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An Exotic Orientable 4-Manifold

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1. Introduction

Suppose that two compact smooth manifolds are known to be homeomorphic. It is still possible that they are not diffeomorphic. Examples of such "exotic smoothings" have been known since the 1950's, beginning with Milnor's discovery of exotic 7-spheres. Exotic smoothings are known and well understood in dimensions 5 and higher. In dimensions 3 and below, there are no exotic smoothings; homeomorphic manifolds are diffeomorphic. In dimension 4 (as usual) the situation has been more difficult to understand.

The first example of an exotic smoothing in dimension 4 was the fake $\mathbb{R}P^4$ of Cappell and Shaneson [CS]. This manifold is now known to be homeomorphic to $\mathbb{R}P^4$ (by Freedman's work [F2]), but it cannot be diffeomorphic to $\mathbb{R}P^4$. Other, related examples have been constructed. All of these examples, however, have been nonorientable. Exotic smoothings on orientable compact 4-manifolds have been difficult to find.

Perhaps the best candidate for an exotic orientable (closed) 4-manifold has been Scharlemann's manifold M, which is homeomorphic to $M' = S^3 \times S^1 \# S^2 \times S^2$. Unfortunately, the $S^2 \times S^2$ summand has blocked any attempt to distinguish M from M'. (In fact, compact orientable manifolds which are homeomorphic must become diffeomorphic after connected sum with copies of $S^2 \times S^2$ [G2], making this problem seem exceedingly difficult. Perhaps M and M' are diffeomorphic.)

In the present paper, we exhibit two compact orientable manifolds (with boundary), M_1 and M_1', which are homeomorphic, but not diffeomorphic. (Note that this is automatically rel ∂, by uniqueness of smoothings in dimension 3.) These are constructed as smooth submanifolds of M and M', respectively. Roughly speaking, we obtain M_1 and M_1' by cutting away the problematic homology of $S^2 \times S^2$ in M and M', using Freedman theory. The construction is explicit enough that we may describe M_1' completely, up to certain ramification. In fact, M_1' is

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diffeomorphic to $S^3 \times S^1 \# H$, where H is the 2-handlebody on a certain zero-framed link L (namely, a ramified 5-fold double of the Hopf link). In particular, L is topologically slice, and the intersection form of H is trivial.

Recently, Donaldson has announced [D] an exotic smoothing of a closed, simply connected 4-manifold. Donaldson's example represents a much more subtle phenomenon than that described here. In particular, his proof requires extensive use of gauge theory to distinguish the smoothings, whereas the present paper uses only Rohlin's theorem.

2. The Construction

We work in the smooth category except when otherwise stated.

Our goal is to prove the following:

Theorem. *There are two compact, orientable 4-manifolds with boundary which are homeomorphic but not diffeomorphic. One manifold has the form $S^3 \times S^1 \# H$, where H is a 2-handlebody with trivial intersection form.*

We begin the construction with Scharlemann's candidate M for a fake oriented 4-manifold [S]. This is constructed as follows: Let Σ denote the Poincaré homology sphere, $SU(2)/G$, where G is the binary icosahedral group. In the interior of the manifold $\Sigma \times I$, perform a single surgery to kill the fundamental group. (Almost any element of G may be used for this.) Now identify the two boundary components, obtaining the closed manifold M with $\pi_1(M) = \mathbb{Z}$. M has the homology of $M' = S^3 \times S^1 \# S^2 \times S^2$. (In fact, M and M' are homeomorphic, by Freedman's s-cobordism theorem for fundamental group $\mathbb{Z}$ [F2].) However, M contains a copy of Σ representing a generator of H_3, which suggests that M and M' may not be diffeomorphic (due to the nontrivial Rohlin invariant of Σ).

By construction, $M - \Sigma$ is a simply connected manifold whose intersection form is hyperbolic with rank 2. Thus, by Casson's embedding theorem [C] we may represent $H_2(M - \Sigma)$ by two Casson handles. That is, there is a 4-ball B in $M - \Sigma$, and a pair of Casson handles CH_i $(i = 1, 2)$ in $M - \Sigma - \operatorname{int} B$, such that the Casson handles are attached with framing zero to a Hopf link in ∂B. The open set $U = \operatorname{int} B \cup CH_1 \cup CH_2$ carries $H_2(M)$.

We now surger away $H_2(M)$. By Freedman's main theorem [F1] any Casson handle is homeomorphic to an open 2-handle, so U is homeomorphic to $S^2 \times S^2$ minus a point. In particular, $H_2(U)$ is carried by a compactum X homeomorphic to $S^2 \vee S^2$, with $U - X$ homeomorphic to $S^3 \times \mathbb{R}$. For later purposes, it will be convenient to require X to lie in a certain compact submanifold T of U. If T_i denotes the first 5 stages of CH_i, then T consists of $T_1 \cup T_2$ glued to a 4-ball $B' \subset B$. By Freedman's reimbedding theorems, any 5-stage tower contains a Casson handle, so we may find a flat $S^2 \vee S^2$ in $\operatorname{int} T$. If X denotes this compactum, we still have $U - X$ homeomorphic to $S^3 \times \mathbb{R}$. (Proof: Both ends of $U - X$ are topologically collared by $S^3 \times \mathbb{R}$, so we may compactify to obtain S^4, by Freedman's topological Poincaré conjecture [F1].)

Let M_0 denote the open manifold $M - X$. Let M_* be its one-point compactification. This is a topological manifold (since the end of M_0 is collared by $U - X$).

In fact, M_* is homeomorphic to $S^3 \times S^1$, by standard surgery theory and Freedman's s-cobordism theorem over $\mathbb{Z}$.

There is a manifold R homeomorphic to $\mathbb{R}^4$, and a smooth embedding $i: U - X \hookrightarrow R$ onto a neighborhood of the end of R, such that the outward-pointing end of $U - X$ in M_0 is mapped to the outward-pointing end of $i(U - X)$ in R. This is a standard argument, which was first used to construct exotic $\mathbb{R}^4$'s (see, for example, [G1]). Specifically, consider the standard handle decomposition of $S^2 \times S^2$, with two 2-handles. Since every Casson handle has a canonical smooth embedding in the standard 2-handle [C], we immediately obtain a canonical embedding $i: U \hookrightarrow S^2 \times S^2$. The manifold $R = S^2 \times S^2 - i(X)$ is contractible and collared topologically by $i(U - X)$, homeomorphic to $S^3 \times \mathbb{R}$. Thus, by Freedman [F1], R is homeomorphic to $\mathbb{R}^4$. (However, a recent result of Taubes implies that R cannot be diffeomorphic to $\mathbb{R}^4$.)

Let $M'_0 = S^3 \times S^1 \# R$. Then i is a diffeomorphism between neighborhoods of the ends of M_0 and M'_0. The one-point compactification M'_* of M'_0 is clearly homeomorphic to $S^3 \times S^1$, so there is a homeomorphism $h: M_* \to M'_*$. We may assume that $h \circ i^{-1}$ preserves orientation. Note that if x denotes the point at infinity in M_*, then $U_* = (U - X) \cup \{x\}$ is homeomorphic to $\mathbb{R}^4$, and i extends to a homeomorphism $i_*: U_* \to U'_* \subset M'_*$.

Next, we perturb h to obtain a homeomorphism $f_*: M_* \to M'_*$ which agrees with i_* near x. In fact, we require $f_* = i_*$ near the compact set $T_* = (T - X) \cup \{x\}$ in U_*. This is possible by the stable homeomorphism theorem of Quinn [Q]. In particular, let B_* be a flat topological 4-ball in U_*, with $T_* \subset \operatorname{int} B_*$. We may perturb h so that $h(B_*) = i(B_*)$ (via Quinn's annulus conjecture). Now note that any two orientation-preserving homeomorphisms of $\mathbb{R}^4$ ($\approx \operatorname{int} B_*$) may be isotoped to agree on a preassigned compact set, via a topological isotopy with compact support. [Simply smooth them near 0 (with respect to the standard smooth structure) by Quinn's theorem, then force them to agree near 0. Conjugating by a dilation enlarges the region of agreement to include the compact set.]

Now we restrict f_* to $f_0: M_0 \to M'_0$. By construction, f_0 is a homeomorphism and $f_0 | T - X = i | T - X$ is a diffeomorphism. In particular, ∂T and $f_0(\partial T)$ are smooth submanifolds. Thus, $M_1 = M_0 - \operatorname{int}(T - X)$ and $M'_1 = M'_0 - \operatorname{int} f_0(T - X)$ are smooth compact manifolds with boundary, and the restricted map $f_1: M_1 \to M'_1$ is a homeomorphism. We will see that M_1 and M'_1 cannot be diffeomorphic.

By construction, M_1 contains an embedded copy of the homology sphere Σ, which represents a generator of $H_3(M_1) \cong H_3(M_*) \cong \mathbb{Z}$. This lifts to a smooth embedding into the universal cover $\tilde{M}_*$ of M_*. (Note that $\tilde{M}_*$ is homeomorphic to $S^3 \times \mathbb{R}$, and smooth except at the lifts of the singular point x.) Similarly, $H_3(M'_1)$ is represented by a smoothly embedded S^3. If M_1 and M'_1 were diffeomorphic, we would obtain disjoint smooth embeddings of Σ and S^3 in $\tilde{M}_*$, each representing a generator of $H_3(\tilde{M}_*)$. The region W bounded by Σ and S^3 would be a compact homology cobordism, smooth except at a finite number of singularities in its interior. Each singularity would have a deleted neighborhood diffeomorphic to $U - X$, so we could resolve the singularities by replacing them with copies of X. (Downstairs in M_*, this procedure gives back the original manifold M.) We would obtain a smooth compact manifold $\hat{W}$ homeomorphic to a connected sum of W with copies of $S^2 \times S^2$. In particular, $\hat{W}$ would be a cobordism between Σ and S^3, and be spin with signature zero, contradicting Rohlin's theorem.

We conclude with a description of the manifold M_1'. Recall that $M_1' = S^3 \times S^1 \# R - \operatorname{int} i(T - X) = S^3 \times S^1 \# H$, where $H = S^2 \times S^2 - \operatorname{int} i(T)$. But T has the form $B' \cup T_1 \cup T_2$ where T_1 and T_2 are 5-stage towers glued with framing zero to a Hopf link in the boundary of the 4-ball B'. We may write $S^2 \times S^2$ as $i(B')$ $\cup$(two 2-handles) $\cup$4-handle, with the 2-handles glued to the same framed Hopf link in $\partial i(B')$. The embedding $i: T \hookrightarrow S^2 \times S^2$ is determined by the standard embedding of each T_i into the corresponding 2-handle h_i. This standard embedding has the following description [C]: Find a certain "ramified collection of 5-fold Whitehead curves" in a neighborhood of the belt circle of h_i. This is an unlink in ∂h_i, so it has a canonical family of slice disks $\{D_k\}$ in h_i. Removing a tubular neighborhood of $\cup D_k$ from h_i leaves T_i.

This description of i shows that $H = S^2 \times S^2 - \operatorname{int} i(T)$ is a 2-handlebody, namely the 4-handle of $S^2 \times S^2$ (upside down) together with closed neighborhoods of the disks D_k in each h_i. The intersection form of H vanishes, since it lies in the manifold R homeomorphic to $\mathbb{R}^4$. The same observation shows that the 2-handles of H are attached with framing zero to a link L which is topologically slice. In fact, the link L is formed from a Hopf link, by replacing each component with the collection of 5-fold Whitehead curves associated to the corresponding T_i.

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