BRANCHED SHADOWS AND COMPLEX STRUCTURES ON 4-MANIFOLDS

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ABSTRACT. We define and study branched shadows of 4-manifolds as a combination of branched spines of 3-manifolds and of Turaev's shadows. We use these objects to combinatorially represent 4-manifolds equipped with $Spin^c$-structures and homotopy classes of almost complex structures. We then use branched shadows to study complex 4-manifolds and prove that each almost complex structure on a 4-dimensional handlebody is homotopic to a complex one.

CONTENTS

1. INTRODUCTION

Shadows were defined by V. Turaev at the beginning of the nineties in [28] as a method for representing knots alternative to the standard one based on knot diagrams and Reidemeister moves. The theory was developed in the preprint "Topology of shadows", later included as a revised version in [26]; a short account of the theory was also published in [27]. Since then, only few applications of shadows were studied. Among these we recall the use of shadows made by U. Burri in [10] and A. Shumakovitch in [24] to study Jones-Vassiliev invariants of knots and the study of "Interdependent modifications of links and invariants of finite degree" developed by M.N. Goussarov in [18]. More recently, in [16] and [17] D.P. Thurston and the author used shadows to study 3-manifolds and their geometry and to prove a quadratic upper bound on the minimal complexity of a 4-manifold whose boundary is a given 3-manifold.

On the other side, *branched polyhedra* (and in particular branched spines of 3-manifolds) were studied in depth by Benedetti and Petronio since 1997 in [7]. They showed how branched spines encode 3-manifolds equipped with additional topological structures as $Spin^c$-structures, vector fields and a more refined structure they called *concave traversing fields*. In [8], they provided a calculus for these objects, i.e. a finite set of local modifications connecting any two branched spines of the same manifold respecting the extra-structure. Later, in [9], using these objects they extended the definition of the Reidemeister-Turaev refined torsion to 3-manifolds with arbitrary boundary. Branched spines were also used by Benedetti and Baseilhac to define quantum hyperbolic invariants ([3], [4] and [5]) .

The present paper is devoted to combine the notions of shadow of a 4-manifold and branched polyhedron: the resulting theory is a generalization to dimension 4 of a set of results which hold for branched spines of 3-manifolds. The main interest of the new theory is that a series of these results can be further developed and studied in new directions which are not visible in dimension 3. In particular, here we study the study of the relations between branched shadows and complex structures.

After an introductory section in which we recall the notion of shadow and how to thicken it to a 4-manifold, we show how a branching encodes a $Spin^c$-structure on that manifold. Then we prove in Theorem 4.12 a result that we summarize as follows:

Theorem 1.1. *Let M be a 4-dimensional handlebody (i.e. a 4-manifold admitting a handle decomposition without 3 and 4-handles). Each $Spin^c$-structure and each homotopy class of almost complex structures on M can be encoded by a branched shadow of M.*

Hence branched shadows are a key tool for a combinatorial approach to a series of 4-dimensional problems related with $Spin^c$-structures and almost complex structures. We further develop the theory in the last section by studying complex structures and branched shadows. It turns out that a branched shadow embedded in a complex manifold behaves like an embedded oriented real surface. We translate to the world of shadows a series of classical definitions and results due to Chern, Spanier, Bishop and Lai on indices of embedded real surfaces. We prove a shadow version of the well known result of Harlamov and Eliashberg which allows, under suitable conditions, to annihilate pairs of complex points in embedded real surfaces. In the end, as an application, we prove the following:

Theorem 1.2. *Let M be a 4-dimensional handlebody. Each homotopy class of almost complex structures on M can be represented by an integrable complex structure.*

Our proof is based on the machinery of shadows and one the ideas used in [21]. We also point out that the above statement is a special case of a result P. Landweber in [22] (proved through different techniques) and also by M.L. Gromov [19]. In [15], we study sufficient combinatorial conditions assuring that a branched shadow determines a Stein domain.

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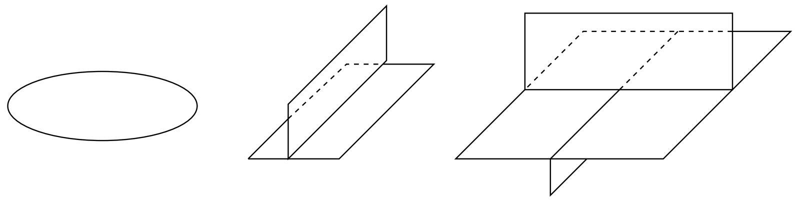

FIGURE 1. The three local models of a simple polyhedron.

2. Shadows of 4-manifolds

In this section we recall the notion of shadow of a 4-manifold. We refer to [14] for an introduction to this topic and to [26] for a complete account of it. From now on, all the manifolds and homeomorphisms will be smooth unless explicitly stated.

Definition 2.1. A simple polyhedron P is a 2-dimensional CW complex whose local models are those depicted in Figure 1; the set of points whose neighborhoods have models of the two rightmost types is a 4-valent graph, called *singular set* of the polyhedron and denoted by $Sing(P)$. The connected components of $P - Sing(P)$ are the *regions* of P. A simple polyhedron whose singular set is connected and whose regions are all discs is called *standard*.

Definition 2.2 (Shadow of a 4-manifold)**.** Let M be a smooth, compact and oriented 4-manifold. $P \subset N$ is a *shadow* for M if:

(1) P is a closed polyhedron embedded in M so that $M - P$ is diffeomorphic to $\partial M \times (0, 1]$;
(2) P is *flat* in M, that is for each point $p \in P$ there exists a local chart (U, ϕ) of M around p such that $\phi(P \cap U)$ is contained in $\mathbb{R}^3 \subset \mathbb{R}^4$ and in this chart the pair $(\mathbb{R}^3 \cap \phi(U), \mathbb{R}^3 \cap \phi(U \cap P))$ is diffeomorphic to one of the models depicted in Figure 1.

Remark 2.3. Note that the original definition of shadows was given in the PL setting by Turaev in [26] . But in four dimensions the smooth and the PL setting are equivalent, that is for each PL-structure on a compact manifold there exists a unique compatible (in a suitable sense) smooth structure. Definition 2.2 is the natural translation of the notion of shadow to the smooth setting.

A necessary and sufficient condition (see [13]) for a 4-manifold M to admit a shadow is that M is a 4-handlebody, that is M admits a handle decomposition without 3 and 4-handles. In particular, ∂M is a non empty connected 3-manifold. From now on, *all the manifolds will be 4-handlebodies and all the polyhedra will be standard and flat* unless explicitly stated.

Can we reconstruct the neighborhoods of a polyhedron P in a manifold M from its combinatorics? Let us first understand the easier 3-dimensional case, where the polyhedron

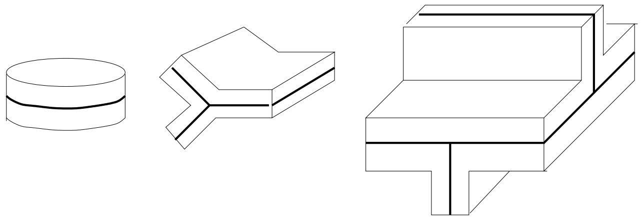

FIGURE 2. The three type of blocks used to thicken a spine of a 3-manifold.

is embedded in an oriented 3-manifold N and its combinatorics allows one to reconstruct its regular neighborhoods in the following way. Any decomposition of P into the local patterns of Figure 1, induces a decomposition of any of its regular neighborhoods into blocks as those of Figure 2. These can be reglued to each other according to the combinatorics of P. That way, a polyhedron embedded in a 3-manifold, determines its regular neighborhoods in the 3-manifold. It is known [11] that any 3-manifold with non-empty boundary can be reconstructed that way, as a neighborhood of some embedded standard polyhedra, called *spine*.

Let us now pass to the 4-dimensional case. Suppose that P is a surface embedded in a oriented 4-manifold M. In general we cannot reconstruct the tubular neighborhoods of P by using only its topology, since their structure depends on the self-intersection number of P in M. To state it differently, the tubular neighborhood of a surface in a 4-manifold is homeomorphic to the total space of a disc bundle over the surface (its normal bundle), and the Euler number of this bundle is a necessary datum to reconstruct its topology.

Hence, we see that to encode the topology of a neighborhood of P in M we need to "decorate" P with some additional information; when P is an oriented surface, the Euler number of its normal bundle is a sufficient datum.

We describe now the basic decorations we need for a general standard polyhedron P, for a more detailed account see [26]. Let us denote $\mathbb{Z}[\frac{1}{2}]$ the group of integer multiples of $\frac{1}{2}$. There are two canonical *colorings* on the regions of P, i.e. assignments of elements of $\mathbb{Z}_2$ or $\mathbb{Z}[\frac{1}{2}]$, the second depending on a flat embedding of P in an oriented 4-manifold. They are:

The $\mathbb{Z}_2$-gleam of P, defined as follows. Let D be the (open) 2-cell associated to a given region of P and $\overline{D}$ be the natural compactification $\overline{D} = D \cup S^1$ of the (open) surface represented by D. The embedding of D in P extends to a map $i : \overline{D} \to P$ which is injective in $int(\overline{D})$, locally injective on $\partial\overline{D}$ and which sends $\partial\overline{D}$ into $Sing(P)$. Using the map i we can "pull back" a small open neighborhood of D in P and construct a simple polyhedron $N(D)$ collapsing on $\overline{D}$ and such that the map i extends as a local homeomorphism $i' : N(D) \to P$ whose image is contained in a small neighborhood of the closure of D in P. When i is an embedding of $\overline{D}$ in P, then $N(D)$ turns out to be homeomorphic to a neighborhood of D in P and i' is its inclusion in P. In general, $N(D)$ collapses over a polyhedron which is obtained by gluing D to the core of an annulus or of a Möbius strip: we define the $\mathbb{Z}_2$-gleam

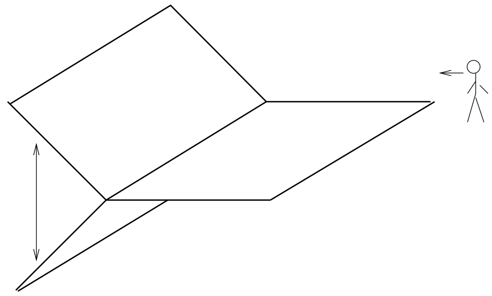

FIGURE 3. The picture sketches the position of the polyhedron in a 3-dimensional slice of the ambient 4-manifold. The direction indicated by the vertical double arrow is the one along which the two regions touching the horizontal one get separated.

of D in P as 0 in the former case and 1 in the latter. This coloring only depends on the combinatorial structure of P.

The gleam of P, defined as follows. Let us now suppose that P is flat in an oriented 4-manifold M, with D, $\overline{D}$ and $i : \overline{D} \to P$ as above. Pulling back through i a small neighborhood of $i(N(D))$ in M, we obtain a 4-dimensional neighborhood of $N(D)$. This is an oriented ball B on which we fix an auxiliary riemannian metric. Since $N(D)$ is locally flat in B, $N(D) - D$ well defines a line normal to $\overline{D}$ in B along $\partial\overline{D}$ and hence a section of the projectivized normal bundle of $\overline{D}$ (see Figure 3). Let then $gl(D)$ be equal to $\frac{1}{2}$ times the obstruction to extend this section to the whole $\overline{D}$; such an obstruction is an element of $H^2(\overline{D}, \partial\overline{D}; \pi_1(S^1))$, which is canonically identified with $\mathbb{Z}$ since B is oriented. Note that the gleam of a region is an integer if and only if its $\mathbb{Z}_2$-gleam is zero.

Using the fact that the $\mathbb{Z}_2$-gleam is always defined, Turaev generalized [26] the notion of gleam to non-embedded polyhedra as follows:

Definition 2.4. A *gleam* on a simple polyhedron P is a coloring on the regions of P with values in $\mathbb{Z}[\frac{1}{2}]$ such that the color of a region is an integer if and only if its $\mathbb{Z}_2$-gleam is zero.

Theorem 2.5 (Reconstruction Theorem [26])**.** *Let P be a polyhedron with gleams gl; there exists a canonical reconstruction map associating to (P, gl) a pair (M_P, P) where M_P is a smooth, compact and oriented 4-manifold, and $P \subset M$ is a shadow of M (see Definition 2.2). If P is a standard polyhedron flat in a smooth and oriented 4-manifold and gl is the gleam of P induced by its embedding, then M_P is diffeomorphic to a compact neighborhood of P in M.*

The proof is based on a block by block reconstruction procedure similar to the one used to describe 3-manifolds by means of their spines. Namely, for each of the three local patterns of Figure 1, we consider the 4-dimensional thickening given by the product of an interval with the corresponding 3-dimensional block shown in Figure 2. All these thickenings are glued to each other according to the combinatorics of P and its gleam.

By Theorem 2.5, to study 4-manifolds one can either use abstract polyhedra equipped with gleams or embedded polyhedra. The latter approach is more abstract, while the former is purely combinatorial; we will use both approaches in the following sections. The translation in the combinatorial setting of the definition of shadow of a 4-manifold is the following:

Definition 2.6 (Combinatorial shadow)**.** A polyhedron equipped with gleams (P, gl) is said to be a *shadow* of the 4-manifold M if M is diffeomorphic to the manifold associated to (P, gl) by means of the reconstruction map of Theorem 2.5.

Given a shadow (P, gl) of a 4-manifold M, it is possible to modify it by a series of local modifications called "moves". The three most important are shown in Figure 4. To

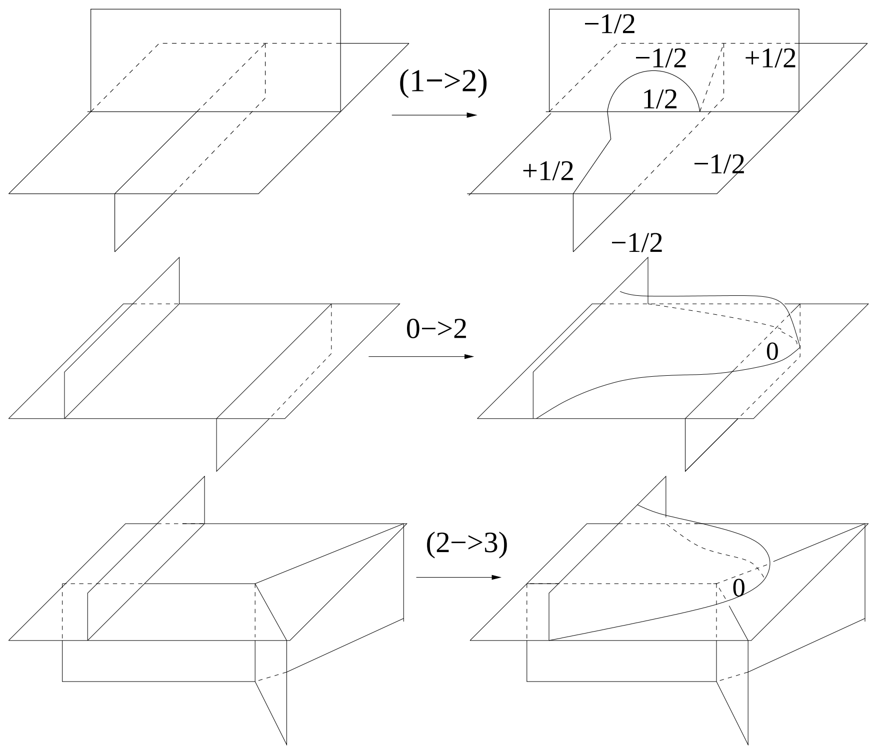

FIGURE 4. The three shadow equivalences.

visualize these moves, imagine that a region of P slides over some other regions, producing a polyhedron which differs from the initial one only in a collapsible subpolyhedron (drawn in the left part of the figure). These moves are called respectively $1 \to 2$, $0 \to 2$ (or lune or finger-move) and $2 \to 3$-moves (or Matveev-Piergallini move), because of their effect on the number of vertices of the polyhedra.

Let us analyze first the $0 \to 2$-move. This move acts in a 4-ball contained in M and containing the part of P shown in the left part of the picture. After the move, we replace this part of P with that drawn in the right and obtain a shadow P' of M. The whole move can be performed in a 3-dimensional slice of the 4-ball in M. The same comments apply to the case of the $2 \to 3$-move.

The case of the $1 \to 2$-move is slightly different. This move applies to a neighborhood of a vertex and slides a region (the vertical lower one in the left part of the figure) over the vertex thus creating a new one. It is a good exercise to visualize this sliding: it cannot be performed in $\mathbb{R}^3$.

The above comments apply also to the inverses of the moves. We now clarify the meaning of the numbers written on the regions of the polyhedra in the figure. Each move represents a modification of the embedded polyhedron, and produces a new polyhedron whose gleam is induced by the embedding in the ambient manifold as explained in the definition of the gleam. Each region of this polyhedron corresponds to one of the initial polyhedron except the disc created by the positive moves and contained in the right part of the pictures. The numbers in the figure represent the changes in the gleams of the regions; the gleam of the new disc is explicited in the pictures. The gleam of the regions which do not appear in the figure do not change since their embedding in M does not.

All these comments can be summarized in the following proposition:

Proposition 2.7. *Let (P, gl) be a shadow of a 4-manifold M and let (P', gl') be obtained from (P, gl) through the application of any sequence of $1 \to 2$, $0 \to 2$ and $2 \to 3$ moves and their inverses. Then (P', gl') is a shadow of M.*

3. Branched shadows

Given a simple polyhedron P we define the notion of *branching* on it as follows:

Definition 3.1 (Branching condition)**.** A branching b on P is a choice of an orientation for each region of P such that for each edge of P, the orientations induced on the edge by the regions containing it do not coincide.

Remark 3.2. This definition corresponds to the definition of "orientable branching" of [7].

We say that a polyhedron is *branchable* if it admits a branching and we call *branched polyhedron* a pair (P, b) where b is a branching on P.

Definition 3.3. Let (P, gl) be a shadow of a 4-manifold M. We call *branched shadow* of M the triple (P, gl, b) where (P, gl) is a shadow and b is a branching on P. When this will not cause any confusion, we will not specify the branching b and we will simply write (P, gl).

Proposition 3.4. *Any 4-manifold admitting a shadow admits also a branched shadow.*

Proof of 3.4. We sketch the idea of the proof which is an adaptation of Theorem 3.4.9 of [7]. We note that a branched shadow P' is obtained from a shadow P via an algorithmic procedure. Orient arbitrarily all the regions of P; if for each edge of $Sing(P)$ the three orientations induced on it by the regions containing it do not coincide, then we have already found a branching on P. Let us suppose then that on an edge e is induced three times the same orientation, and that the endpoints of e lie on two different vertices of P (we can always find a shadow for which all the edges have this property by applying a suitable sequence of the moves of Figure 4 on (P, gl)). The basic idea of the proof is to apply a $2 \to 3$-move along the edge e to "blow up" it and create a new region whose orientation we can choose arbitrarily: indeed a $2 \to 3$-move makes an edge disappear and creates a new region whose boundary is formed by three edges (see Figure 4). Choosing appropriately the orientation of the new region and keeping unchanged the orientations of the other ones, we can assure that no edge is induced three times the same orientation from the regions containing it. In some particular cases, one of the edges touched by the new region does not satisfy the branching condition, hence we apply again a $2 \to 3$-move on it. In [7] an accurate analysis of the possible cases is performed showing that this process eventually ends with a branched polyhedron. It is important to stress that the above proof can be adapted to our case because it does not use any assumption on the $\mathbb{Z}_2$-gleam and all the moves used can be performed in the ambient 4-manifold. $\boxed{3.4}$

A branching on a shadow allows us to smooth its singularities and equip it with a smooth structure as shown in Figure 5. This smoothing can be performed also inside the ambient manifold obtained by thickening the shadow; the shadow locally appears as in Figure 5, where the two regions orienting the edge in the same direction approach each other so that, for any auxiliary riemannian metric on the ambient manifold, all the derivatives of their distance go to zero while approaching the edge.

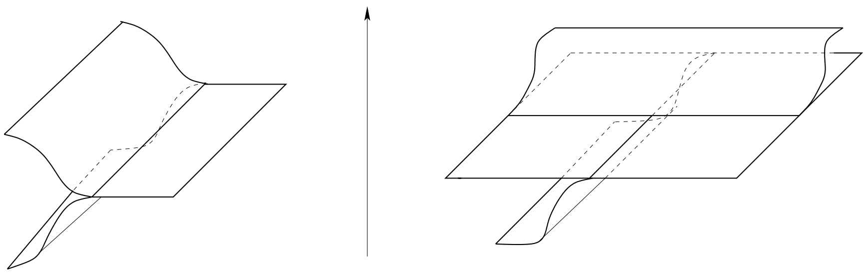

FIGURE 5. How a branching allows a smoothing of the polyhedron: the regions are oriented so that their projection on the "horizontal" (orthogonal to the drawn vertical direction) plane is orientation preserving.

If (P, gl) is a branched shadow of a 4-manifold, and we apply to it one of the moves of Figure 4, we get another shadow P' of the same manifold containing one region more than P. Each region of P naturally corresponds to a region of P' and the region of P' which does not correspond to one of P is the small disc created by the move (see Figure 4). Hence the branching on P induces a choice of an orientation on each region of P' except on that disc and these orientations satisfy the branching condition on all the edges of P' not touched by that disc. Analogously, if P' is obtained from P through the inverse of a basic move, then each region of P' corresponds to a region of P and the branching on P induces an orientation on each region of P'.

Definition 3.5. A basic move $P \to P'$ applied on a branched polyhedron P is called *branchable* if it is possible to choose an orientation on the disc created by the move which, together with the orientations on the regions of P' induced by the branching of P, defines a branching on P'. Analogously, the inverse of a basic move applied to P is branchable if the orientations induced by the branching of P on the regions of P' define a branching.

A branching is a kind of loss of symmetry on a polyhedron and this is reflected by the fact that each move has many different branched versions. To enumerate all the possible branched versions of the moves, one has to fix any possible orientation on the regions of the left part of Figure 4 and complete these orientations in the right part of the figure by fixing one orientation on the region created by the move; by Definition 3.5, one obtains a branched version of a basic move when the branching condition is satisfied both in the left and in the right part of the figure. Fortunately, many of the possible combinations are equivalent up to symmetries of the pictures; we show in Figure 6 all the branched versions of the $0 \to 2$-move and in Figure 7 those of the $2 \to 3$-move. In these figures we split these branched versions in two types namely the *sliding*-moves and the *bumping*-moves; this differentiation will be used later.

We will also use often the following terminology:

Definition 3.6. Let e be an edge of a branched polyhedron P and let R_i,R_j and R_k be the regions of P containing it in their boundary. Then R_i is said to be the *preferred region of* e if it induces the opposite orientation on e with respect to those induced by R_j and R_k.

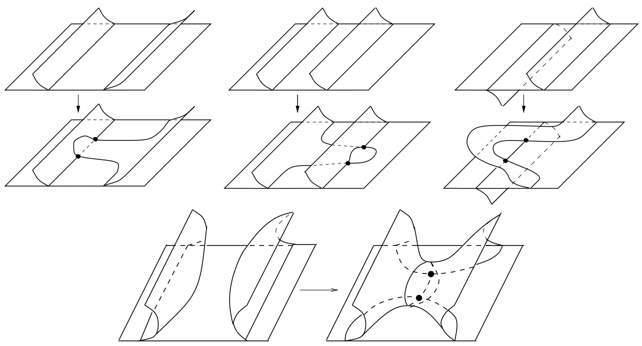

FIGURE 6. In the upper part of this figure we show the three branched versions of the lune-move called "sliding"-moves. In the bottom part we show the version called "bumping"-move.

It has been proved in [7][Chapter 3] that any lune and $2 \to 3$-move is branchable, but some of their inverses are not. Regarding the $1 \to 2$-move and its inverse, the following holds:

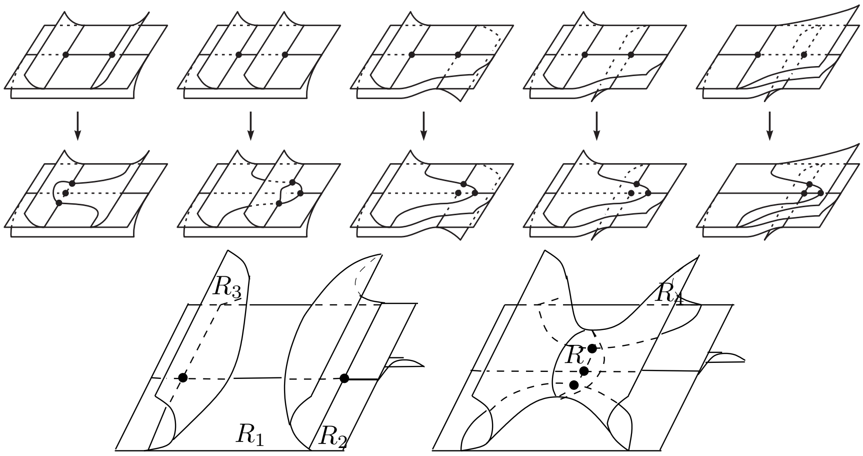

FIGURE 7. In the upper part of the figure we show the 5 branched versions of the 2 → 3-move called “sliding”-moves. The bottom part of the figure represents the “bumping”-move.

Lemma 3.7. *Each* $1 \to 2$*-move or its reverse is branchable.*

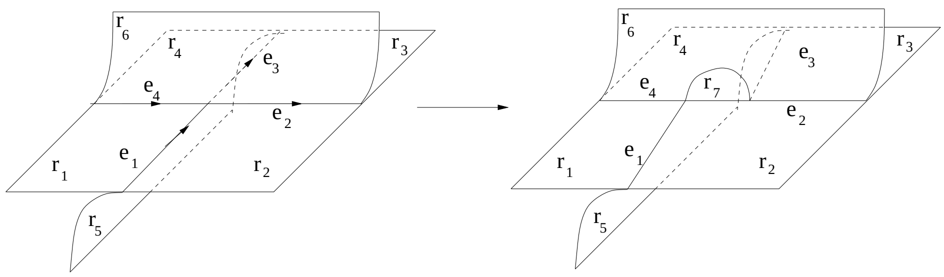

FIGURE 8. In this picture we fix the notation we use in the proof of Lemma 3.7.

Proof of 3.7. Let P be a branched polyhedron on which a $1 \to 2$-move acts. To fix the notation, we use the right-hand rule in Figure 8 and give a sign to the each region of the polyhedron involved in the $1 \to 2$-move: $+$ if the region is positively oriented with respect to the upward direction and $-$ otherwise. That way, we identify a branching near the vertex with a six-uple of signs. There are 24 possibilities but since the opposite of a branching is a branching we reduce to study the 12 branchings where the sign of the region r_1 is $+$. A branching induces an orientation on the edges $e_i,\ 1 = 1, 2, 3, 4$ touching the vertex; in the case represented in the left part of the figure all the signs are $+$.

We need to show that, for each branching (and then for each six-uple of signs) near the vertex in the left part of Figure 8, there is a choice of an orientation (and so a sign) for the region r_7 in the right part of the figure so that no edge $e_i,\ i = 1, 2, 3, 4, 5, 6$ is induced three times the same orientation from the regions containing it. This is proved in the following table where, in the left column, we list the possible branchings before the move, and in the right column we write the signs corresponding to the compatible orientations of r_7. Note that there are cases where both the orientations of r_7 give a branching to P.

$(r_1, r_2, r_3, r_4, r_5, r_6)$	r_7
$(+, +, +, +, +, +)$	$+$
$(+, +, +, +, +, -)$	$\pm$
$(+, +, +, +, -, +)$	$\pm$
$(+, +, +, +, -, -)$	$-$
$(+, +, +, -, +, +)$	$+$
$(+, +, -, -, +, +)$	$+$

$(r_1, r_2, r_3, r_4, r_5, r_6)$	r_7
$(+, +, -, +, -, +)$	$\pm$
$(+, -, +, +, -, -)$	$-$
$(+, +, -, -, -, +)$	$+$
$(+, -, -, +, -, -)$	$-$
$(+, -, -, +, -, +)$	$-$
$(+, -, -, -, -, +)$	$\pm$

The case of the $(1 \to 2)^{-1}$-move is simpler: if the polyhedron is branched before the inverse move, then it is after, since no new edge is created during the move and no regions merge. $\boxed{3.7}$

The following proposition is a consequence of the fact that a 4-handlebody retracts onto its shadows.

Proposition 3.8. *Let P be a branched shadow of a 4-manifold M, and let $R_i,\ i = 1, \ldots, n$ and $e_j, j = 1, \ldots, m$ be respectively the regions and the edges of P oriented according to the branching of P. Then $H_2(M; \mathbb{Z})$ is the kernel of the boundary application ∂ : $\mathbb{Z}[R_1, \ldots, R_n] \to \mathbb{Z}[e_1, \ldots, e_m]$. Moreover $H^2(M; \mathbb{Z})$ is the abelian group generated by the cochains $\hat{R}_i,\ i = 1, \ldots, n$ dual to the regions of P subject to the relations generated by the coboundaries of the edges having the form $\delta(\hat{e}_j) = -\hat{R}_i + \hat{R}_j + \hat{R}_k$ where R_i is the preferred region of e_j.*

Given a shadow (P, gl) of a 4-manifold M, there are three cochains representing classes in $H^2(M; \mathbb{Z})$ naturally associated to (P, gl). The first one is the *Euler cochain* of P, denoted $Eul(P)$ and constructed as follows. Let m be the vector field tangent to P (using the smooth structure given by the branching) which near the center of the edges points inside the preferred regions; we extend m in a neighborhood of the vertices as shown in Figure 9.

The field constructed above is the *maw* of P. For each region R_i of P, the maw gives a vector field defined near ∂R_i, so it is possible to extend this field to a tangent field on the whole R_i having isolated singularities of indices ± 1; let n_i be the algebraic sum of these indices over the region R_i. The Euler cochain is defined as $Eul(P) = \sum_i n_i \hat{R}_i$; its meaning will be analyzed in the next section.

4. Branched shadows and almost complex structures

As before, let M be an oriented 4-handlebody and let g be a fixed auxiliary riemannian metric on M. In this section we show that a branched shadow determines a pair $(M, [J])$ where $[J]$ is a homotopy class of almost complex structures on M suitably compatible with g, and that for each such class, there exists a branched shadow of M encoding it.

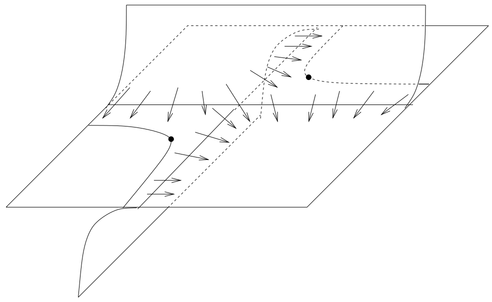

FIGURE 9. In this figure we show how the maw behaves near a vertex of a branched polyhedron.

Definition 4.1. An almost complex structure J on an oriented 4-manifold M is a smooth morphism $J : TM \to TM$ such that for each point $p \in M$ it holds $J^2 = -Id$. We say that J is *positive* if at each point $p \in M$ there exists a positive (with respect to the orientation of M) basis of T_pM of the form $(x, J(x), y, J(y))$. We say that J is orthogonal with respect to g if for each $p \in M$ the map $J : T_pM \to T_pM$ is g-orthogonal.

Let P be a branched shadow of M. In each of the local blocks used to reconstruct M from P as in Theorem 2.5 (these blocks are the products of an interval with the 3-dimensional blocks of Figure 2), P is smoothly embedded in a non symmetrical way (see Figure 5). As in Figure 5, we choose an *horizontal* 2-plane and orient it according to the branching of P, so that each of the basic building blocks of M is equipped with a distribution of oriented 2-planes, denoted as $T(P)$.

Let $V(P)$ be the field of oriented *vertical* 2-planes of TM which are pointwise positively g-orthogonal to the planes of $T(P)$.

We define an almost complex structure J_P by requiring that its restriction to the two fields $T(P)$ and $V(P)$ acts pointwise as a $\frac{\pi}{2}$ positive rotation and we extend this action linearly to the whole TM. By construction J_P is positive and g-orthogonal. Note that the choice of g was arbitrary. The following lemma assures us that our constructions are well defined up to homotopy.

Lemma 4.2. *Let M be an oriented 4-handlebody and g a riemannian metric on M. If P is a branched shadow of M and g' is another riemannian metric on M, the almost complex structures on J_P and J'_P constructed as above using respectively g and g' are homotopic. Moreover, each homotopy class of almost complex structures on M contains a g-orthogonal representative.*

Proof of 4.2. Let J_0 be an arbitrary almost complex structure on M. We first prove the second statement and split its proof in two steps:

(1) there exists a riemannian metric g_0 on M such that J_0 is g_0 orthogonal;
(2) J_0 is homotopic to a g-orthogonal almost complex structure J_1.

Step 1. The space of the scalar metrics on $\mathbb{R}^4$ with respect to which a fixed complex structure J on $\mathbb{R}^4$ is orthogonal is a non-empty convex set: indeed in a fixed basis the orthogonality condition can be written as $J^t g J = g$ where J is the complex structure and g is the symmetric matrix representing the scalar product. In particular, the fiber bundle over M of pointwise scalar products with respect to which J_0 is pointwise orthogonal has collapsible fiber, and hence admits a section g_0.

Step 2. The homotopy of metrics on M given by $g_t = (1-t)g_0 + tg$ connects g_0 to g: we show how to lift it to a homotopy J_t of g_t-orthogonal almost complex structures connecting J_0 to a g-orthogonal almost complex structure.

Consider the bundle of J_0 complex 2-planes on M; its fiber is $\mathbb{CP}^1 = S^2$ and so, since M retracts onto a 2-dimensional polyhedron, we can find a section and choose a field F of J_0-complex tangent 2-planes on M. For each point $p \in M$ we define J_t to be the $\frac{\pi}{2}$ rotation with respect to the metric g_t both on F_p and on its g_t-orthogonal 2-plane. This gives us a g_t-orthogonal almost complex structure J_t connecting J_0 to a g-orthogonal almost complex structure concluding the proof of the second statement. To prove the first statement, use Step 2 with $F = T(P)$ and $g_0 = g'$. $\boxed{4.2}$

Corollary 4.3. *Let M be a smooth 4-manifold admitting a shadow. The restriction to the branched shadows of M of the reconstruction map of Theorem 2.5 can be refined to a map whose image is contained in the set of pairs $(M, [J])$ where $[J]$ is a homotopy class of positive almost complex structures on M.*

From now on, we will only use positive and g-orthogonal almost complex structures. Our construction splits the tangent bundle of M as the sum of two linear complex bundles $V(P)$ and $T(P)$, hence the first Chern class of TM, viewed as a complex bundle using the almost complex structure J_P, is equal to $c_1(T(P)) + c_1(V(P))$. The following proposition (whose proof is identical to that of Proposition 7.1.1 of [7]) is a recipe to calculate $c_1(T(P))$:

Proposition 4.4. *The class in $H^2(P;\mathbb{Z})$ represented by the Euler cochain $Eul(P)$ coincides with the first Chern class of the horizontal plane field $T(P)$ of P in M.*

We define the *gleam cochain* $gl(P)$ as $gl(P) = \sum_i gl(R_i)\hat{R}_i$, where $\hat{R}_i$ is the cochain dual to R_i, the coefficient $gl(R_i)$ is the gleam of the region R_i and the sum ranges over all the regions R_i of P. Note that since the gleams can be half-integers, it is not a priori obvious that the gleam cochain represents an integer class in $H^2(M;\mathbb{Z}) \cong H^2(P;\mathbb{Z})$.

Lemma 4.5. *The gleam cochain $gl(P)$ represents in $H^2(M;\mathbb{Z})$ the first Chern class of the field of oriented 2-planes $V(P)$.*

Proof of 4.5. The normal bundle of each region of P is given by $V(P)$. Call $PV(P)$ the projectivisation of $V(P)$ and let v be a generic section of $V(P)$ on P. The number of zeros of the projection of v in $PV(P)$ on a 2-cycle is twice the number of zeros of v on the same cycle; indeed, the index of each zero of the projection of v in $PV(P)$ is the double of the index of the corresponding zero of v. By the construction of the gleam of a region, 2-times the gleam cochain represents the first Chern class of $PV(P)$. In particular, the number of zeros of the projection of v in $PV(P)$ on a 2-cycle is 2 times the evaluation of the gleam cochain on the cycle. Hence the number of zeros of v on the same cycle equals the evaluation of the gleam cochain on the cycle. $\boxed{4.5}$

Corollary 4.6. *The first Chern class of the almost complex structure on M associated with P is represented by the cochain $c_1(P) = Eul(P) + gl(P)$.*

4.1. **Comparing different almost complex structures.** Corollary 4.3 does not tell us which homotopy classes $[J]$ of almost complex structures can be reconstructed by means of suitable branched shadows of M: we prove in the subsequent subsections that the reconstruction map is surjective on the pairs $(M, [J])$. To do this, we need to "compare" different homotopy classes of almost complex structures on M. This can be performed through a shortcut based on the theory of $Spin^c$-structures.

Theorem 4.7 ([21])**.** *Let M be a smooth, compact and oriented 4-manifold admitting a shadow and equipped with an auxiliary orthogonal riemannian metric g.*

(1) *There exists a canonical bijection b between the set $\mathbb{J}$ of homotopy classes of orthogonal, positive, almost complex structures on M and the set S of $Spin^c$-structures on M.*
(2) *Fixed an arbitrary $Spin^c$-structure s on M, any other such structure s' is isomorphic to $s \otimes l$ where l is a complex line bundle over M; this equips the set of $Spin^c$-structures (and hence of homotopy classes of almost complex structures) on M with a structure of affine space over $H^2(M;\mathbb{Z})$ where $s' - s = c_1(l)$.*
(3) *To each $Spin^c$-structure s one can associate an element of $H^2(M;\mathbb{Z})$ called its Chern class and denoted by $c_1(s)$ such that if s and s' are as above then $c_1(s') - c_1(s) = 2c_1(l)$ and such that $c_1(b(J)) = c_1(J)$ for each almost complex structure J.*

Corollary 4.8. *If $H^2(M;\mathbb{Z})$ has no 2-torsion, then two almost complex structures are homotopic if and only if their Chern classes coincide.*

Proof of 4.7. All the results on $Spin^c$-structures are standard, see [21]. In that paper a construction is exhibited which associates to each pair (s, ψ) where s is a $Spin^c$-structure on M and ψ a positive spinor on it, an almost complex structure defined out of the zeros of ψ. If ψ is generic the almost complex structure is defined out of a finite set.

If M collapses onto a 2-dimensional polyhedron, the spinor can be choosen to be non-zero and each two such spinors can be connected through a path of non-zero spinors. This gives the map b^{-1} from $Spin^c$-structures into the set of almost complex structures; it is a standard fact that it is surjective since each almost complex structure J allows one to explicitly construct the $Spin^c$-structure $b(J)$; also the last statement comes directly from the construction of $b(J)$ from J. $\boxed{4.7}$

In particular, a branched shadow P can be used to reconstruct a $Spin^c$-structure on M instead of the homotopy class $[J_P]$. A direct consequence of Theorem 4.7 is that the set $\mathbb{J}$ of homotopy classes of almost complex structures on M is equipped with a structure of an affine space over $H^2(M;\mathbb{Z})$ or, by Poincaré duality, on $H_2(M, \partial M;\mathbb{Z})$ and one can calculate the "difference" $\alpha(J_2, J_1)$ between two classes $[J_1]$ and $[J_2]$ by considering the difference between the corresponding $Spin^c$-structures. More explicitly, we define $\alpha(J_2, J_1) = PD(b(J_2) - b(J_1))$, where PD stands for "Poincaré dual". The following holds:

Proposition 4.9. *Let J_1 and J_2 be in general position with respect to each other. The class $\alpha(J_2, J_1) \in H_2(M, \partial M; \mathbb{Z})$ is represented by the properly embedded surface $S = \{x \in M | J_1 = -J_2 \text{ in } x\}$ suitably oriented. The properties of α are:*

1. *$\alpha(J_2, J_1) = -\alpha(J_1, J_2)$;*
2. *$\alpha(\cdot, J_1) : \mathbb{J} \to H_2(M, \partial M; \mathbb{Z})$ is bijective for any J_1;*
3. *$\alpha(J_3, J_2) + \alpha(J_2, J_1) = \alpha(J_3, J_1)$.*

Proof of 4.9. Let us associate to each J_i a self dual 2-form $\omega(J_i)$ on M as follows. At each point $p \in M$ consider a basis of T_pM of the form $(x_1, J_i(x_1), x_2, J_i(x_2))$ and stipulate that the 2-form $\omega(J_i)$ in p is represented by $x_1 \wedge J_i(x_1) + x_2 \wedge J_i(x_2)$. The so obtained 2-form is self dual and has everywhere norm $\sqrt{2}$. The forms $\omega_i = \omega(J_i),\ i = 1, 2$ are (up to normalization) sections of the (2-dimensional) unit bundle of Λ^+M (the bundle of self dual two forms) which we will denote by $U\Lambda^+M$. Hence they define embeddings M_i of M into the total space of a 6-dimensional bundle. Since J_i are in general position with respect to each other, the set of points of M where $\omega_2 = -\omega_1$ is an orientable surface $(S, \partial S)$ properly embedded in $(M, \partial M)$. Since M is oriented, also M_i can be oriented by pulling back the orientation of M through the projection of the fiber bundle $U\Lambda^+M$ on M. Also the total space of the bundle $U\Lambda^+M$ can be oriented using the "horizontal" orientation of TM and the orientation on the fiber fixed by stipulating that, if e_1, e_2, e_3, e_4 is a positive orthonormal basis of T_p^*M, the basis of Λ^+M given by $e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4, e_1 \wedge e_4 + e_2 \wedge e_3$ is positive.

Now that all our starting objects are oriented, we equip S with an orientation depending only on the ordered pair (J_1, J_2) as follows. Note that S is the projection of the surface $-M_1 \cap M_2$, where $-M_1$ is the embedding of M in Λ^+M given by $-\omega_1$. Let us orient $-M_1 \cap M_2$ locally around a point q as follows. Let $x_1, x_2, x_3, x_4, x_5, x_6$ be a local system of coordinates of $U\Lambda^+M'$ around q such that:

1. the basis $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}$ is an oriented basis of the tangent space at q of the total space of $U\Lambda^+M$;
2. the vectors $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}$ form a positive basis of T_qM_2;
3. the vectors $\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}$ form a positive basis of $T_q(-M_1)$.

We stipulate that a positive basis of $T_q(-M_1 \cap M_2)$ is given by $\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}$. This construction yields a well defined orientation on $-M_1 \cap M_2$ and hence on S.

To each component of S, we now assign an integer index equal to ± 1. Let S_1 be one component of S and consider a small disc D transverse to S_1 and intersecting it in a single point p; we orient D so to complete the orientation of S_1 to the orientation of M. By construction $J_2 = -J_1$ on p and not on $D - \{p\}$. Locally, on D, we can identify the section of $U\Lambda^+M$ given by J_1 with the "north pole" of the sphere bundle. Then the image of D through the other section, given by $-J_2$, is a small disc surrounding the north pole. The index of S_1 is defined to be 1 if this small disc is oriented as the spherical fiber and -1 otherwise.

The so obtained class $[S] \in H_2(M, \partial M)$, which we will also call the *comparison class* between $[J_2]$ and $[J_1]$, represents the obstruction to the homotopy between J_2 and J_1 and so $[S] = \alpha(J_2, J_1)$. The properties of $\alpha(J_2, J_1)$ descend from the definition and the properties of $Spin^c$-structures recalled in Theorem 4.7. They could also be deduced by independent arguments, as, for instance, the definition of the analogue of the Pontrjagin-move for almost complex structures and surfaces in M (see [7], Chapter VI).

$\boxed{4.9}$

4.2. **Surjectivity of the reconstruction map.** Next we prove that all the homotopy classes of almost complex structures can be obtained via the reconstruction map of Corollary 4.3.

We first calculate the difference $\alpha(J_{P'}, J_P)$ when P' and P are two branched shadows of M connected by a branched move. Let us note that we can combinatorially calculate $2\alpha(J_{P'}, J_P)$: indeed, by Theorem 4.7 and the last statement of Proposition 4.9, we have $2\alpha(J_{P'}, J_P) = PD(c_1(J_{P'}) - c_1(J_P))$ and using Corollary 4.6 both $c_1(J_{P'})$ and $c_1(J_P)$ are represented by the sum of the Euler and gleam cochains of P' and P respectively. When there is no 2-torsion in $H^2(M;\mathbb{Z}) \cong H_2(M, \partial M;\mathbb{Z})$ we are done since we can divide by two the above equality. In the following lemma, we calculate the values of $[\Delta Eul] := [Eul(P')] - [Eul(P)]$, $[\Delta gl] := [gl(P')] - [gl(P)] \in H^2(M;\mathbb{Z})$, and $2\alpha(J_{P'}, J_P) \in H_2(M, \partial M;\mathbb{Z})$ when P' is obtained from P by the application of a $0 \to 2$-move or a $2 \to 3$-move.

Lemma 4.10. *If P' is obtained from P through the application of a sliding $0 \to 2$-move or $2 \to 3$-move (i.e. one of those drawn in the upper parts of Figures 6 and 7) then $[\Delta Eul] = [\Delta gl] = 2PD(\alpha(J_{P'}, J_P)) = 0$; if the $0 \to 2$ or $2 \to 3$-move is a bumping one (shown in the bottom parts of Figures 6 and 7) then $[\Delta Eul] = -2\hat{R}_4$, $[\Delta gl] = 0$ and hence $2\alpha(J_{P'}, J_P) = -2PD(\hat{R}_4)$ where R_4 is the upper-right region in our figures of the bumping moves, its orientation is given by the branching and $PD(\hat{R}_4)$ is the Poincaré dual of the element of $H^2(M;\mathbb{Z})$ which is represented by the cochain dual to R_4 (this element can be represented by a properly embedded disc intersecting R_4 transversely once).*

Proof of 4.10. The proof is a straightforward computation based on Proposition 3.8 and Corollary 4.6. As an example, we consider the calculation for the bumping $2 \to 3$-move, shown in the lower part of Figure 7. Since P and P' describe the same manifold, the presentations of $H^2(M;\mathbb{Z})$ they provide through Proposition 3.8 are equivalent, and so, in what follows, we use the presentation provided by P'.

Since the maw is fixed on the boundary of the polyhedron shown in the figure, its behavior on the boundary of the regions does not change after the move except for R_1, R_2 and R_3. For instance, before the move R_1 is the preferred region of the central edge while after the move it is the preferred region of no edge in the figure. A similar phenomenon can be observed on ∂R_2 and ∂R_3. It can be checked that in all these cases the result is the addition of a full positive or negative twist to the maw on the boundaries of the regions, so the coefficients of $\hat{R}_1$, $\hat{R}_2$ and $\hat{R}_3$ in the Euler cochain change respectively of 1, -1 and -1.

Finally, the maw on the boundary of the new region R points always outwards and hence the index of the singularity on the interior of R is 1. So the difference between the cochains

$Eul(P')$ and $Eul(P)$ is $\hat{R} + \hat{R}_1 - \hat{R}_2 - \hat{R}_3$; but since the relations in the presentation of $H^2(M;\mathbb{Z})$ induced by the edges between R_1 and R_2 and R and R_3 are respectively $\hat{R}_2 - \hat{R}_1 - \hat{R}_4$ and $\hat{R}_3 - \hat{R} - \hat{R}_4$, the total change of the Euler cohomology class is $-2\hat{R}_4$. To finish, note that the cohomology class represented by the gleam cochain is not changed by the move because the gleams of the regions involved in the move do not change and the region R has zero gleam. $\boxed{4.10}$

To perform the calculations in the case of $1 \to 2$-moves we first set up the notation. By Lemma 3.7, each $1 \to 2$-move is branchable and has 32 branched versions, one for each of the 24 initial branching of the vertex where the move acts, with some cases which split since any orientation of the region created by the move (R_7 in Figure 8) gives a branching. Up to diffeomorphisms of the ball where the move acts, we reduce to the 16 cases examined in the proof of Lemma 3.7. Using the same convention, we summarize in the following table how do the classes given in $H^2(M;\mathbb{Z})$ by the Euler cochain, the gleam cochain and the Chern class, change after a branched $1 \to 2$-move $P \to P'$ in each of the 16 cases. We use the presentation of $H^2(M;\mathbb{Z})$ given by P'.

Case	$(R_1, R_2, R_3, R_4, R_5, R_6)$	R_7	$[\Delta\mathrm{Eul}]$	$[\Delta\mathrm{gl}]$	$[\Delta c_1]$
1	$(+,+,+,+,+,+)$	$+$	0	0	0
2a	$(+,+,+,+,+,-)$	$+$	$-\hat{R}_6$	$-\hat{R}_6$	$-2\hat{R}_6$
2b	$(+,+,+,+,+,-)$	$-$	$-\hat{R}_5$	$-\hat{R}_5$	$-2\hat{R}_5$
3a	$(+,+,+,+,-,+)$	$+$	$-\hat{R}_5$	$-\hat{R}_5$	$-2\hat{R}_5$
3b	$(+,+,+,+,-,+)$	$-$	$-\hat{R}_6$	$-\hat{R}_6$	$-2\hat{R}_6$
4	$(+,+,+,+,-,-)$	$-$	0	0	0
5	$(+,+,+,-,+,+)$	$+$	$\hat{R}_4$	$-\hat{R}_4$	0
6	$(+,+,-,-,+,+)$	$+$	$\hat{R}_5$	$-\hat{R}_5$	0
7a	$(+,+,-,+,-,+)$	$+$	$-\hat{R}_4$	$-\hat{R}_4$	$-2\hat{R}_4$
7b	$(+,+,-,+,-,+)$	$-$	$-\hat{R}_2$	$-\hat{R}_2$	$-2\hat{R}_2$
8	$(+,-,+,+,-,-)$	$-$	$\hat{R}_2$	$-\hat{R}_2$	0
9	$(+,+,-,-,-,+)$	$+$	0	0	0
10	$(+,-,-,+,-,-)$	$-$	$\hat{R}_6$	$-\hat{R}_6$	0
11	$(+,-,-,+,-,+)$	$-$	0	0	0
12a	$(+,-,-,-,-,+)$	$+$	$-\hat{R}_2$	$-\hat{R}_2$	$-2\hat{R}_2$
12b	$(+,-,-,-,-,+)$	$-$	$-\hat{R}_4$	$-\hat{R}_4$	$-2\hat{R}_4$

The above computations give the value of $2\alpha(J_{P'}, J_P) = PD([\Delta c_1])$ where $[\Delta c_1] = c_1(J_{P'}) - c_1(J_P)$. Note that this value has always the form $2PD(\hat{R})$ for a suitable region R and so $PD(\hat{R})$ is a natural candidate for $\alpha(J_{P'}, J_P)$.

Theorem 4.11. *Let P and P' be two branched shadows of the same manifold connected by a $0 \to 2$, $2 \to 3$ or $1 \to 2$ branched move, and let $c_1(J_{P'}) - c_1(J_P) = 2\hat{R}$ for a suitable region R be the difference of the Chern classes of the associated almost complex structures, calculated as explained above. Then $\alpha(J_{P'}, J_P) = PD(\hat{R})$; similarly, if $c_1(J_{P'}) - c_1(J_P) = 0$ then $\alpha(J_{P'}, J_P) = 0$.*

Proof of 4.11. Let T and T' be the collapsible subpolyhedra respectively of P and P' on which the branched move acts. Let also $\pi : M \to P$ be a collapse of M onto P and $B = \pi^{-1}(T)$ be the regular neighborhood in M of T. Since $P \setminus T = P' \setminus T'$ in M, B is also a regular neighborhood of T'. Moreover since T is collapsible B is a 4-ball; let us decompose ∂B into the "vertical part" $\partial_v B = \pi^{-1}(\partial T)$ and the "horizontal" part $\partial_h B = \pi^{-1}(int(T)) \cap \partial B$; clearly $\partial B = \partial_h B \cup \partial_v B$ and $\partial T = \partial T' \subset \partial_v B$.

Since the branched move acts only near T and T', the structures J_P and $J_{P'}$ are modified only in B and the class $\alpha(J_{P'}, J_P) \in H_2(M, \partial M; \mathbb{Z})$ is represented by a properly embedded (possibly empty) oriented surface $S \subset B$ (see Proposition 4.9) with $\partial S \subset \partial_h B$. In particular, S represents a cycle in $H_2(B, \partial_h B; \mathbb{Z})$ and, through the inclusion $i : (B, \partial_h B) \hookrightarrow (M, \partial M)$, the element $\alpha(J_{P'}, J_P) \in H_2(M, \partial M; \mathbb{Z})$. Hence, to compute $\alpha(J_{P'}, J_P)$ we will compute $[S] \in H_2(B, \partial_h B; \mathbb{Z})$ (which depends only on B), and then consider its image in $H_2(M, \partial M; \mathbb{Z})$ through i_*.

We can choose a set of generators for $H_2(B, \partial_h B; \mathbb{Z})$ by considering the classes $\mathbf{R_i}$ of the "vertical discs", i.e. the discs of the form $\pi^{-1}(pt)$ where pt is a point inside the region R_i of T and orienting them in order to complete the orientations of R_i to that of M; in particular, it holds $PD([\hat{R}_i]) = i_*(\mathbf{R_i})$ (where $\hat{R}_i \in H^2(M; \mathbb{Z})$ is the cochain dual to R_i). The relations between these generators have the form $\mathbf{R_i} = \mathbf{R_j} + \mathbf{R_k}$ where R_i, R_j and R_k share an edge in T for which R_i is the preferred region.

Now, to compute $[S] \in H_2(B, \partial_h B; \mathbb{Z})$, let us study the case when M is the manifold obtained by doubling B along $\partial_v B$, whose shadow is obtained by doubling T along $T \cap \partial B_v$. In this case the theorem is true: indeed by Lemma 4.10, we need to find the class $X \in H_2(M, \partial M; \mathbb{Z})$ such that $2X = 2PD(\hat{R})$ but it is easy to check that $i_* : H_2(B, \partial_h B; \mathbb{Z}) \to H_2(M, \partial M; \mathbb{Z})$ is an isomorphism and that $H_2(M, \partial M; \mathbb{Z})$ has no 2-torsion so that we can divide by two and get $X = PD(\hat{R})$. But clearly it holds $PD(\hat{R}) = i_*(\mathbf{R})$ and so $[S] = \mathbf{R} \in H_2(B, \partial_h B; \mathbb{Z})$. Similarly if $c_1(J_{P'}) - c_1(J_P) = 0$ then $X = [S] = 0$.

Now that we found $[S]$, if we consider a generic manifold M containing B, letting again $i : B \hookrightarrow M$ be the inclusion, we have that $\alpha(J_{P'}, J_P)$ is the class represented by $i_*([S]) = i_*(\mathbf{R}) \in H_2(M, \partial M; \mathbb{Z})$, and this class equals $PD(\hat{R})$ since, by construction, it is represented by a properly embedded vertical disc intersecting geometrically R.

$\boxed{4.11}$

Theorem 4.12. *The refined reconstruction map from branched shadows of M to pairs $(M, [J])$ with $[J]$ homotopy class of positive almost complex structures on M, is surjective.*

Proof of 4.12. Fix an auxiliary riemannian metric g on M. We limit ourselves to give an idea of the proof since it is an adaptation of that of Theorem 4.6.4 of [7]. Let $[J]$ be as in the statement, P be a branched shadow of M (which exists by Proposition 3.4) and let $[J_P]$ the homotopy class associated to P by Corollary 4.3. We want to show that there exists a branched shadow P' such that $[J_{P'}] = [J]$. If $[J] = [J_P]$ we are done. Otherwise, let us write the Poincaré dual of $\alpha(J, J_P)$ as $\sum_i k_i \hat{R}_i$ where $\hat{R}_i$ is the cochain dual to the region R_i of P (using the presentation of $H^2(M; \mathbb{Z})$ given by P). The idea of the proof is to apply a suitable sequence of $0 \to 2$ and $2 \to 3$-moves to P to modify it and get a new branched shadow of M carrying the homotopy class $[J]$. For that, it is sufficient to exhibit a sequence that "decreases" the difference α between J and J_P by a chain whose Poincaré dual is cohomologous to one of the form $\pm\hat{R}_i$ for any region R_i of P. For instance, suppose that we want to get a shadow carrying a homotopy class which differs from J_P by $-\hat{R}_i$, and that the boundary of the region R_i contains an edge e_j whose preferred region is not R_i; then, by Theorem 4.11 a self $0 \to 2$-move is a bumping move producing a branched shadow P' such that $PD(\alpha(J_{P'}, J_P)) = -\hat{R}_i$. If for all the edges in ∂R_i the preferred region is R_i, let e_j be an edge of ∂R_i and R_k and R_l be other regions containing it. Since $\hat{R}_i = \hat{R}_k + \hat{R}_l$ it is sufficient to apply the above moves both to R_k and R_l to get the wanted shadow. The sequence of moves producing a difference of the form $+\hat{R}_i$ is more complicated: we refer to [7] for a complete account and give here a sketch of the construction. The idea is to first create a region R'_i such that $\hat{R}'_i = -\hat{R}_i$ and then to apply the preceding arguments to R'_i. One can suppose that there is an edge in ∂R_i along which R_i is not the preferred region. Then, to create R'_i, one first applies a local $0 \to 2$-move on that edge to slide R_i over a small disc $D \subset R_i$; it is not hard to check that $\hat{D} = 0$. Then, letting R_i slide over a disc $D' \subset D$ through another $0 \to 2$-move, one sees that $\hat{D}' = -\hat{R}_i$ and so one can put $R'_i = D'$. $\boxed{4.12}$

5. Branched shadows and complex structures

Definition 5.1. An almost complex structure J on a smooth 4-manifold M is said to be *integrable* or *complex* if for each point p of M there is a local chart of M with values in $\mathbb{C}^2$ transforming J into the complex structure of $\mathbb{C}^2$.

5.1. **Branched shadows in complex manifolds.** In this subsection, supposing that M is equipped with an integrable structure J, we adapt to the case of branched shadows a series of classical definitions and results of Bishop ([2]), Chern and Spanier ([12]), Lai ([23]) and Harlamov and Eliashberg ([20]) concerning invariants of embeddings of real surfaces in complex manifolds.

Let P be a branched shadow embedded in M. Up to perturbing the embedding of P through a small isotopy we can suppose that there is only a finite number of points $p_1, \dots, p_n$ and $q_1, \dots, q_m$, contained in the regions of P where $T_{p_i}P$ (resp. $T_{q_j}P$) is a complex plane such that the orientations induced by the branching of P and by the complex structure coincide (resp. do not coincide).

Definition 5.2. The points $p_1, \dots, p_n$ are called *positive complex* points of P or simply *positive* points. Analogously, the points $q_1, \dots, q_m$ are called *negative complex* points of P or *negative* points. All the other points of P are called *totally real.*

To each complex point p of a region R_i of P we can assign an integer number called its *index*, denoted $i(p)$, as follows. Fix a small disc D in R_i containing p and no other complex point and let N be the radial vector field around p. The field $J(N)$ on ∂D is a vector field transverse to P since no point on $D - p$ is complex. Let $\pi(J(N))$ be the projection of this field onto the normal bundle of D in M. Since D is collapsible, this bundle is trivial and we can count the number $\nu(p)$ of twists performed by $\pi(J(N))$ while following ∂D (D and M are oriented). The index of p is: $i(p) = \nu(p) + 1$. Moreover, we define $\nu(R_i)$ as the sum over all the complex points p of R_i of $\nu(p)$.

Up to a small perturbation by an isotopy of the embedding of P in M we can assume that all the indices of the complex points of P are equal to ± 1.

Definition 5.3. A complex point p of P whose index is equal to 1 is *elliptic*, if its index is -1 it is *hyperbolic*.

We define the "Chern index" $c_1(p)$ of a complex point p of a region R_i of P as follows. Let D and N be as above and complete N on ∂D to a basis of TD by using the field $T = T\partial D$ tangent to the boundary of D. The pair of fields (N, T) gives a basis of TD in each point q of ∂D, and, since no such point is complex, they can be completed to a positive complex basis of $T_q M$ given by $(N, J(N), T, J(T))$. Let now $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial w}$ be two vector fields defined on a neighborhood of D in M such that $(\frac{\partial}{\partial z}, \frac{\partial}{\partial w})$ is pointwise a complex basis of TM. Then, on each point q of ∂D we can compare the two complex bases given by $(N + J(N), T + J(T))$ and $(\frac{\partial}{\partial z}, \frac{\partial}{\partial w})$ by considering the determinant det_q of the change of basis from the latter to the former basis. The value of the index of det_q around 0 in $\mathbb{C}$ while q runs across ∂D according to the orientation of D, is defined to be $c_1(p)$. We define $c_1(R_i)$ as the sum of $c_1(p)$ over all the complex points p of R_i.

The following result was proved for the case of surfaces in $\mathbb{C}^2$ by Chern and Spanier ([12]) and in a general framework by Lai [23] (see also [1]); its proof in the case of surfaces with boundary is a straightforward adaptation of their techniques so we omit it.

Theorem 5.4. *Let R_i be a surface with boundary contained in a complex manifold M such that ∂R_i does not contain complex points and let $I^+(R_i) = \sum_{i=1,\ldots,n} i(p_i)$ and $I^-(R_i) = \sum_{j=1,\ldots,m} i(q_j)$, where p_i (resp. q_j) are the positive (resp. negative) complex points of R_i. Then the following holds:*

$$I^+(R_i) = \frac{1}{2}(\chi(R_i) + \nu(R_i) + c_1(R_i))$$

$$I^-(R_i) = \frac{1}{2}(\chi(R_i) + \nu(R_i) - c_1(R_i))$$

We will need the following, due to Harlamov and Eliashberg [20]:

Theorem 5.5 (Annihilation theorem)**.** *Let S be an oriented real surface embedded in a complex manifold M and let p_1 and p_2 be two complex points of S of the same sign (i.e. both positive or negative) and belonging to the same connected component of S. Let α be an arc in S connecting p_1 and p_2 and containing no other complex point of S and suppose that $i(p_1) = 1 = -i(p_2)$. There exists on S a small isotopy $\phi_t,\ t \in [0, 1]$ which is the identity out of a small neighborhood $U(\alpha)$ of α and such that $\phi_1(U(\alpha))$ contains no complex points.*

The following is the analogous in the world of shadows of Theorem 5.5:

Lemma 5.6. *Let R_i, R_j and R_k be three regions of P adjacent along a common edge $e \in Sing(P)$ so that R_i is the preferred region of e. There exists an isotopy $\phi_t : P \to M,\ t \in [0, 1]$*

whose support is contained in a small ball B around the center of e such that $\phi_1(B \cap P)$ contains three more complex points p_i, p_j and p_k all of the same sign, respectively in R_i, R_j and R_k such that $i(p_i) = \pm 1$, $i(p_j) = \mp 1$ and $i(p_k) = \mp 1$.

Proof of 5.6. By modifying with a C^0-small isotopy the position of R_i in a neighborhood of an interior point and applying Theorem 5.5, we can assume that near the edge e there is a pair of complex points p_i and p'_i in R_i of opposite indices. Roughly speaking, we now "slide R_j over" p'_i and show that both on R_k and R_j two complex points are created by this sliding.

Consider an arc γ embedded in R_i whose endpoints are p'_i and a point q on e. Let D_j and D_k be small disc neighborhoods of q respectively in R_j and R_k and let D' a neighborhood of $D_j \cup \gamma$ in $R_i \cup R_j$ (since P is branched, D' is a smooth disc). Consider the isotopy that fixes every point of $P \setminus (D' \cup D_k)$ and moves D_j by letting it slide over γ and pass over p'_i. This isotopy acts in a 4-ball and changes only the regions R_i and R_k along their borderlines. The point p'_i passes from one side to the other one, becoming a complex point p_k in R_k; we are left to prove that a J-complex point has been created on R_j having the same index as p_k. This is proved by using Theorem 5.4. Consider the image D'' of the bigger disc D' after the isotopy: it is a disc whose boundary is made only of totally real points and $i(D'') = i(D')$ since $\partial D'' = \partial D'$ by construction. Then we finish by observing that $i(D') = i(p'_i)$ since D' by construction contains only p'_i. $\boxed{5.6}$

The above lemma suggests the following:

Definition 5.7. The *positive index* and *negative index* cochains of P, denoted respectively $I^+(P)$ and $I^-(P)$ are the 2-cochains given by $\Sigma_i I^{\pm}(R_i)\hat{R}_i$, where i ranges over all the regions of P.

Theorem 5.8. *The cohomology classes $[I^{\pm}(P)] \in H^2(M;\mathbb{Z})$ are invariants of the embedding of P in M up to isotopy.*

Proof of 5.8. Let $\phi_t, t \in [0,1]$ be an isotopy of P in M so that $\phi_0 = id$ and $\phi_1(P) = P'$. Up to slightly perturbing ϕ, we can suppose that the following holds:

(1) the number of creation/annihilations of complex points with opposite indices during the isotopy is finite;
(2) no creation/annihilation of complex points at time t happens on $\phi_t(Sing(P)), t \in [0,1]$;
(3) the complex points of $\phi_t(P)$ cross $\phi_t(Sing(P))$ only a finite number of times and transversally in the interior of the edges of $\phi_t(Sing(P))$.

Then we have to check the invariance of the classes $I^{\pm}(P)$ under two kinds of catastrophes: when a pair of complex points is created or annihilated in the interior of a region of P and when a complex point crosses $\phi_t(Sing(P))$. The invariance is trivial in the first case, by the very definition of $I^{\pm}(P)$, while it follows immediately from Lemma 5.6 in the second case. $\boxed{5.8}$

We now compare the almost complex structure carried by a branched shadow P with the ambient complex structure J.

Proposition 5.9. *The following holds: $\alpha(J, J_P) = PD([I^-(P)])$ where PD denotes the Poincaré dual and $[I^-(P)]$ is the cohomology class represented by the negative index cochain of P. Moreover, if $\alpha(J, J_P) = 0$ there exists an isotopy $\phi_t,\ t \in [0,1]$ of P in M such that ϕ_0 is the identity and $\phi_1(P)$ contains no negative J-complex points.*

Proof of 5.9. Up to homotopy, we can suppose that J_P is generic with respect to J and represent $\alpha(J, J_P)$ as an oriented properly embedded surface S in $(M, \partial M)$ whose components are equipped with indices equal to ± 1. Consider the 2-cochain $\beta = \sum n_i \hat{R}_i$ where i ranges over all the regions R_i of P and n_i is equal to the sum of the indices of all the intersection points between S and R_i. By the construction of S (see Proposition 4.9), $P \cap S$ is the set of negative J-complex points of P equipped with indices equal to those of the corresponding components of S, hence $\beta = I^-(P)$. On the other side, by Proposition 4.9, $\beta = PD(\alpha(J, J_P))$.

If $\alpha(J, J_P) = 0$, then $[I^-(P)] = 0$ in $H^2(M; \mathbb{Z})$ and its expression as a cochain can be reduced to 0 by means of the relations given by the edges of P (see Proposition 3.8). To conclude it is sufficient to prove that, given an arbitrary edge e of $Sing(P)$ inducing a relation of the form $\hat{R}_i = \hat{R}_j + \hat{R}_k$ on the three regions containing it, it is possible to find an isotopy ϕ_e of P in M which has the effect of eliminating a point p_i' of index ± 1 on R_i and creating a pair of points of the same index p_k and p_j, respectively on R_k and on R_j. This is the statement of Lemma 5.6. $\boxed{5.9}$

5.2. **Integrable representatives of almost complex structures.** This subsection is devoted to prove Theorem 1.2:

Proof of 1.2. Let $[J]$ be a homotopy class of almost complex structures on M. By Theorem 4.12 there exists a branched shadow P of M such that the homotopy class of the almost complex structure J_P associated to P is $[J]$. To construct an integrable representative of $[J]$ we now immerse M in a suitable complex manifold and pull back its complex structure to M through the immersion (the pull-back through a local diffeomorphism of a complex structure is a complex structure). We then prove that the so obtained complex structure belongs to $[J]$. The branched shadow P is crucial both in the construction of the immersion of M and in the control of the homotopy class of the complex structure.

First, since $Sing(P)$ is a graph, a regular neighborhood N of it in P can be embedded in $\mathbb{C}^2$ equipped with coordinates (z, w). Moreover, we can suppose that the projection over the plane $w = 0$ has surjective and positive differential (i.e. the image of the orientation of TP coincides with that of the plane $w = 0$) in every point of N. This implies that the image of N in $\mathbb{C}^2$ contains no negative complex points. Let us now extend arbitrarily the so constructed immersion of N in $\mathbb{C}^2$ to an immersion of the whole P: this can be done since each boundary component of N coincides with a boundary component of a region of P and each loop in $\mathbb{C}^2$ bounds an immersed disc.

That way we construct an immersion of P in $\mathbb{C}^2$ but we have to solve two problems:

(1) modify the immersion to delete all the negative complex points of the image;
(2) extend the new immersion to the whole M.

We solve both problems with the same technique, but to use it we first fix the notation. Up to isotopy of the immersion we can suppose that for each region R_i of P, the image of R_i contains two small discs D_i^+ and D_i^- parallel to the $w = 0$ plane and oriented by R_i respectively in the positive and in the negative way with respect to the complex orientation of the plane. Let p_i^+ and p_i^- points in D_i^+ and D_i^- respectively and let C' be the complex 4-manifold obtained by deleting small neighborhoods of $p_i^\pm$ of the form $p_i^\pm + \{(z,w)|||z|| \leq \epsilon_i, ||w|| \leq \epsilon_i\}$ from $\mathbb{C}^2$. If we glue back to C' along ∂D_i^- a 2-handle equipped with a complex structure of the form $\{(z,w)|\ ||z|| \leq \epsilon_i, ||w|| \leq \frac{\epsilon_i}{2}\}$ though the map $(z,w) \to p_i^- + (z, z^{n_i}w)$ and acting analogously for D_i^+ (using an exponent m_i), we obtain an immersion of P in a new complex manifold by sending $D_i^\pm$ to the cores of the 2-handles. If we slightly perturb this new immersion we see that the points $p_i^\pm$ are isolated complex points (respectively of positive and negative type) whose indices are m_i and n_i. The above recipe allows us to modify the immersion in small neighborhood of points of each region (changing also the codomain of the immersion) and change arbitrarily the total indices $I^\pm$ of the image of each region. Then, we can fix $I^-(R_i)$ to be zero on each region, and, after applying a suitable number of times Theorem 5.5, we can suppose that each R_i contains no negative points.

Let us now solve the second problem; the pull back of a neighborhood of the image of P through the so obtained immersion gives a thickening of P but a priori it is not true that such a thickening coincides with M. It is so if the gleams induced on the regions of P by this thickening (recall the definition of the gleam) are the same as those induced by M on P. Changing suitably the coefficients m_i in the above construction we can change arbitrarily the gleams induced on each region R_i without adding negative complex points; indeed the gleam is a relative version of the self-intersection number of each region and adding a twist to the attaching map of the 2-handle around p_i^+ changes by one this self intersection number.

Then, up to suitably reattaching to C' all the 2-handles corresponding to the points $p_i^\pm$, the immersion of P can be extended to an immersion of M in C'' and the pull back of the complex structure of C'' to M is a complex structure such that P contains no negative point. By Proposition 5.9 we conclude that the so obtained complex structure on M belongs to the homotopy class carried by P and we are done. $\boxed{1.2}$

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