The 12th problem

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In the year 2000, exactly one hundred years after David Hilbert posed his now famous list of 23 open problems, The Clay Mathematics Institute (CMI) announced its seven Millennium Problems. (http://www.claymath.org/millennium). The Gazette has asked leading Australian mathematicians to put forth their own favourite 'Millennium Problem'. Due to the Gazette's limited budget, we are unfortunately not in a position to back these up with seven-figure prize monies, and have decided on the more modest 10 Australian dollars instead.
In this issue Paul Norbury will explain his favourite open problem that should have made it to the list.

Lawson conjecture for minimal surfaces

The round three-sphere consists of all unit vectors in $\mathbb{R}^4$

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{1}$$

As a topological manifold — it is locally the same as $\mathbb{R}^3$ — it is the three-sphere. Geometrically (1) has more structure since we can measure length, area and volume of subsets. The extra structure is a metric on the three-sphere known as the round metric, so the name *round three-sphere* refers to the topology *and* the geometry of the vectors satisfying (1) .

From the topology we know that for any closed path, or circle, there is a disk in the three-sphere with boundary given by that path. The disk may have self-intersections. Plateau's problem asks for such a disk with smallest area. A solution to Plateau's problem is known as a *minimal surface*. More generally, a minimal surface is a surface in the three-sphere that locally minimises area. In other words, any small circle on a minimal surface bounds a small disk on the surface and that disk is the solution to Plateau's problem for the small circle.

The interesting topology of the three-sphere allows compact, embedded minimal surfaces without boundary. An example is given by any equator, also known as a great two-sphere. Thus, the claim is that the minimal area surface that bounds a small circle on a great two-sphere lives in the two-sphere. It is counter-intuitive if you think of two-spheres in $\mathbb{R}^3$ since a plane in $\mathbb{R}^3$ that meets a two-sphere in a small circle contains a smaller area disk than the disk on the two-sphere. This is where we see the round nature of the three-sphere, compared to the flatness of $\mathbb{R}^3$. In fact the only minimal two-spheres in the three-sphere are great two-spheres.

Another example of an embedded minimal surface in the round sphere is the torus

$$x_1^2 + x_2^2 = \frac{1}{2} = x_3^2 + x_4^2 \tag{2}$$

which is known as a Clifford torus. Just as there are many equators — in fact a three-dimensional space of them — there is a four-dimensional space of Clifford tori, obtained from (2) by acting on $\mathbb{R}^4$ with the six-dimensional group $O(4)$ that preserves the three-sphere and its metric, and moves Clifford tori.

Lawson Conjecture. *Any embedded minimal torus in the round three-sphere is a Clifford torus.*

The embeddedness in the statement of the conjecture is important since there are infinitely many immersed minimal tori [2]. The conjecture was posed by Lawson 35 years ago out of his construction of minimal surfaces in the round three-sphere [3]. It was published as a conjecture in Yau's survey [7] and the problems section of that proceedings.

While mentioning the age of the Lawson conjecture, let us compare it to the ages of the Clay millenium problems. Thirty-five years is the age of the youngest millenium problem — the P vs NP problem — if we do not count the Yang-Mills and Mass Gap problem which was essentially formulated when the seven problems were announced. The oldest millenium problems are the Riemann Hypothesis — 145 years — and the Poincare conjecture — 100 years — the latter which has most likely now been solved.

An inviting aspect of this problem, and more generally the study of minimal surfaces in the round three-sphere, is the different approaches that it allows (although one may feel that an unsolved problem has thus far allowed little.) It can be studied using PDEs, Morse theory, Riemann surfaces and smooth topology.

PDEs arise because, as one might expect, the notion of locally minimising area can be expressed infinitesimally, or in other words in terms of derivatives at each point of the surface. (We shift our view slightly and think of a surface mapping to the three-sphere instead of simply living inside the three-sphere.) From this perspective a minimal surface is a surface with vanishing mean curvature, and the mean curvature — given by the sum of the principle curvatures — is a second order non-linear PDE satisfied by the map. See for example [6] where you can also read about the related Willmore conjecture.

Morse theory relates the topology of a manifold and the critical points of a function on the manifold. One might deduce topological information about a manifold from more easily obtained critical point information, or alternatively the information may go the other way. Consider the infinite-dimensional space of embedded tori in the round three-sphere, and a function on this space given by area. Minimal surfaces are the critical points of the area function and the existence of minimal surfaces and their local properties can be deduced from knowledge of the topology the space of embedded tori which is studied using homotopy theory. See for example [4].

Hitchin [1] showed that harmonic maps of the torus to the round three-sphere, of which minimal tori are examples, can be studied in terms of hyperelliptic Riemann surfaces. In fact, a harmonic map can be completely determined by the hyperelliptic Riemann surface equipped with a line bundle, two meromorphic differentials and constraints on their periods — objects from the study of classical Riemann surfaces. The Lawson conjecture is proven when the genus of the hyperelliptic curve is less than 2 and greater than 5.

Any two tangent minimal surfaces must meet at a saddle point. This is known as the maximum principle for minimal surfaces. It uses PDEs, but once the result is known, one can work without PDEs. Thus, if topology is more to your taste than geometry, one can simply study smooth embedded tori in the round three-sphere that meet the great spheres and the Clifford tori in saddle points, without mention of vanishing mean curvature. The tangent plane at any point of an embedded minimal torus is tangent to a unique great two-sphere, and hence saddle to that sphere, and tangent to a one-dimensional family of Clifford tori, and also saddle to each of them. For an example of such arguments see [5].

The Clifford torus is flat since it is (geometrically) the product of two circles. The Lawson conjecture is proven for flat tori, where we mean to take the induced metric on the embedded torus. It has also been proven in other special cases. It is proven when the minimal embedded torus has extra symmetry, say it is symmetric with respect to a circle action, or various finite groups, and it follows from an affirmative answer to a conjecture of Yau relating the first eigenfunction of the Laplacian on the minimal surface and its embedding.

References

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