $\overline{\phi}: X \to L$. By the Aleksandrov–Hurewicz theorem, the property $X\tau S^n$ is equivalent to the property $\dim X \leqslant n$.

We define a partial ordering on $\mathcal{S}$ by the following rule: $L \leqslant K$ if $X\tau L$ implies $X\tau K$, for every space $X \in \mathcal{C}$. Thus, $S^n \leqslant S^m$, for every $n \leqslant m$. This partial order defines an equivalence relation on $\mathcal{S}: L \overset{e}{\sim} K$, L is extensionally equivalent to K if they have the same sets of subordinate complexes. The homotopy extension theorem implies that a homotopy equivalence $L \overset{h}{\sim} K$ implies the extensional equivalence $L \overset{e}{\sim} K$. The converse is not true, take, for example, $S^n \overset{e}{\sim} S^n \vee S^{n+1}$. We call a class of extensionally equivalent complexes by *extension type* and we denote by $[L]$ the class of $L \in \mathcal{S}$. Let $\mathcal{E}$ be the set of all extension types. Then $\mathcal{E}$ inherits the partial order from $\mathcal{S}$.

Definition 2.1. We say that X is at most *L-dimensional* (or just *L-dimensional*) if the property $X\tau L$ holds.

Definition 2.2. The *extensional dimension* of $X \in \mathcal{C}$, $\operatorname{Dim} X$, is the minimal element $[L] \in \mathcal{E}$ such that X is L-dimensional.

Theorem 2.3.
(1) *For every $X \in \mathcal{C}$, the class $\operatorname{Dim} X \in \mathcal{E}$ is well-defined.*
(2) *For every $[L] \in \mathcal{E}$, there exists a countably dimensional (i.e., a countable union of finite-dimensional compacta) compactum X with $\operatorname{Dim} X = [L]$.*

To distinguish $\operatorname{Dim} X$ from the covering dimension $\dim X$, we call it the *extensional dimension* of X. We recall that the *cohomological dimension* $\dim_G X$ of a compactum X over a group G is defined for an arbitrary abelian group G as follows: $\dim_G X \leqslant n$ if and only if $X\tau K(G,n)$, where $K(G,n)$ is the Eilenberg–MacLane complex. The Aleksandrov cohomological dimension theorem can now be written as the extensional equivalence:

$$S^n \overset{e}{\sim} K(\mathbb{Z},n).$$

The following is the *generalized* Aleksandrov theorem:

Theorem 2.4 [13]. *For every abelian group G, every integer $n > 1$ we have*

$$M(G,n) \overset{e}{\sim} K(G,n)$$

whereas for $n = 1$ we have $K(G,1) \leqslant M(G,1)$.

Here $M(G,n)$ is the Moore space. Note that $S^n = M(\mathbb{Z}, n)$.

Problem 2.5. Does the inequality $M(G,1) \leqslant K(G,1)$ also hold?

Next, we state the fundamental theorem of extension theory:

Theorem 2.6 [13]. *Let L be a simple complex and X a finite-dimensional compactum. Then the following assertions are equivalent*:

(1) $X\tau L$;
(2) *for every integer* $k \geqslant 1$, $\dim_{\pi_k(L)} X \leqslant k$; *and*
(3) *for every integer* $k \geqslant 1$, $\dim_{H_k(L)} X \leqslant k$.

Remark 2.7. The proof in [13] is given for the case of a 1-connected L. However, it works without changes also for simple L. Note also that the condition

$$\dim_{G_k} X \leqslant k, \quad \text{for every integer } k \geqslant 1,$$

can be reformulated as

$$\operatorname{Dim} X \leqslant \bigvee_{k=1}^{\infty} K(G_k, k).$$

The fundamental theorem of extension theory can be reformulated in view of Theorem 2.4 as follows:

Theorem 2.8. *For every simple complex L, the following extension equivalences hold*:

$$\begin{aligned} L &\overset{e}{\sim} \bigvee_i K(\pi_i(L), i) \overset{e}{\sim} \bigvee_i K(H_i(L), i) \\ &\overset{e}{\sim} \bigvee_i M(\pi_i(L), i) \overset{e}{\sim} \bigvee_i M(H_i(L), i), \end{aligned}$$

where $\bigvee_i$ can be replaced by $\overrightarrow{\prod_i}$.

In the 1950s Bockstein solved a long-standing open problem in cohomological dimension theory—he found a countable basis of abelian groups:

$$\sigma = \{\mathbb{Q}, \mathbb{Z}_{(p)}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}\}_{p \text{ prime}}$$

which enables one to compute the cohomological dimension of a space with respect to every other (abelian) group. This theory can be summarized in the following *generalized Bockstein* theorem:

Theorem 2.9 [13]. *For every simple complex L, there exist numbers $n_L(G)$, $G \in \sigma$, such that*

$$L \overset{e}{\sim} \bigvee_{G\in\sigma} K(G, n_L(G)) \overset{e}{\sim} \overrightarrow{\prod_{G\in\sigma}} K(G, n_L(G)).$$

Remark 2.10. Observe that the complex $\prod_{G\in\sigma}^{\rightarrow} K(G, n_L(G))$ is simple. We may assume that $n_L(G)$ satisfies the Bockstein inequalities:

$$n_L(\mathbb{Q}) \leqslant n_L(\mathbb{Z}_{(p)}), \quad n_L(\mathbb{Z}_p) \leqslant n_L(\mathbb{Z}_{(p)}), \quad \text{etc.}$$

Proof of Theorem 2.3. (1) Let $\bigvee_\alpha L_\alpha$ be the wedge of all $L_\alpha \in \mathcal{S}$ such that $X\tau L_\alpha$. It is clear that $X\tau(\bigvee_\alpha L_\alpha)$. By Theorem 2.9, we have a countable simple complex $L \overset{e}{\sim} (\bigvee_\alpha L_\alpha)$. It then follows that $\operatorname{Dim} X = [L] \in \mathcal{E}$.

(2) Note that the condition $X\tau(\bigvee_\sigma K(G, n_L(G)))$ is equivalent to the system of inequalities $\dim_G X \leqslant n_L(G)$, for every group $G \in \sigma$. Since $\{n_L(G)\}$ satisfies the Bockstein inequalities [5] there exists a compactum X with $\dim_G X = n_L(G)$, for every group $G \in \sigma$. Hence

$$\operatorname{Dim} X = \left[\bigvee_\sigma K(G, n_L(G))\right]. \qquad \square$$

Lemma 2.11. *For every compactum X of positive dimension,* $\operatorname{Dim}(X \times I)$ *is simply connected.*

Proof. Since every nonzerodimensional compactum contains a nondegenerate continuum it suffices to prove the simple connectedness of $\operatorname{Dim}(X \times I)$ in the case when X is connected. Consider the following subset S of the product $X \times Y$, presented by the union $(X \times S^0) \cup (\{x_0, x_1\} \times I)$, where $\{x_0, x_1\}$ is a pair of different points of X and $S^0 = \{0, 1\}$ is the boundary of I. Let f be the mapping of S onto $S^1 = \sum S^0$, defined so that $f(x, i) = i$ for $i = 0, 1$ and $f(x_i, t) = (i, t)$ for $0 < t < 1$ and $i = 0, 1$ (where 0 and 1 are the vertices of the suspension).

Suppose that a complex L is such that $(X \times I)\tau L$. Let $g : S^1 \to L$ be any loop. To prove its contractibility let us consider the composition gf. It is extendable over $X \times I$. So fix such an extension F. Without loss of generality one can assume X lying in the Hilbert cube Q and F is defined over the product $U \times I$ of some open neighborhood U in Q, where f is defined over $U \times S^0$ by the same formula $f(x, i) = i$. Let us choose an arc A in U, joining x_0 and x_1, so that the internal boundary of the 2-cell $\partial(A \times I)$ coincides with $(A \times S^0) \cup (\{x_0, x_1\} \times I)$. In this case one obtains a degree one map $f : \partial(A \times I) \to S^1$. And since gf (being extendable over a 2-cell) is null-homotopic, the same must be true for g. So the simple connectedness of L (and hence of $\operatorname{Dim}(X \times I)$) is thus proved. $\square$

Lemma 2.12. *If a complex L is simply connected, then it is extensionally equivalent to the suspension $L \overset{e}{\sim} \sum N$ for $N \in \mathcal{S}$.*

Proof. By Theorem 2.8,

$$L \overset{e}{\sim} \bigvee_{i>1} M(\pi_i(L), i) \cong \bigvee_{i>1} \left(\sum M(\pi_i(L), i-1)\right)$$
$$\cong \sum \left(\bigvee_{i>1} M(\pi_i(L), i-1)\right). \qquad \square$$

Theorem 2.13 [13]. *For every pair of compacta X and Y, the following inequality holds*:

$$\mathrm{Dim}(X \times Y) \leqslant \mathrm{Dim}\, X \wedge \mathrm{Dim}\, Y.$$

We will apply Theorem 2.13 for $Y = [0,1]$ and conclude that:

$$\mathrm{Dim}\,\big(X \times [0,1]\big) \leqslant \sum(\mathrm{Dim}\, X).$$

For this case the proof of Theorem 2.13 is easy and it can be found in [7]. The operation of smash product on extension types is well-defined [13] and the proof of this fact is based on the union and the decomposition theorems stated below:

Theorem 2.14 [19]. *Suppose that the compactum X can be written as $X = Z \cup Y$, where Z is K-dimensional and Y is M-dimensional. Then X is $(K * M)$-dimensional.*

Theorem 2.15 [10]. *Let $L = K * M$ and let X be L-dimensional. Then $X = Z \cup Y$, where Z is K-dimensional and Y is M-dimensional, and either Z or Y can be assumed to be F_σ.*

The last one is applied to prove the following lemma.

Lemma 2.16. *Suppose that X is a compactum with simply connected* $\mathrm{Dim}\, X$. *Then $X = (\bigcup_{i=1}^{\infty} Z_i) \cup X'$, where* $\mathrm{Dim}(X' \times I) \leqslant \mathrm{Dim}\, X$ *and* $\mathrm{Dim}\, Z_i = 0$, *for every $i \in \mathbb{N}$.*

Proof. Let $L \in \mathrm{Dim}\, X$. By Lemma 2.12 we have

$$L \overset{E}{\sim} \sum N \overset{h}{\sim} S^0 * N.$$

By Theorem 2.15, we have a decomposition $X = (\bigcup Z_i) \cup X'$, with $\dim Z_i = 0$, and $\mathrm{Dim}\, X' \leqslant n$. From Theorem 2.13 it then follows

$$\mathrm{Dim}(X' \times I) \leqslant \mathrm{Dim}\, X' \wedge S^1 \leqslant N \wedge S^1 = \sum N = L = \mathrm{Dim}\, X. \qquad \square$$

The two other important theorems of extension theory are the completion and the *countable union* theorems:

Theorem 2.17 [27]. *For every countable complex L and every L-dimensional separable metric space X, there exists an L-dimensional completion $\overline{X}$.*

Theorem 2.18 [8]. *Let $X = \bigcup_{i=1}^{\infty} X_i$ be a countable union of L-dimensional compacta. Then X is also L-dimensional.*

Corollary 2.19. *Let $U \subset X$ be an open subset of X and let both U and $X \setminus U$ be L-dimensional. Then X is also L-dimensional.*

The family of compacta in $\mathcal{C}$ having the extensional dimension $\mathrm{Dim}\, X$, will be denoted by $\mathrm{DIM}\, X$. We note that $\mathrm{Dim}\, X = \mathrm{Dim}\, Y$ if and only if $\mathrm{DIM}\, X = \mathrm{DIM}\, Y$.

In the sequel we shall need the following theorem on *test spaces*:

Theorem 2.20. *For every integer m and every abelian group G, there exists a compactum T_G^m of* $\dim T_G^m = m$ *such that for all compacta X of* $\dim X \leqslant m$ *we have*

$$\dim(X \times T_G^m) = m + \dim_G X.$$

Lemma 2.21. *If for every compactum Z,* $\dim(X \times Z) \leqslant \dim(Y \times Z)$, *then* $\operatorname{Dim} X \leqslant \operatorname{Dim} Y$.

Proof. Assume the contrary. Then by Theorem 2.9 there exists $G \in \sigma$ such that $\dim_G X \geqslant \dim_G Y$. Then

$$\dim(X \times T_G^m) = \dim_G X + m \leqslant \dim_G Y + m = \dim(Y \times T_G^m),$$

for a test space T_G^m of sufficiently large dimension. Contradiction. □

Theorem 2.22 [16]. *For every pair of compacta X and Y, the following assertions are equivalent*:

(1) $\operatorname{Dim} X = \operatorname{Dim} Y$;
(2) $\operatorname{DIM} X = \operatorname{DIM} Y$; *and*
(3) $\dim(X \times Z) = \dim(Y \times Z)$, *for every compactum Z.*

Let $\operatorname{Dim} X = [L]$. By $[L] + n$, $n \in \mathbb{Z}$, we denote the extensional type $\overrightarrow{\prod}_{G\in\sigma} K(G, n_L(G) + n)$, whenever this makes sense (i.e., when $n_L(G) + n > 0$, for all G). For $n > 0$,

$$[L] + n = [L \wedge S^n].$$

Lemma 2.23. *For every compactum X of dimension* $\dim X > 1$, *the following assertions are equivalent*:

(1) $\operatorname{Dim} X$ *is simply connected*;
(2) $\operatorname{Dim} X = \operatorname{Dim}(X \vee I^2)$;
(3) $\dim_G X > 1$, *for every abelian group G*; *and*
(4) *there exists a compactum X' such that* $\operatorname{Dim} X = \operatorname{Dim}(X' \times I)$.

Proof. (1) $\Rightarrow$ (2) First note that $\operatorname{Dim}(X \vee I^2) \geqslant \operatorname{Dim} X$. Since $\operatorname{Dim} X$ is simply connected, it follows that $I^2 \,\tau \operatorname{Dim} X$ and hence $\operatorname{Dim}(X \vee I^2) \leqslant \operatorname{Dim} X$.

(2) $\Rightarrow$ (3) Since I^2 is 2-dimensional for all coefficients.

(3) $\Rightarrow$ (4) $\operatorname{Dim} X = \overrightarrow{\prod}_{G\in\sigma} K(G, n(G))$ with $n(G) > 1$. Consider

$$L = \overrightarrow{\prod_{G\in\sigma}} K\bigl(G, n(G) - 1\bigr).$$

By Theorem 2.3, there exists X' with $\operatorname{Dim} X' = L$. Therefore, $\operatorname{Dim}(X' \times I) = \operatorname{Dim} X$.

(4) $\Rightarrow$ (1) It follows that $\dim X' > 0$, hence by Lemma 2.11, $\operatorname{Dim}(X' \times I)$ is simply connected. □

Lemma 2.24. *Every family $\{L_\alpha\} \subset \mathcal{E}$ has the supremum $L \in \mathcal{E}$.*

Proof. Consider $\mathcal{M} = \{M \in \mathcal{E} \mid L_\alpha \leqslant M$, for every $\alpha\}$. First observe that $M \neq \emptyset$ since $[pt] \in \mathcal{M}$. Consider $L = \overrightarrow{\prod}_{M\in\mathcal{M}} M$. Since

$$X\tau \overrightarrow{\prod_{M\in\mathcal{M}}} \Leftrightarrow X\tau M, \quad \text{for every } M \in \mathcal{M} \Rightarrow X\tau L_\alpha, \text{ for every } \alpha,$$

it follows that $L_\alpha \leqslant L$, for every α. Assume that $L_\alpha \leqslant L'$, for every α. Then

$$L \overset{e}{\sim} \bigvee_{M\in\mathcal{M}} M \leqslant L'$$

since $L' \in \mathcal{M}$. $\square$

We conclude by the *generalized* Hurewicz formula:

Theorem 2.25. *Let $f : X \to Y$ be a map between finite-dimensional compacta X and Y. Then* $\operatorname{Dim} X \leqslant \sup_{y\in Y} \operatorname{Dim}(Y \times f^{-1}(y))$. *In particular,* $\dim X \leqslant \sup_{y\in Y} \dim(Y \times f^{-1}(y))$.

Proof. (Since we are going to apply this theorem in the case of dimensionally full-valued compacta Y, we prove here only this particular case.) Let $\overline{Y}$ be such that $\operatorname{Dim} \overline{Y} = \sup_{y\in Y}\{\operatorname{Dim}(Y \times f^{-1}(y))\}$. Such $\overline{Y}$ exists because of Lemma 2.24 and Theorems 2.3 and 2.4. Consider a composition $\bar{f} : X \times Z \to Y$ of the projection $X \times Z \to X$ and the map f. Note that $\bar{f}^{-1}(y) = f^{-1}(y) \times Z$. By the classical Hurewicz theorem,

$$\dim(X \times Z) \leqslant \dim Y + \sup_{y\in Y} \dim(f^{-1}(y) \times Z) = \dim \overline{Y} \times Z.$$

Since the choice of compactum Z was arbitrary, Lemma 2.21 implies that $\operatorname{Dim} X \leqslant \operatorname{Dim} \overline{Y}$. $\square$

3. Approximation theorem

Definition 3.1. A compactum $X \subset \mathbb{R}^n$ is said to be *negligible* with respect to a compactum X or shortly *X-negligible* if the subspace $\{f \in C(X, \mathbb{R}^n) \mid f(X) \cap Y = \emptyset\}$ is dense in $C(X, \mathbb{R}^n)$.

We begin by proving the *Negligibility* criterion [6].

Theorem 3.2 [6]. *Let a compactum $Y \subset \mathbb{R}^n$ be tame,* $\dim Y \leqslant n - 3$. *Then Y is X-negligible if and only if* $\dim(X \times Y) < n$.

Proof. A compactum Y is negligible with respect to $X \Leftrightarrow X\tau(U \setminus Y)$, for every open ball $U \subset \mathbb{R}^n \Leftrightarrow$ (by Theorem 2.6) $\dim_{H_k(U\setminus Y)} X \leqslant k$, for every k and every $U \Leftrightarrow \dim_{H_c^{n-k-1}(Y\cap U)} X \leqslant k$, for every k and every $U \Leftrightarrow H_c^{k+1}(V; H^{n-k-1}(Y \cap U)) = 0$, for every k and for every open subset $V \Leftrightarrow \sum H_c^{k+1}(V; H_c^{n-k-1}(Y \cap U)) = H_c^n(V \times (U \cap Y); \mathbb{Z}) = 0$, for every k, every U, and every open $V \subset X \Leftrightarrow \dim_{\mathbb{Z}}(X \times Y) < n \Leftrightarrow \dim(X \times Y) < n$. $\square$

Remark. The easy part of this criterion, namely ($\Rightarrow$), has an elementary proof which can be found in Section 4. Also, conditions of tameness of Y and $\dim Y \leqslant n-3$ can be removed.

Next, we shall prove three useful lemmas: on *density of nonlowering* Dim *mappings* [16], on X-nets [18], and on *isotopies of* 0-*dim* compacta.

Lemma 3.3. *There exists a dense G_δ subset of $C(X, \mathbb{R}^n)$, consisting of maps $f : X \to \mathbb{R}^n$ which do not lower* $\operatorname{Dim} X$.

Lemma 3.4. *For every compactum $X \subset \mathbb{R}^n$ there exists a countable union $X^\sigma = \bigcup_{i=1}^\infty X_i$ of compacta $X_i \subset \mathbb{R}^n$ such that:*
(1) *for every i, $\operatorname{Dim} X = \operatorname{Dim} X_i$; and*
(2) *every compactum $Y \subset \mathbb{R}^n \setminus X^\sigma$ is X-negligible.*

Lemma 3.5. *Suppose that $\{Z_i\}_{i\in\mathbb{N}}$ is a countable family of 0-dimensional tame compacta in $\mathbb{R}^n$. Then for every $\varepsilon > 0$, there exists a homeomorphism $h_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$d(h_\varepsilon, \mathrm{id}_{\mathbb{R}^n}) < \varepsilon \quad \textit{and} \quad (\mathrm{pr} \circ h_\varepsilon)|_{\bigcup_{i=1}^\infty Z_i} : \bigcup_{i=1}^n Z_i \to \mathbb{R}$$

is one-to-one, where $\mathrm{pr} : \mathbb{R}^n \to \mathbb{R}$ *is the orthogonal projection.*

Proof. A subset of $\mathbb{R}^n$ is called *horizontal* if it is projected to a point by the orthogonal projection. Without losing generality we can assume that Z_i's are nested—just let $Z_i' = \bigcup_{j \leqslant i} Z_j$. Consider the space of all homeomorphisms $\mathcal{H}(\mathbb{R}^n, \mathbb{R}^n)$. First consider one Z_i. Call it Z. We must establish control. So we cover Z by cells without intersecting interiors,

$$Z = \coprod_{i=1}^n \operatorname{Int} C_i, \qquad \operatorname{diam} C_i < \varepsilon, \qquad C_i \cap C_j = \emptyset \quad (i \neq j)$$

(here we need the tameness hypothesis). Take one of C_i, say C_1, and the horizontal Cantor set inside it. Get a homeomorphism which is the identity on ∂C_1 and maps $Z \cap C_1$ onto the Cantor set. Extend over $\mathbb{R}^n$. Now do this for all other C_i's in such a way that corresponding horizontal sets are chosen on the different levels, i.e., have different (hence disjoint) images under pr. Denote the constructed homeomorphism of $\mathbb{R}^n$ by h_ε'. This homeomorphism does not belong to the subspace $\mathcal{H}_\varepsilon^1$, which is defined below.

Let $Z_i' = \bigcup_{j \leqslant i} Z_j$. Look at $\mathcal{H}(\mathbb{R}^n, \mathbb{R}^n)$. Look at the subspace $\mathcal{H}_\varepsilon^i$, the space of all h such that there exists a point t such that $\mathrm{pr} \circ (h|_{Z_i'}(t))$ has $\operatorname{diam} h^{-1} \geqslant \varepsilon$.

Claim. *$\mathcal{H}^i_\varepsilon$ is closed.*

Proof. Follows immediately since Z'_i is compact by construction. To see this just take a converging sequence and find the limit. □

Claim. *$\mathcal{H}^i_\varepsilon$ is nowhere dense.*

Proof. Let h be any homeomorphism of $\mathbb{R}^n$. Apply the above construction of h'_ε to $h(Z'_i)$ instead of Z. Then the composition $h'_\varepsilon \circ h$ is ε-close to h and does not belong to $\mathcal{H}^i_\varepsilon$.

It now follows by the Baire category theorem that $\bigcup_{\substack{i\geqslant 1\\ n\geqslant 1}} \mathcal{H}^i_{1/n}$ is nowhere dense. Thus take any ε-close to identity homeomorphism from $\mathcal{H} \setminus \bigcup_{\substack{i\geqslant 1\\ n\geqslant 1}} \mathcal{H}^i_{1/n}$. This is then the desired h_ε. □

Next, we shall prove the lemmas on *transversal hyperplane sections* and the *embedding of supremum.*

Lemma 3.6. *Suppose that $X \subset \mathbb{R}^n$ is a tame compact subset such that* $\operatorname{Dim} X$ *is simply connected. Then for every $\varepsilon > 0$, every orthogonal projection* $\mathrm{pr}:\mathbb{R}^n \to \mathbb{R}$ *there exists a homeomorphism $h_\varepsilon:\mathbb{R}^n \to \mathbb{R}^n$ such that h_ε is ε-close to the identity and such that* $\operatorname{Dim}((h_\varepsilon(X) \cap \mathrm{pr}^{-1}(t)) \times I) \leqslant \operatorname{Dim} X$, *for every $t \in \mathbb{R}$.*

Proof. Write $X = X' \cup (\bigcup Z_i)$ using Lemma 2.16, so that $\operatorname{Dim}(X' \times I) \leqslant \operatorname{Dim} X$ and $\dim Z_i = 0$. Next, since X is tame it follows that also Z_i are tame, so by Lemma 3.5 there exists h_ε such that the cardinality $|h_\varepsilon(\bigcup Z_i) \cap \mathrm{pr}^{-1}(t)| \leqslant 1$. Finally, we must show that this h_ε is the desired one. By Corollary 2.19,

$$\operatorname{Dim}\left(h_\varepsilon(X) \cap \mathrm{pr}^{-1}(t)\right) \quad \text{and} \quad \operatorname{Dim}\left(h_\varepsilon(X') \times I\right) \leqslant \operatorname{Dim} X \quad \text{and}$$
$$h_\varepsilon(X) \cap \mathrm{pr}^{-1}(t) = \left(h_\varepsilon(X') \cap \mathrm{pr}^{-1}(t)\right) \cup \left(\left(\bigcup Z_i\right) \cap \mathrm{pr}^{-1}(t)\right),$$

since $(\bigcup Z_i) \cap \mathrm{pr}^{-1}(t)$ is either empty or a point. Thus,

$$\operatorname{Dim}\left(h_\varepsilon(X) \cap \mathrm{pr}^{-1}(t)\right) = \operatorname{Dim}\left(h_\varepsilon(X') \cap \mathrm{pr}^{-1}(t)\right) \leqslant \operatorname{Dim} h_\varepsilon(X') \leqslant \operatorname{Dim} X - 1$$

and hence

$$\operatorname{Dim}\left(h_\varepsilon(X) \cap \mathrm{pr}^{-1}(t)\right) \leqslant \operatorname{Dim} X - 1. \qquad \square$$

Lemma 3.7. *Let $\{X_t\}_{t\in T}$ be a family of compacta in $\mathbb{R}^n$. Then there exists a compactum $\overline{X} \subset \mathbb{R}^n$ such that* $\operatorname{Dim} \overline{X} = \sup_{t\in T} \operatorname{Dim} X_t$.

Proof. By Proposition 2.4 from [16], there exists a countable subset $S \subset T$ such that $\sup_{t\in T} \operatorname{Dim} X_t = \sup_{t\in S} \operatorname{Dim} X_t$. Consider a sequence of balls $\{B_k\} \subset \mathbb{R}^n$ converging to a point $x_0 \in \mathbb{R}^n$. Let $\varphi:\mathbb{N} \to S$ be a one-to-one mapping of integers. In every B_k fix a compactum X_k, homeomorphic to $X_{\varphi(k)}$. Then $\overline{X} = \bigcup_{k\in\mathbb{N}} X_k \cup \{x_0\}$ is a compact space of the type

$$\operatorname{Dim}\overline{X} = \sup_{k\in\mathbb{N}} \operatorname{Dim} X_k = \sup_{t\in S} \operatorname{Dim} X_t = \sup_{t\in T} \operatorname{Dim} X_t. \qquad \square$$

Definition 3.8. For every compactum X of dimension n and every integer $k > n$, denote

$$k - \operatorname{Dim} X = \sup\bigl\{ \operatorname{Dim} Y \mid \dim(Y \times X) = k \bigr\}.$$

Compactum Y such that $\operatorname{Dim} Y = k - \operatorname{Dim} X$ is said to be *dimensionally k-complementary* to X. The following proposition summarizes those properties of k-complements which we shall use in the sequel.

Proposition 3.9. *If X is a compactum such that* $\operatorname{Dim} X_n^* = n - \operatorname{Dim} X$ *then the following holds*:

(1) $\dim X_n^* < n$ *if* $\dim X > 0$;
(2) $\dim(X_n^* \times X) = n$;
(3) $\dim(Y \times X) \leqslant n$ *implies* $\operatorname{Dim} Y \leqslant \operatorname{Dim} X_n^*$;
(4) $\operatorname{Dim} X_{n+k}^* = \operatorname{Dim}(X_n^* \times I^k)$; *and*
(5) $n - (n - \operatorname{Dim} X) = \operatorname{Dim} X$.

For a proof of (5) see [16], (1) follows from Section 2 while (2)–(4) trivially follow from the definition. We now prove a lemma on *embedding of complements*:

Lemma 3.10. *If a compactum $X \subset \mathbb{R}^n$ has dimension* $\dim X \leqslant n-3$ *then there exists a compact space $X^* \subset \mathbb{R}^n$ such that* $\operatorname{Dim} X^* = (n-1) - \operatorname{Dim} X$.

Proof. Fix a tame embedding $X \subset \mathbb{R}^n$. Consider an X-net generated by X (as in Lemma 3.4). Denote it by X^σ. Then X^σ is a countable union of tame compacta and $\operatorname{Dim} X^\sigma = \operatorname{Dim} X$. By negligibility criterion X^σ is $X^*_{(n-1)}$-negligible (where $\operatorname{Dim} X^*_{n-1} = (n-1) - \operatorname{Dim} X$). Hence in the space of mappings $C(X^*_{n-1}, \mathbb{R}^n)$ there is a G_δ-dense subset of mappings missing X^σ and therefore not raising Dim (all subsets in the complement of X^σ being X-negligible have Dim less or equal to $\operatorname{Dim} X^*_{n-1}$ by negligibility criterion).

On the other hand there is a G_δ-dense subset of light mappings in $C(X^*, \mathbb{R}^n)$ which do not lower Dim as follows from the generalized Hurewicz formula. Hence there is a mapping in the intersection and its image has the required Dim. $\square$

Lemma 3.11. *Let $X \subset \mathbb{R}^n$ be a compactum such that* $\operatorname{Dim} X$ *is simply connected and* $\dim X \leqslant n-3$. *Then there is a compactum $X' \subset \mathbb{R}^{n-1}$ such that* $\operatorname{Dim} X' = \operatorname{Dim} X - 1$.

Proof. By Lemma 3.6 one may consider X as having the following property with respect to the linear projection $\pi : \mathbb{R}^n \to \mathbb{R}$:

$$\operatorname{Dim}\bigl(\pi^{-1}(t) \cap X\bigr) \leqslant \operatorname{Dim} X - 1, \quad \text{for every } t \in \mathbb{R}.$$

By the generalized Hurewicz formula,

$$\operatorname{Dim} X \leqslant \sup_{t\in\mathbb{R}} \operatorname{Dim}\bigl(\pi^{-1}(t)\cap X\bigr) + \operatorname{Dim}\pi(X).$$

Since $\operatorname{Dim}\pi(X) \leqslant 1$, one obtains that

$$\sup \operatorname{Dim}\bigl(\pi^{-1}(t)\cap X\bigr) = \operatorname{Dim} X - 1.$$

But all sections $\pi^{-1}(t)\cap X$ are (topologically) contained in $\mathbb{R}^{n-1}$ as $\pi^{-1}(t)\cong\mathbb{R}^{n-1}$. Hence by Lemma 3.7 one concludes there is a subspace $X'\subset\mathbb{R}^{n-1}$ such that

$$\operatorname{Dim} X' = \sup \operatorname{Dim}\bigl(\pi^{-1}(t)\cap X\bigr) = \operatorname{Dim} X - 1. \qquad \square$$

We are now ready for the theorem on *embeddings of* Dim-*types*.

Theorem 3.12 [10]. *For every compactum X of dimension n there is a compactum $X'\subset\mathbb{R}^{n+2}$ such that* $\operatorname{Dim} X' = \operatorname{Dim} X$.

Proof. Embed X into $\mathbb{R}^{2n+1}$. Then by Lemma 3.10, the $2n$-complement of X is represented by a compact space $X^*_{2n}\subset\mathbb{R}^{2n+1}$. Since $\operatorname{Dim} X^*_{2n} = \operatorname{Dim}(X^*_{n+1}\times I^{n-1})$ one obtains, by applying Lemma 3.11 $(n-1)$-times by induction, that X^*_{n+1} can be realized up to Dim in $\mathbb{R}^{n+2}$. Moreover, in this case X^*_{n+1} is realized in $\mathbb{R}^{n+3}$. Since $\dim X^*_{n+1}\leqslant n$ one can apply Lemma 3.10 and conclude that the $(n+2)$-complement of X^*_{n+1} is contained in $\mathbb{R}^{n+3}$. But such a complement has dimension equal to $\operatorname{Dim}(X\times I)$. By applying Lemma 3.11 to this complement one then realizes X in $\mathbb{R}^{n+2}$. $\square$

For the proof of the main result of Section 3, we shall need the following *approximation* lemma.

Lemma 3.13. *Suppose that $X^*_{n-1}\subset\mathbb{R}^n$. Then the space $C(X,\mathbb{R}^n)$ contains a dense G_δ set of maps $f: X\to\mathbb{R}^n$ such that* $\operatorname{Dim} f(X) = \operatorname{Dim} X$.

Proof. We begin by $(X^*_{n-1})^\sigma$ (cf. Lemma 3.4). Since $\operatorname{Dim} X$ is simply connected, we have $\dim X^*_{n-1}\leqslant n-3$. Embed X^*_{n-1} into $\mathbb{R}^n$ as a tame set. Take $(X^*_{n-1})^\sigma$. This will again be tame. Apply Theorem 1.10 and conclude that X is removable from this set. So every map can be pushed into the complement. By Lemma 3.4, the complement has dimension $\leqslant \operatorname{Dim} X$. Note this is true for a dense G_δ set. By Lemma 3.3, there is another G_δ dense subset, for which we have dimension $\geqslant \operatorname{Dim} X$. Taking the intersection, get the assertion. $\square$

Finally, here is the *approximation* theorem.

Theorem 3.14 [10]. *Let X be a compactum of dimension* $\dim X\leqslant n-3$ *and suppose that* $\operatorname{Dim} X$ *is simply connected. Then every mapping $f: X\to\mathbb{R}^n$ can be approximated by a mapping $f': X\to\mathbb{R}^n$ such that* $\operatorname{Dim} f'(X) = \operatorname{Dim} X$.

Proof. The hypothesis that $\operatorname{Dim} X$ simply connected implies that the $(n-1)$-complement of X is of codimension 3. By the Realization theorem [16] such complement, denoted by X^*, can be tamely embedded into $\mathbb{R}^n$. Consider the X^*-net generated by X^* and denoted by X^*_σ. This net being X-negligible has the property, by Negligibility criterion, that all subsets in $\mathbb{R}^n \setminus X^*_\sigma$ have Dim less than or equal to $\operatorname{Dim} X$. So in the space of mappings $\mathcal{C}(X, \mathbb{R}^n)$ there is a dense G_δ-set consisting of mappings missing X^*_σ which do not rise the dimensional type.

On the other hand, there is a G_δ-dense subset of light mappings in $\mathcal{C}(X, \mathbb{R}^n)$ which do not lower Dim. The intersection of these two sets is G_δ-dense and so it has both properties. □

4. Intersection formula

Transversal intersection formula of Theorem 1.10 and its relative version of Theorem 1.12 have similar proofs. To unify them we shall prove a general theorem on $\mathcal{F}, \mathcal{G}$-transversality. Throughout this chapter we shall denote by $\mathcal{F}$ and $\mathcal{G}$ two complete (i.e., complete as metric spaces with respect to the metric generated by the sup-norm) classes of mappings of compacta X and Y into the Euclidean space $\mathbb{R}^n$, respectively.

Lemma 4.1. *The subset $\mathcal{D}_k$ of $\mathcal{F} \times \mathcal{G}$, consisting of the pairs f, g such that* $\dim(f(X) \cap g(Y)) \leqslant k$, *is a G_δ-set, for every k.*

Proof. The subspace $\mathcal{D}^\varepsilon_k$ of $\mathcal{F} \times \mathcal{G}$, consisting of the mappings f with the property $a_k(f(X) \cap g(Y)) < \varepsilon$, where a_k denotes the Aleksandrov k-dimensional *width* (which is defined as the minimum of ε such that the set admits an ε-translation to a k-dimensional polyhedron), is an open subset. This is easy to see and is well known. The intersection $\bigcap_n \mathcal{D}^{1/n}_k$ of the sequence of such subspaces with $\varepsilon = 1/n$ gives us exactly $\mathcal{D}_k$. □

This lemma has an important corollary:

Corollary 4.2. *If two mappings $f \in \mathcal{F}$ and $g \in \mathcal{G}$ stably intersect (with respect to $\mathcal{F}$, $\mathcal{G}$), then for every $\varepsilon > 0$, there are mappings $f' \in \mathcal{F}$ and $g' \in \mathcal{G}$ which intersect transversally (with respect to $\mathcal{F}$, $\mathcal{G}$) and are ε-close to f and g, respectively.*

Proof. Denote by $\mathcal{B}_k$ the boundary of the closure of $\mathcal{D}_k$. In this case, it follows by Lemma 4.1 that the set of transversal pairs coincides with the complement of $\bigcup_k \mathcal{B}_k$. □

Definition 4.3. A mapping class $\mathcal{H}: Z \to \mathbb{R}^m$ is said to *k-stably intersect* a subset $Y \subset \mathbb{R}^m$ at a mapping $h \in \mathcal{H}$, if $\dim h'(Z) \cap Y \geqslant k$, for all mappings h' which are sufficiently close to h.

Hence, *k-unstability* coincides with *k-removability*. We shall say that $\mathcal{H}$ is *stably k-removable* from Y at h, if every mapping of $\mathcal{H}$, sufficiently close to h, is k-removable.

Remark 4.4. By Lemma 4.1, one can define *transversality* as a combination of k-stability and stable $(k+1)$-removability, for some k.

Lemma 4.5. *$\mathcal{F}$ is stably k-removable from Y if and only if $\mathcal{F}$ k-unstably intersects Y at every f' which is sufficiently close to f.*

Proof. The last condition implies local density at f of mappings which satisfy the condition $\dim(f'(X) \cap Y) < k$. However, by Lemma 4.1 such a set is G_δ. This shows that $\mathcal{F}$ is stably k-removable. In the opposite direction the proof is trivial. □

Lemma 4.6. *If a mapping $f : X \to \mathbb{R}^n$ is stably removable from every k-plane with respect to the class $\mathcal{F}$ then for almost all mappings from $\mathcal{F}$, which are sufficiently close to f, it follows that* $\dim f'(X) < n - k$.

Proof. Let us fix a δ so small that every f' from the δ-neighborhood $O_\delta(f)$ of f in $\mathcal{F}$, unstably intersects every k-plane. For every $\varepsilon > 0$ one can choose a locally finite sequence of k-planes $L_1, L_2, \ldots$, with Aleksandrov $(n-k)$-width of the complement of its union to be less than ε (just take the set of all points with at least k rational coordinates with denominator $< n/\varepsilon$). Then for almost all elements of $O_\delta(f)$, the image lies in the complement of the union of the sequence, hence has the $(n-k)$-width less than ε. Since $O_\delta(f)$ is complete and the width inequality defines an open set, one concludes that the image of almost all mappings of $O_\delta(f)$ has the Aleksandrov $(n-k)$-width zero. □

The following is a *correct* version of the Čogošvili theorem:

Corollary 4.7. *A compactum $X \subset R^n$ is stably removable from every k-plane if and only if* $\dim X < n - k$.

Proof. Since the inclusion $X \subset \mathbb{R}^n$ cannot be approximated by mappings with the dimension of image less than $\dim X$, one obtains from Lemma 4.6 the proof in the *if* direction. To prove the opposite direction, one has to consider polyhedral approximations and use standard general position arguments. □

The following result can be considered as the *generalized* Čogošvili theorem.

Theorem 4.8. *If a mapping $f : X \to \mathbb{R}^m$ is stably k-removable from every n-plane with respect to a complete class $\mathcal{F} : X \to \mathbb{R}^n$ then $\dim f'(X) < m - n + k$, for almost all $f' \in \mathcal{F}$, which are sufficiently close to f.*

Proof. According to Lemma 4.6 above it is enough to prove the stable removability of f from every $(n-k)$-plane L. Let us consider any n-plane containing $L \subset L^n$.

Let ε be so small that every mapping from $\mathcal{F}$, ε-close to f, is k-removable from L^n. Then we can approximate such an f' arbitrary closely by f'' so that $\dim(f''(X) \cap L) < k$. But every less than k-dimensional compactum unstably intersects every $(n-k)$-dimensional plane. Consequently, f'' unstably intersects L. □

Definition 4.9. A compact subset $X \subset \mathbb{R}^n$ is called *stably* k-removable from another subset $Y \subset \mathbb{R}^n$ if the class $C(X, \mathbb{R}^n)$ is stably k-removable from Y at the inclusion $X \subset \mathbb{R}^n$.

The default value of k in this terminology is 0. One says *unstably* intersects instead of 0-*unstably* intersects and *stably removable* instead of stably 0-removable.

Definition 4.10. Mapping classes $\mathcal{F}$ and $\mathcal{G}$ are called k-*stably* intersecting at a pair $f \in \mathcal{F}$, $g \in \mathcal{G}$ if $\dim(f'(X) \cap g'(Y)) \geqslant k$, for all pairs f', g', sufficiently close to f, g.

Definition 4.11. An affine plane $L^n \subset \mathbb{R}^n \times \mathbb{R}^n$ is called *skew* if it satisfies the following condition: $p_1(L^n) = p_2(L^n) = \mathbb{R}^n$, where by $p_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ we denote the projection on the ith factor.

Let us denote by $\mathrm{Auth}(\mathbb{R}^n, \mathbb{R}^n)$ the space of all affine automorphisms of $\mathbb{R}^n$. Then skew planes are in one-to-one correspondence with the graphs of elements from $\mathrm{Auth}(\mathbb{R}^n, \mathbb{R}^n)$. In particular, the diagonal of the product corresponds to the identity isomorphism. If we denote the graph of such an automorphism H by G_H one has the following natural homeomorphism:

$$(X \times Y) \cap G_H \simeq H(X) \cap Y.$$

Characteristics of the intersection of classes $\mathcal{F}$ and $\mathcal{G}$, such as stability and its relative versions (removability and transversality), correspond to the analogous characteristics to the intersection of the product $\mathcal{F} \times \mathcal{G}$ with the diagonal. In particular, one obtains:

Lemma 4.12. *For every $H \in \mathrm{Auth}(\mathbb{R}^n, \mathbb{R}^n)$, the following assertions are equivalent:*

(1) *the product $\mathcal{F} \times \mathcal{G}$ k-stably intersects intersects the graph G_H at $f \times g$; and*

(2) *$H \circ \mathcal{F}$ k-stably intersects $\mathcal{G}$ at $H \circ f, g$.*

Lemma 4.13. *If $\mathcal{F} \times \mathcal{G}$ is stably k-removable from every skew n-plane at f, g, then $\dim(f'(X) \times g'(Y)) < n + k$, for almost all pairs f', g' from $\mathcal{F}, \mathcal{G}$ which are sufficiently close to the pair f, g.*

Proof. Since every n-plane can be approximated by a skew plane the stable removability from skew planes implies the same property for all n-planes. Now the proof can be accomplished by applying Lemma 4.8. □

A mapping class $\mathcal{F}$ is called *almost light* if almost all of its elements are light mappings. Moreover, it is called *affinely invariant* if $h \circ \mathcal{F} = \mathcal{F}$, for every $h \in \mathrm{Auth}(\mathbb{R}^n, \mathbb{R}^n)$.

Lemma 4.14. *If $\dim(X \times Y) \geqslant n + k$, the classes $\mathcal{F} : X \to \mathbb{R}^n$, $\mathcal{G} : Y \to \mathbb{R}^n$ are almost light, and $\mathcal{F}$ is affinely invariant, then $\mathcal{F}$ and $\mathcal{G}$ k-stably intersect at some f, g.*

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ contain a dense G_δ subset of light mappings, the same is true for $\mathcal{F} \times \mathcal{G}$. Since light mapping do not lower dimension, we have $\dim(f(X) \times g(X)) \geqslant n+k$, for almost all f, g. By Lemma 4.13, there is a skew n-plane G_H from which $\mathcal{F} \times \mathcal{G}$ is not stably k-removable at some f, g. Since $\mathcal{F}$ is affinely invariant, one has that $H \circ \mathcal{F} = \mathcal{F}$. Therefore, $\mathcal{F} \times \mathcal{G}$ is not stably removable from the diagonal at $H \circ f, g$. Hence, there exist $f' \in \mathcal{F}$, close to $H \circ f$ and $g' \in \mathcal{G}$, close to g, such that $\mathcal{F} \times \mathcal{G}$ k-stably intersects the diagonal at f', g'. Now the proof is accomplished by invoking Lemma 4.12. □

Definition 4.15. Let Y', Y be a compact pair ($Y' \subset Y$) of subsets of $\mathbb{R}^n$. By a Y'-*relative mapping* we shall mean any continuous mapping of Y into $\mathbb{R}^n$, which is identical over Y'. The class of Y'-relative mappings will be denoted as $\mathcal{C}(Y \operatorname{rel} Y', \mathbb{R}^n)$. The stability, removability and transversality with respect to this class will be called the *relative* stability, *relative* transversality, etc.

Applying Lemma 4.14 above to the case $\mathcal{F} = \mathcal{C}(X, \mathbb{R}^n)$ and $\mathcal{G} = \mathcal{C}(Y \operatorname{rel} Y', \mathbb{R}^n)$, one deduces (since $h \circ \mathcal{F} = \mathcal{F}$) the following theorem on *k-stable intersections*.

Theorem 4.16. *If* $\dim(X \times Y) \geqslant n + k$, *for a compactum* X *and a compact subset* $Y \subset \mathbb{R}^n$, *then for every closed subset* $Y' \subset Y$, *there exist a mapping* $f: X \to \mathbb{R}^n$ *and a* Y'*-relative mapping* $g: Y \to \mathbb{R}^n$ *which k-stably intersect relatively to* Y'.

When $Y' = Y$, Theorem 4.16 implies Theorem 1.8. In the case when $\mathcal{F} = \mathcal{C}(X, \mathbb{R}^n)$ and $\mathcal{G} = \mathcal{C}(Y, \mathbb{R}^n)$, one obtains from Lemma 4.14 the following theorem:

Theorem 4.17. *If* $\dim(X \times Y) \geqslant n + k$, *for compacta* X *and* Y, *then there exists mappings* $f: X \to \mathbb{R}^n$ *and* $g: Y \to \mathbb{R}^n$ *which k-stably intersect.*

The following theorem represents the hard part of the transversal intersection formula of Theorem 1.7.

Theorem 4.18 [11]. *Suppose that a compactum* $Y \subset \mathbb{R}^n$ *is tame, and of codimension* > 2 *and that* X *is compactum such that* $\dim(X \times Y) < k + n$. *Then for every mapping* $f: X \to \mathbb{R}^n$, *the intersection of* f *and* Y *is k-unstable.*

We begin by the following lemma:

Lemma 4.19. *If* $\dim(X \times Y) < n + k$, *for a tame compactum* $Y \subset \mathbb{R}^n$ *and a compactum* X *of dimension* $< n + 2$, *then for every linear plane* L^{n-k}, *one has* $\dim X \times (h(Y) \cap L^{n-k}) < n$, *for almost all homeomorphisms* $h \in \mathcal{H}(\mathbb{R}^n, \mathbb{R}^n)$.

Proof. Let us denote by $\mathcal{H}'$ the set of all homeomorphisms satisfying the condition above. Lemma 4.13 implies that $\mathcal{H}'$ is of type G_δ. So it is enough to prove the density of $\mathcal{H}'$. Since $\mathcal{H}$ is a topological group it suffices to prove that $\mathcal{H}'$ intersects every neighbourhood of the identity.

Let us chose an increasing sequence of linear subspaces $\{L^{n-i}\}_{0 \leqslant i \leqslant k}$ starting with the chosen plane and finishing with the whole space $L^n = \mathbb{R}^n$ (indices of these planes correspond to their dimensions).

Let us fix a $(k+1)$-dimensional simplex T, intersecting transversally all planes of the sequence. Therefore, $\dim(T \cap L^{n-i}) = (k+1) - i$ and $\dim(h(T) \cap L^{n-i}) \geqslant (k+1) - i$, for all $h \in \mathcal{H}$ sufficiently close to the identity. Let us consider $Y' = Y \cup T$. The choice of T implies simple connectedness of $\operatorname{Dim}(Y' \cap L^{n-i})$, for all $i < k$, as well as of $\operatorname{Dim}(h(Y') \cap L^{n-i})$, for all h close to the identity. Since $\dim X < n-2$, one has $\dim(X \times T) < n+k$, hence $\dim(X \times Y') < n+k$.

Now we apply, step by step, Lemma 3.6 to Y' and construct a sequence of homeomorphisms $h_i : L^{n-i} \to L^{n-i}$, close to the identity, such that

$$\operatorname{Dim}\left(h_i h_{i-1} \cdots h_0(Y')\right) \cap L^{n-i} = \operatorname{Dim} Y' - i, \quad \text{for all } i < k.$$

The local contractibility of the homeomorphisms group of $\mathbb{R}^n$ allows us to extend every h_i to a homeomorphism $H_i \in \mathcal{H}$, so that H_i, as well as their superposition, belong to any prechosen neighbourhood of the unity in $\mathcal{H}$. Denote by h the composition $H_{k-1}H_{k-2}\cdots H_0$. Then $\operatorname{Dim}(h(Y') \cap L^{n-k}) = \operatorname{Dim} Y' - k$, hence

$$\dim\left(X \times (Y \cap L^{n-k})\right) \leqslant \left(X \times (Y' \cap L^{n-k})\right) < n. \qquad \square$$

By the *Nöbeling net* of dimension k in $\mathbb{R}^n$ we mean any countable union of k-dimensional planes such that its complement has dimension $n-k-1$. The main example is the *rational* Nöbeling net which consists of all points with at least k rational coordinates.

Lemma 4.20. *If N is a k-dimensional Nöbeling net in $\mathbb{R}^n$, then* $\dim X \leqslant \dim(X \cap N) + n - k$, *for every compactum $X \subset \mathbb{R}^n$.*

Proof. The proof immediately follows from the classical Urysohn–Menger formula for the dimension of the union. $\square$

Proof of Theorem 4.18. Let us consider the Nöbeling net N of all $(n-k)$-dimensional planes. By Lemma 4.19, one applies the Baire theorem to the space $\mathcal{H}$ of all autohomeomorphisms to get the existence of a homeomorphism h of $\mathbb{R}^n$, close to the identity and such that $\dim((h(Y) \cap N) \times X) < n$. In this case, X being removable from the intersection of $h(Y)$ with any plane of N, is removable from the whole intersection $h(Y) \cap N$, by virtue of the same Baire theorem. Hence for every mapping $f : X \to \mathbb{R}^n$, there exist arbitrary close mapping f', whose images miss $N \cap h(Y)$ and therefore the dimension of the intersection $f'(X) \cap h(Y)$ is not greater than $\dim(\mathbb{R}^n \setminus N) = k-1$.

However, the set $h^{-1}f'(X) \cap Y$, being homeomorphic to $f'(X) \cap h(Y)$, has dimension $\leqslant k-1$, and the composition $h^{-1}f'$ is also close to f, since h is close to the identity. So the required k-unstability of intersections of arbitrary f with Y is established. $\square$

Transversality with respect to $C(Y \operatorname{rel} Y', \mathbb{R}^n)$ will be called rel Y'-*transversality*. Now both transversality formulas of the introduction, Theorems 1.10 and 1.12, follow from the following *unified* intersection formula for subsets.

Theorem 4.21. *If compact subsets $X, Y \subset \mathbb{R}^n$ with dimensionally homogeneous product $X \times Y$, intersect* rel Y'*-transversally (where $Y' \subset Y$ is a tame compactum) and* $\dim Y < n-2$, *then*:

$$\dim(X \cap Y) = \dim(X \times Y) - n.$$

Proof. Let us denote $k = \dim(X \times Y) - n$. The hard part of the intersection formula of Theorem 4.18 implies $(k+1)$-unstability of the intersection. So one has the inequality $\dim(X \cap Y) \leqslant k$.

Suppose that $\dim(X \cap Y) < k$. The transversality condition implies the existence of an $\varepsilon > 0$ such that the inequality $\dim(f(X) \cap g(Y)) < k$ holds for almost all ε-translations f of X and Y'-relative ε-translations g of Y. Pick a point x in the intersection $X \cap Y$. Denote by X_ε, Y_ε the intersections of X and Y, respectively, with the closed $\varepsilon/2$-neighbourhood of x. By the dimensional homogeneity of $X \times Y$, one has that $\dim(X_\varepsilon \times Y_\varepsilon) = k + n$. Now apply Theorem 4.16, for the case $\mathcal{F} = \mathcal{C}(X_\varepsilon, O_\varepsilon(x))$ and $\mathcal{G} = \mathcal{C}(Y_\varepsilon \operatorname{rel} Y', O_\varepsilon X)$ (where $O_\varepsilon(x)$) is the open ε-neighbourhood of x which is homeomorphic to $\mathbb{R}^n$), to obtain a mapping $f_\varepsilon : X_\varepsilon \to O_\varepsilon(x)$ which k-stably (rel Y') intersects Y_ε. Use f_ε to get $f'_\varepsilon : X \to \mathbb{R}^n$, extending f'_ε to all X such that it is ε-close to the identity. Thus we get a contradiction to our hypothesis since f'_ε k-stably intersects $Y \operatorname{rel} Y'$, being an ε-translation. □

Finally let us prove the *mapping* transversal intersection formula.

Theorem 4.22. *Let $f : X \to \mathbb{R}^n$ and $g : Y \to \mathbb{R}^n$ be transversally intersecting mappings of compacta of dimensions $< n - 2$. Then*

$$\dim\left(f(X) \cap g(Y)\right) = \dim(X \times Y) - n.$$

Lemma 4.23. *Theorem* 4.22 *is true if* $\operatorname{Dim} X$ *and* $\operatorname{Dim} Y$ *are simply connected.*

Proof. Let $k = \dim(X \times Y) - n$. Let us now prove that there is no pair f, g from the given mapping classes with $(k+1)$-stable intersection. Since $\operatorname{Dim} Y'$ and $\operatorname{Dim} X'$ are simply connected, one can apply Approximation Theorem 3.14 to conclude that for almost all mapping pairs F, G from X', Y' to $\mathbb{R}^n$, one has $\operatorname{Dim} G(Y') = \operatorname{Dim} Y'$ and $\operatorname{Dim} F(X') = \operatorname{Dim} X'$. Let us fix such a pair F, G, ε-close to our f, g. Applying Theorem 4.18 to the images $F(X)$, $G(Y)$, one finds two ε-translations f', g' such that

$$\begin{aligned}\dim\left(f'(F(X)) \cap g'(G(Y))\right) &\leqslant \dim\left(F(X) \times G(Y)\right) - n \\ &= \dim(X \times Y) - n = k.\end{aligned}$$

So we find another pair, 2ε-close to the pair f, g (namely $f' \circ F$, $g' \circ G$), with the dimension of the intersection less than $k+1$. This proves the nonexistence of $(k+1)$-stable intersections.

To prove the rest of the lemma it is enough to demonstrate the density of mapping pairs with k-stable intersections among all stably intersecting mappings. Let us consider an arbitrary stably intersecting mapping pair f, g and a positive ε. Fix a pair of points $(x, y) \in (X \times Y)$ with coinciding images $f(x) = g(y)$. Choose compact neighborhoods O_x and O_y so small that their images lie in an open ε-ball $B \subset \mathbb{R}^n$. The dimension homogeneity of $X \times Y$ implies $\dim(O_x \times O_y) = n + k$ hence Lemma 4.14 applied in the case $\mathcal{F} = \mathcal{C}(O_x, B)$, $\mathcal{G} = \mathcal{C}(O_y, B)$ yields a pair f', g' with a k-stable intersection.

Extend f' and g' over X and Y so that their images remain in B. Choose a pair of continuous real functions $\varphi: X \to [0,1]$ and $\psi: Y \to [0,1]$ such that $\varphi(x) = 1 = \psi(y)$, if $x \in O_x$ and $y \in O_y$ and $\varphi(x) = 0 = \psi(y)$, if $f(x) \notin B$ and $g(y) \notin B$. In this case the linear combinations $f(1-\varphi) + f'\varphi$ and $g(1-\psi) + g'\psi$ are ε-close to f, g and they k-stably intersect. □

Proof of Theorem 4.22. Suppose that $\dim X \leqslant \dim Y$. Let us consider the compacta $X' = X \cup D$ and $Y' = Y \cup D$, homeomorphic to the disjoint union of X and Y, respectively with the two-dimensional disk D. Then the condition on codimension implies unstability of the intersections of X as well as Y with D. Hence the dimension of the transversal intersection of X' and Y' is the same as for X and Y. Let us now compare the dimensions of the products $X \times Y$ and $X' \times Y'$. If they coincide then all ingredients of the intersection formulas for both pairs are the same and it suffices to prove the formula for X', Y' which are of simply connected dimension. If $\dim(X \times Y) < \dim(X' \times Y')$, then

$$\dim(X' \times Y') = \dim(Y' \times D) = \dim Y + 2 < n.$$

So in this case there are no transversal intersections of X' and Y' and consequently none of X and Y. □

Corollary 4.24 [10]. *Every pair of mappings of compacta X and Y into $\mathbb{R}^n$ such that $\dim(X \times Y) < n$, $\dim X < n-2$, and $\dim Y < n-2$ unstably intersects.*

Proof. The existence of a stably intersecting pair f, g leads to the existence of a transversely intersecting pair, according to Corollary 4.2. But the dimension of transversal intersection calculated by Theorem 4.22 is negative. Contradiction. □

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