COMPLEXITY OF 4-MANIFOLDS

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Abstract. We define and study a notion of complexity for smooth, closed and orientable 4-manifolds. This notion, based on the theory of Turaev shadows, represents the 4-dimensional analogue of Matveev's complexity of 3-manifolds. We classify complexity 0 and 1 four manifolds and provide examples of manifolds of higher complexity.

Contents

1. Introduction

A natural notion of complexity of a PL n-dimensional manifold is the minimal number of highest-dimensional simplexes in a triangulation of the manifold. Such a complexity is an integer valued function and is finite (for each $k \geq 0$ there are only finitely many manifolds whose complexity is up to k). In order to find all the n-manifolds of complexity k, one has to identify all the possible ways of gluing k copies of the n-simplex such that the link of each point is a $n-1$-sphere. Hence, producing lists of low-complexity n-manifolds can be a difficult task if $n \geq 3$ because of the sphere recognition problem. In dimension 3, S. Matveev ([13]) defined an alternative notion of complexity which, for "most" 3-manifolds is equivalent to the above defined one. Matveev's complexity is based on a combinatorial description of 3-manifolds by means of 2-polyhedra (their "spines") and turns out to be strictly related to the topological properties of the manifolds: for instance, it is additive under connected sums and is finite when restricted to irreducible manifolds. Its combinatorial nature, makes it a computable invariant: using the stratification of the set of 3-manifolds induced by Matveev's complexity it is possible to produce a list of 3-manifolds up to complexity 10 by means of computer-based computations ([16]). More than this, the new techniques and tools set up ([14]) to study the topology of 3-manifolds in order to produce these lists, allowed the creation of computer programs "recognizing" 3-manifolds ([15]).

On the other side, the existence of "exotic" spaces makes smooth topology of 4-manifolds an intriguing and still rather mysterious subject. It is our hope that a combinatorial "complexity-based" approach could produce new examples of 4-manifolds, sufficiently "simple" to be studied directly. Hence, we define a notion of complexity of 4-manifolds based on the theory of Turaev's shadows ([18]), which represents an analogue in dimension 4 of Matveev's complexity. Roughly speaking, shadows of 4-manifolds can be viewed as simple polyhedra equipped with integer colorings on the regions, which can be canonically thickened to smooth (or, equivalently, PL) 4-manifolds. To clarify the reasons why we use shadows instead of triangulations in order to define a complexity of closed 4-manifolds, let us note that a triangulation contains a full description of a handle decomposition of the manifold itself, while it is known that the union of the handles of index up to 2 is sufficient to reconstruct the manifold. Hence, in a sense, the information contained in a triangulation is redundant; on the contrary, it can be shown that a shadow of a 4-manifold encodes combinatorially only the union of handles of index up to 2. Moreover, as in the 3-dimensional case with Matveev's complexity, it is straightforward to prove that shadow complexity is sub-additive under connected sum; whilst the same is not obvious a priori for the triangulation-based complexity.

In Section 2 we recall the basic definitions and results on shadows which we will need later; no new result is proved in that section. In Section 3, we introduce two notions of complexity of a closed 4-manifold X: the *complexity*, denoted $c(X)$ and the *special complexity*, denoted $c^{\mathrm{sp}}(X)$. The former represents the direct analogue of Matveev's complexity and has the drawback that infinitely many 4-manifolds have complexity 0: this is related to the problem of restricting to irreducible (in a smooth sense!) manifolds; further comments on these aspects will be provided in Section 3. On the other-side special complexity, obtained by restricting the set of shadows used to encode 4-manifolds, turns out to be finite. In particular we prove the following (Theorem 3.10 below):

Theorem 1.1. *If a closed, smooth and orientable 4-manifold X has 0 special complexity then X is diffeomorphic to one of the following manifolds: $S^4, \mathbb{CP}^2, \overline{\mathbb{CP}}^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, $S^2\times S^2$, $\mathbb{CP}^2\#\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2\#\overline{\mathbb{CP}}^2$. Moreover, there are no closed 4-manifolds with special complexity 1.*

It is interesting to stress that, because of Freedman's Theorem, from the point of view of classification up to homeomorphism, the above statement is not surprising since low complexity special polyhedra carry only low-rank homology. So, the non-trivial content of the above result is given by the fact that it proves that no exotic versions of the complexity 0 manifolds exist in complexity 1. In Subsection 3.3 we provide examples of higher-complexity and estimate the complexity of the elliptic surfaces $E(n)$. We then use this to provide upper estimates for the answer to the following:

Question 1.2. *What is the minimal (special) complexity of a pair of homeomorphic but non-diffeomorphic 4-manifolds with/without boundary?*

where the complexity of a pair is defined as the maximum between the complexities of the two manifolds composing the pair.

Acknowledgments. I wish to express all my gratitude to Bruno Martelli with whom many of the ideas and results of the present paper were created and discussed.

2. A crash course on shadows of 4-manifolds

In this section we recall the basic definition and results about shadows; no new result is proven. For a more detailed account, see [18] and [3].

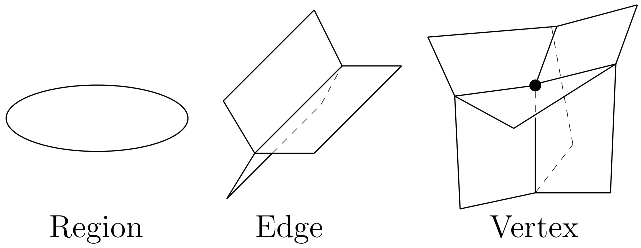

FIGURE 1. The three local models of a simple polyhedron.

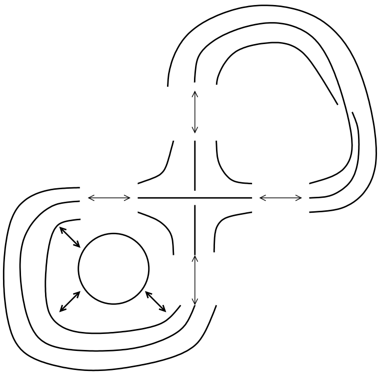

FIGURE 2. How to build up a special polyhedron from its local models.

2.1. **Simple polyhedra.**

Definition 2.1. A simple polyhedron P is a 2-dimensional CW complex whose local models are those depicted in Figure 1; the set of points whose neighborhoods have models of the two rightmost types is a 4-valent graph, called *singular set* of the polyhedron and denoted by $Sing(P)$. The connected components of $P - Sing(P)$ are the *regions* of P. A simple polyhedron whose regions are all discs is called *special*[1]. The *complexity* of a simple polyhedron P, denoted $c(P)$, is its number of vertices.

Standard polyhedra can be described in a combinatorial way by decomposing them into the blocks of Figure 1. One can always "build up" a special polyhedron as exemplified in Figure 2: the central block corresponds to the rightmost block of Figure 1, the curved blocks to the central one of Figure 1 and the regions are discs glued along the boundary of the resulting polyhedron. The resulting diagram, unambiguously defines the initial special polyhedron, but different diagrams could encode the same polyhedron. In Figures 7 and 8, we draw all the possible special polyhedra having at most one vertex (in the figures, discs are to be glued along the boundary components of the polyhedra in order to get special polyhedra).

2.2. **Decorations on polyhedra.** We describe now the basic decorations we need in order to thicken to a 4-manifold a special polyhedron P, for a more detailed account see [18]. Let us denote $\frac{\mathbb{Z}}{2}$ the group of integer multiples of $\frac{1}{2}$. There are two canonical *colorings* on the regions of P, i.e.

[1]According to our definition a polyhedron can be special even if it does not contain any vertices.

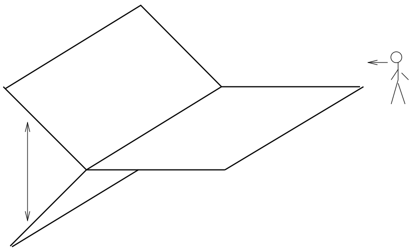

FIGURE 3. The picture sketches the position of the polyhedron in a 3-dimensional slice of the ambient 4-manifold. The direction indicated by the vertical double arrow is the one along which the two regions touching the horizontal one get separated.

assignments of elements of $\mathbb{Z}_2$ or $\frac{\mathbb{Z}}{2}$ (the integer multiples of $\frac{1}{2}$), the second depending on a flat embedding of P in an oriented 4-manifold. They are:

The $\mathbb{Z}_2$-gleam of P, constructed as follows. Let D be a (open) region of P and $\overline{D}$ be the natural compactification of the (open) surface represented by D. The embedding of D in P extends to a map $i : \overline{D} \to P$ which is injective in $int(\overline{D})$, locally injective on $\partial\overline{D}$ and which sends $\partial\overline{D}$ into $Sing(P)$. Using i we can "pull back" a small open neighborhood of D in P and construct a simple polyhedron $N(D)$ collapsing on $\overline{D}$ and such that i extends as a local homeomorphism $i' : N(D) \to P$ whose image is contained in a small neighborhood of the closure of D in P. When i is an embedding of $\overline{D}$ in P, then $N(D)$ turns out to be homeomorphic to a neighborhood of D in P and i' is its embedding in P. In general, $N(D)$ has the following structure: each component of $\partial\overline{D}$ is glued to the core of an annulus or of a Möbius strip and some small discs are glued along half of their boundary on segments which are properly embedded in these annuli or strips and cut transversally once their cores. We define the $\mathbb{Z}_2$-gleam of D in P as the reduction mod 2 of the number of Möbius strips used to construct $N(D)$. This coloring only depends on the combinatorial structure of P.

The gleam of P, constructed as follows. Let us now suppose that P is flat in an oriented 4-manifold M, with D, $\overline{D}$ and $i : \overline{D} \to P$ as above. Pulling back through i a small neighborhood of $i(N(D))$ in M, we obtain a 4-dimensional oriented neighborhood B of $N(D)$ over which we fix an auxiliary riemannian metric. Since $N(D)$ is locally flat in B, $N(D) - D$ well defines a line normal to $\overline{D}$ in B along $\partial\overline{D}$ and hence a section of the projectivized normal bundle of $\overline{D}$ (see Figure 3). Let then $gl(D)$ be equal to $\frac{1}{2}$ times the obstruction to extend this section to the whole $\overline{D}$; such an obstruction is an element of $H^2(\overline{D}, \partial\overline{D}; \pi_1(S^1))$, which is canonically identified with $\mathbb{Z}$ since B is oriented. Note that the gleam of a region is integer if and only if its $\mathbb{Z}_2$-gleam is zero.

Using the fact that the $\mathbb{Z}_2$-gleam is always defined, Turaev generalized [18] the notion of gleam to non-embedded polyhedra as follows:

Definition 2.2 (Gleam)**.** A *gleam* on a simple polyhedron P is a coloring on the regions of P with values in $\frac{\mathbb{Z}}{2}$ such that the color of a region is integer if and only if its $\mathbb{Z}_2$-gleam is zero.

2.3. **The canonical thickening procedure.** We now describe how any simple polyhedron equipped with gleams (P, gl) can be canonically thickened to a smooth 4-manifold collapsing on it. From now on, all the 4-manifolds will be smooth, compact and orientable and all the polyhedra will be flatly embedded unless explicitly stated. Let P' be the the regular neighborhood of $Sing(P)$ in P; when P is special, P' is obtained by puncturing P once along each region. To thicken P:

(1) Thicken P' to a (possibly non-orientable) 3-manifold L collapsing on it.
(2) Thicken L to an oriented 4-manifold H made up only of 0 and 1-handles.
(3) Glue suitable blocks to H corresponding to the regions of P.

Step 1. To thicken P' and get L, glue copies of the two rightmost blocks of Figure 4 according to the combinatorics of P'. The result is a pair (L, P') where P' is a properly embedded copy of P' in L.

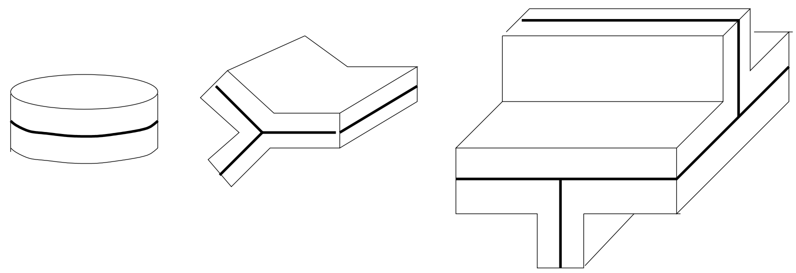

FIGURE 4. In this picture we show the blocks used to thicken a polyhedron to a 3-manifold.

Step 2. To thicken L and get H, one takes the total space of the determinant fiber bundle of L: for instance if L is orientable, $H = L \times [-1, 1]$. More in general, fix an arbitrary orientation on each of the blocks of Figure 4 and glue their products with $[-1, 1]$ using the attaching maps of L along the blocks $\times\{0\}$ and gluing the fibers by multiplying them by -1 iff the gluing maps between the 3-dimensional blocks are orientation preserving.

The resulting manifold H is canonically oriented (it admits an orientation reversing diffeomorphism), collapses over P' (which is properly embedded in it), and so, in particular is made of 0 and 1-handles. Moreover $\partial P'$ is a link in ∂H and has a canonical framing induced by its regular neighborhood in $\partial L \subset \partial H$. Indeed such a neighborhood is a union of bands collapsing on $\partial P'$ so that any curve running parallel to a component c of $\partial P'$ can be described by an integer if the neighborhood of c in ∂L is an annulus and by an half-integer otherwise.

Step 3. To each region R_i we associate the block $R_i \times D^2$ (if R_i is not orientable one chooses the twisted disc bundle over R_i, which is unique since R_i collapses on a graph). Then we glue it along $\partial R_i \times D^2$ on ∂H, by sending $\partial R_i \times \{0\}$ into the corresponding component of $\partial P'$. The gluing map is then completely described once one determines how many twists the image of the framing $\partial R_i \times \{1\}$ performs with respect to the framing of $\partial P' \subset \partial H$, and this is specified by the gleam of R_i.

The above thickening procedure proves part of the following:

Theorem 2.3 (Turaev [18])**.** *Let P be a simple polyhedron and P' be the regular neighborhood of $Sing(P)$ in P. It is possible to canonically thicken P' to an oriented 4-manifold $M_{(P',\emptyset)}$ composed of 0 and 1-handles in which P' is locally flat and properly embedded. If P is equipped with gleams gl, it is possible to extend the thickening to P in a canonical way obtaining a flat embedding in an oriented 4-manifold $M_{(P,gl)}$ collapsing on P. Moreover, if P is embedded in a 4-manifold M, and gl is the gleam induced on P by its embedding (see Subsection 2.2) then $M_{(P,gl)}$ is diffeomorphic to a neighborhood of P in M.*

Example 2.4. If P is a spine of an orientable 3-manifold N, its mod 2 gleam is zero. By performing the construction above, using as gleam on P the 0 gleam over all the regions, we get the manifold $N \times [-1, 1]$.

Example 2.5. To construct a surgery presentation of the pair $(\partial M_{(P',\emptyset)}, \partial P')$ it is sufficient to start from a diagram of P constructed as explained in Subsection 2.1, choose a maximal tree T in $Sing(P) = Sing(P')$, and encircle with 0-framed meridians all the three-tuples of strands running over edges not belonging to the tree (see Figure 5). It is remarkable that the choice of the over/under crossings in the construction does not affect the resulting pair. If P is special, $\partial M_{(P,gl)}$ is then obtained by integral Dehn surgery over the so-constructed pair.

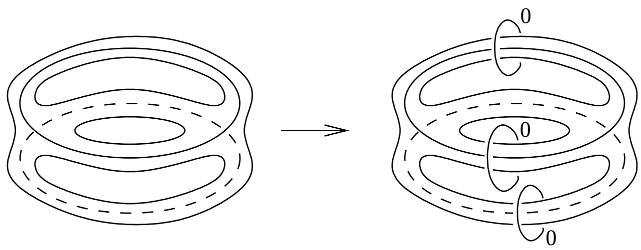

FIGURE 5. How to pass from a diagram of P' to a surgery presentation of $(\partial M_{(P',\emptyset)}, \partial P')$.

Remark 2.6. All the 4-manifolds obtained by thickening the polyhedra equipped with gleams as in Theorem 2.3 are 4-handlebodies, i.e. admit a handle decomposition containing no handles of index greater than 2. It can be shown that also the reverse holds: any 4-handlebody can be obtained by applying Theorem 2.3 to a suitable polyhedron equipped with gleams (see [4]) .

Definition 2.7 (Shadows of 4-handlebodies)**.** A polyhedron equipped with gleam (P, gl) is a *shadow* of a 4-manifold M if M is diffeomorphic to the thickening $M_{(P,gl)}$ of (P, gl) obtained through Theorem 2.3.

2.4. **Shadows of closed 4-manifolds.** By Remark 2.6, shadows can be used to describe combinatorially only a subset of all the smooth 4-manifolds not including closed ones. To obviate to this apparent weakness of the theory, let us recall the following result due to F. Laudenbach and V. Poenaru ([11]):

Theorem 2.8. *Let M be an oriented, smooth and compact 4-manifold with boundary equal to S^3 or to a connected sum of copies of $S^2 \times S^1$. Then, up to diffeomorphisms, there is only one closed, smooth and oriented 4-manifold obtained by "closing M", that is, by attaching to M some 3 and 4-handles.*

Roughly speaking, the above result states that when a manifold is "closable", then it is in a unique way. This allows us to describe all the closed 4-manifolds by means of polyhedra with gleams: given a closed manifold equipped with an arbitrary handle decomposition, considering the union of all handles of index strictly less than 3 we get a new manifold M which admits a shadow and can be described combinatorially as explained in Subsection 2.3. The initial manifold can be then uniquely recovered from M because of Theorem 2.8. With a slight abuse of notation, we then give the following definition:

Definition 2.9 (Shadows of closed manifolds)**.** A polyhedron with gleams (P, gl) is a *shadow* of a closed 4-manifold X if and only if X can be obtained by attaching 3 and 4-handles to the 4-manifold $M_{(P,gl)}$ obtained from P through the reconstruction map of Theorem 2.3.

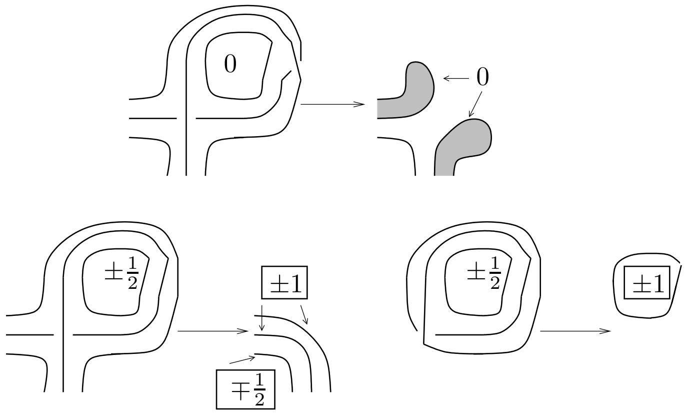

FIGURE 6. Two simplifying tricks removing a vertex.

Hence a necessary and sufficient condition for a pair (P, gl) to be a shadow of a closed 4-manifold, is that $\partial M_{(P,gl)}$ is either S^3 or a connected sum of copies of $S^2 \times S^1$.

We will often use the three simplifying moves of Figure 6 whose effect on a polyhedron P is to produce a simpler polyhedron (possibly with boundary and hence retractible on some sub-polyhedron) whose thickening is diffeomorphic to $M_{(P,gl)}$. The first one produces a region with boundary which can then be collapsed, the remaining two region are locally capped with two 0-gleam discs. The other two, delete a region with gleam $\pm\frac{1}{2}$ changing the gleams of the neighboring regions by the amounts prescribed in the rectangular boxes. The idea of the proof of the fact that these moves do not change the 4-thickening of the polyhedra is based on the observation that the local patterns to which the moves apply may be viewed as the mapping cylinders of the projections into a disc $D^2 \subset B^4$ of some strands of links contained in $S^3 = \partial B^4$; stated using Turaev's notation [18], these patterns are the *shadows projections in* D^2 *of the strands of the links.* Then, the first move corresponds to the application of the second Reidemeister-move to the strands while the other two are instances of the first Reidemeister-move.

3. Complexity of 4-manifolds

Definition 3.1 (Complexity of closed manifolds)**.** Let X be a closed, orientable, smooth 4-manifold. The *c*omplexity of X, denoted $c(X)$, is the least number of vertices in a shadow of X.

The above definition is quite natural and represents the straightforward translation to the 4-dimensional case of Matveev's complexity of 3-manifolds ([13]), based on spines. One of its fundamental properties is indeed shared by this notion:

Proposition 3.2. *Complexity is sub-additive under connected sum, that is if* X_1 *and* X_2 *are closed* 4 *manifolds then* $c(X_1 \# X_2) \leq c(X_1) + c(X_2)$.

Proof of 3.2. Let (P_i, gl_i), $i = 1, 2$ be shadows of X_i. Connecting them through an arc whose endpoints are in the interior of two regions and then "pushing our fingers along the arc", we produce a new (connected) shadow, called $P_1 + P_2$ from Turaev ([18]) and containing $c(P_1) + c(P_2)$ vertices. It is not difficult to guess what the gleam of P should be and to prove then that $M_{(P_gl)} = M_{(P_1,gl_1)} \#_\partial M_{(P_2,gl_2)}$ (where $\#_\partial$ is boundary connected sum), so that closing $M_{(P,gl)}$ produces $X_1 \# X_2$. $\boxed{3.2}$

We will prove that $\mathbb{CP}^2$ has 0-complexity; infact, it can be easily proved that any product $F \times S^2$ or $F \tilde{\times} S^2$ with F orientable surface or $F \tilde{\times} S^2$ (with F non orientable) has complexity 0. As a consequence, the following holds:

Corollary 3.3. *There are infinitely many non-diffeomorphic* 4*-manifolds of complexity* 0*.*

Remark 3.4. The fact that $c(\mathbb{CP}^2) = 0$ also implies that complexity cannot be additive under connected sum: indeed, for each closed 4-manifold X, there exists an integer k such that $X \# k\mathbb{CP}^2$ is diffeomorphic to $n\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2$ for some n, m, and $c(n\mathbb{CP}^2 \# m\overline{\mathbb{CP}}^2) = 0$.

We stress here that the non-finiteness above described is common to Matveev's complexity. Indeed, in dimension 3, there are infinitely many manifolds having 0 complexity, e.g. any connected sums of $L(3,1)$ with himself. The main problem is that in dimension 3 it makes sense to restrict to irreducible 3-manifolds, while in dimension 4, smooth irreducibility is a not yet completely understood property (see [17]). In order to keep complexity finite, the proof of Proposition 3.2 suggests to restrict to special polyhedra. In dimension 3, this is a consequence of restricting to irreducible manifolds, so we ask the following:

Question 3.5. *What is the class of* 4*-manifolds admitting a minimal shadow which is special?*

Even if one restricts to special shadows, it is not obvious that there are only a finite number of 4-manifolds having a fixed complexity. Indeed, a priori, there could exist infinitely many gleams on the same polyhedron P such that $\partial M_{(P,gl)} = S^3 \#_k S^2 \times S^1$, and this is indeed the case! But, fortunately, the following remarkable result of B. Martelli [12] ensures finiteness of complexity on special polyhedra:

Theorem 3.6. *Let N and N' be two closed* 3*-manifolds and $L \subset N$ a framed link. Up to diffeomorphism, there exist only finitely many cobordisms from N to N' constructed by gluing* 2*-handles to N along L.*

We stress here that the above result does not claim that there are finitely many slopes on L surgering over which produces N': it only claims for finiteness of the resulting 4-cobordisms.

Corollary 3.7. *Let P be a special polyhedron, P' be the polyhedron obtained by puncturing once P over each region; let furthermore be $(N, L) \doteq (\partial M_{(P',\emptyset)}, \partial P')$. There are only finitely many closed* 4*-manifolds admitting a shadow whose underlying polyhedron is P.*

In what follows, we restrict to special shadows of 4-manifolds and classify all the 4-manifolds admitting a special shadow with 0 or 1 vertex (Theorem 3.10 below).

Definition 3.8 (Special complexity)**.** Let X be a closed and oriented 4-manifold. The *special complexity* of X, denoted $c^{\text{sp}}(X)$ is the least number of vertices of a special shadow of X.

Theorem 3.9. *If a closed* 4*-manifold X has a shadow with k vertices and r regions which are not discs and whose total Euler characteristic is e, then $c^{\text{sp}}(X) \leq k + 2(r + 2e)$. Moreover the following holds:*

1. *For each integer k there exists only a finite number of smooth, closed and oriented* 4*-manifolds having special complexity $\leq k$.*
2. *If X_1 and X_2 are closed, oriented* 4*-manifolds, then $c^{\text{sp}}(X_1 \# X_2) \leq c^{\text{sp}}(X_1) + c^{\text{sp}}(X_2) + 4$.*
3. *$c^{\text{sp}}(\overline{X}) = c^{\text{sp}}(X)$, where $\overline{X}$ is X with the opposite orientation.*

Proof of 3.9*.* The first statement is a standard fact: it is sufficient to apply some local modifications called "$0 \to 2$-moves" to the initial shadow in order to split the non disc regions into discs. Each of these moves creates 2 vertices, and the total number of these moves is bounded above by $r + 2e$. Fact 1 is a direct consequence the result of Corollary 3.7 and of the fact that there are only finitely many special polyhedra with no more than k vertices. To prove Fact 2, it is sufficient to repeat the construction of the proof of Proposition 3.2 and add two "lune moves" producing 4 new vertices, to ensure that the final polyhedron is special. Fact 3 is a direct consequence of the fact that, if (P, gl) is a shadow of M then, $(P, -gl)$ is a shadow of $\overline{M}$. $\boxed{3.9}$

The main reason why it is interesting to restrict to special shadows is that the number of special polyhedra with less than k vertices is finite for every k. In particular, Figures 7 and 8 summarize respectively the complexity 0 and 1 special polyhedra.

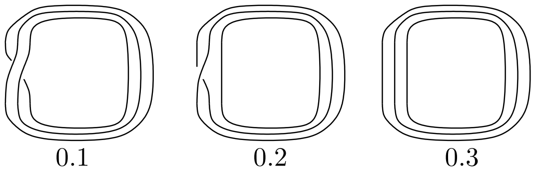

FIGURE 7. The three complexity 0 special polyhedra.

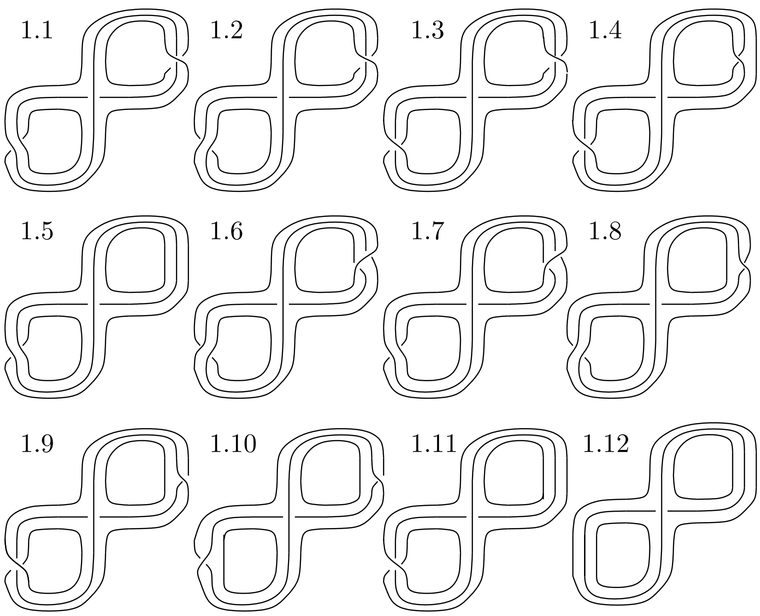

FIGURE 8. The twelve complexity 1 special polyhedra.

Theorem 3.10. *The only closed, smooth* 4*-manifolds having* 0 *special complexity are* S^4,$\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, $S^2 \times S^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, $\mathbb{CP}^2 \# \mathbb{CP}^2$ *and* $\overline{\mathbb{CP}}^2 \# \overline{\mathbb{CP}}^2$. *Moreover, there are no manifolds with special complexity* 1.

Remark 3.11. If one restricts to simply-connected manifolds viewed up to homeomorphism, it is not surprising that there are no new ones in complexity 1: indeed, by Freedman's Theorem, these manifolds are classified up to homeomorphism by their self-intersection form. Hence, since the second homology of a shadow of a 4-manifold surjects onto the second homology of the 4-manifold, and the maximal second Betti number of a special-complexity 1-polyhedron is 4, the possible intersection forms obtainable using complexity one polyhedra are those already carried by complexity 0-manifolds. What is interesting is that no exotic structure on complexity 0-manifolds has been found in complexity one.

To prove Theorem 3.10, for each polyhedron P of Figures 7 and 8, we find all the gleams such that $\partial M_{(P,gl)}$ is $S^3\#_k S^2\times S^1$ for some $k\geq 0$. Then, for each of these gleams, we prove that the closed 4-manifold obtained by closing $M_{(P,gl)}$ belongs to the list above. To do that, we use a series of results ranging from classical topology, to hyperbolic geometry to quantum topology. The next subsection is devoted to recall these results, suitably adapted to our needs.

3.1. **Useful tools.**

3.1.1. *"Classical" facts.*

Proposition 3.12. *Let (P,gl) be a polyhedron, (R_i,gl_i), $i=i\dots n$ be its regions equipped with gleams and oriented arbitrarily, and let $M=M_{(P,gl)}$. The following holds:*

1. $H_*(P,\mathbb{Z})\cong H_*(M,\mathbb{Z})$, $\pi_*(P,x_0)\cong\pi_*(M,x_0)$, *for each basepoint* $x_0\in P$.
2. *If* $H_2(M;\mathbb{Z})=0$, *and* $Tors(H_1(M))\neq 0$ *then* $\partial M\neq S^3,\#_k S^2\times S^1$.
3. *If* $H_2(M;\mathbb{Z})=0$ *and* $H_1(M;\mathbb{Z})$ *is free then* $H_1(\partial M;\mathbb{Z})\cong H_1(M;\mathbb{Z})$.
4. *Each element of* $H_2(M;\mathbb{Z})$ *can be represented in a unique way as a sum* $\sum_i k_iR_i$, *with* $k_i\in\mathbb{Z}$.
5. *Given a basis of* $c^1\dots c^k$ *of* $H_2(M;\mathbb{Z})$ *with* $c^j=\sum_{1\leq i\leq n}c_i^jR_i$, *the self-intersection form of* M *can be represented by an integer matrix* $Q(P,gl)$ *whose* (j,l)*-th entry is given by* $\sum_{1\leq i\leq n}gl(R_i)c_i^jc_i^l$.
6. *Suppose that* $H_1(P;\mathbb{Z})=0$; *then, if* $det(Q(P,gl))\neq 0$ *then* $\#H_1(\partial M;\mathbb{Z})=|det(Q(P,gl))|$, *otherwise* $H_1(\partial M;\mathbb{Z})$ *is infinite.*

Proof of 3.12. Facts 1 and 4 are a consequence of the fact that M retracts on P and P is a CW-complex without 3-cells. Facts 2 and 3 result from $H_3(M;\mathbb{Z})=0$ (P contains no 3-cells), $H_1(M,\partial M;\mathbb{Z})=0$ (P has codimension 2 in M), from the isomorphism $H_2(M,\partial M)\cong Free(H_2(M))\oplus Tors(H_1(M))$, and from the exact homology sequence of the pair $(M,\partial M)$:

$$0\to H_3(M,\partial M)\to H_2(\partial M)\to H_2(M)\to H_2(M,\partial M)\to H_1(\partial M)\to H_1(M)\to 0$$

Fact 5 was proved by Turaev ([18]), and the last is a consequence of 2) and of a classical result of Fox ([6]). $\boxed{3.12}$

We will also use the following strong result due to Gordon and Luecke for the part regarding S^3 ([8]) and to Gabai ([7]) for the part regarding $S^2\times S^1$.

Theorem 3.13. *No integer Dehn filling on a non trivial knot in S^3 produces S^3 or $S^2\times S^1$.*

3.1.2. *Hyberbolic* 3*-manifolds and shadows.* Let P be a special polyhedron containing at least one vertex and let P' be the regular neighborhood of $Sing(P)$ in P. By Theorem 2.3, P' can be thickened (without the need of any gleams!) to a 4-manifold $M_{(P',\emptyset)}$ diffeomorphic to a regular neighborhood of a graph in $\mathbb{R}^4$ so that $\partial P'\subset\partial M$ is a link in $\partial M\cong\#_k S^2\times S^1$ (for a suitable k). For each component of $\partial P'$ we define its $\mathbb{Z}_2$-gleam to be the $\mathbb{Z}_2$-gleam of the region of P containing the component and its *valence* to be equal to the number of vertices of P' touching that region. The following was proved by the author and D.P. Thurston ([5]):

Theorem 3.14. (1) *The* 3*-manifold* $\partial M_{(P',\emptyset)}$ *is a connected sum of* $1-\chi(P')$ *copies of* $S^2\times S^1$ *in which the link* $\partial P'$ *has hyperbolic complement whose volume is* $2|\chi(P')|Vol_{oct}$*, where* Vol_{oct} *is the volume of the regular hyperbolic octahedron.*

(2) *There is a maximal set of sections of the cusps of* $\partial M_{(P',\emptyset)} - \partial P'$ *such that the torus corresponding to a component of* $\partial P'$ *whose valence is* q *is the one depicted in the left part of Figure 9 if its* $\mathbb{Z}_2$*-gleam is zero and in the right part otherwise.*

(3) *The manifold* $M_{(P,gl)}$ *is obtained by attaching* 2*-handles to* $M_{(P',\emptyset)}$ *along* $\partial P'$*, and hence* $\partial M_{(P,gl)}$ *is obtained by an integer Dehn filling of* $\partial M_{(P',\emptyset)} - \partial P'$*.*

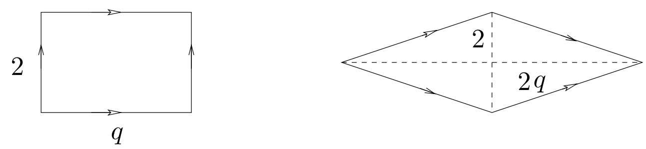

FIGURE 9. The shapes of the section of a cusp of $\partial M_{(P',\emptyset)} - \partial P'$.

Let us recall the following results of I. Agol ([1]) and M. Lackenby ([10]):

Theorem 3.15. *Let* N *be a hyperbolic* 3*-manifold with cusps and let* c *be a fixed section of a cusp of* N*. Gluing a solid torus to* c *through an homeomorphisms sending the meridian of the torus to a geodesic whose length is* >6*, produces a* 3*-manifold* N' *which is hyperbolike.*

Theorem 3.16 ([1])**.** *Let* N *be a hyperbolic* 3*-manifold with cusps,* $C_i,\ i=1,\dots n$ *be embedded sections of all the cusps cutting out of* N *volumes* $v_1,\dots v_n$ *and* $sl_i,\ i=1,\dots n$ *be minimal length geodesics in* C_i*. Let* s *be any subset of* $\{1,\dots n\}$ *and* N_s *be a Dehn filling on* N *along the cusps* $C_i,\ i\in s$*. If for each* $i\in s$ *the distance between the* i*-th slope of the Dehn filling and* sl_i *is greater than* $\frac{18}{v_i}$ *then* N_s *is hyperbolike.*

The following was proved in [2] as a corollary of Theorem 3.14 and Theorem 3.15:

Corollary 3.17 ([2])**.** *Let* (P,gl) *be a standard shadow such that for each region* R *of* P *it holds* $|gl(R)|+v(R)\geq 6$*. Then the manifold* $\partial M_{(P,gl)}$ *is Haken or word hyperbolic, and hence is not* S^3 *or* $\#_k S^2\times S^1$*.*

3.1.3. *State-sum quantum invariants.* Given a shadow (P,gl) it is fairly easy to compute Reshetikhin-Turaev invariants of $\partial M_{(P,gl)}$ as combinatorial state sums. Instead of plunging into the theoretical aspects of these invariants we limit ourselves to define these invariants through explicit coefficients in $\mathbb{C}$ (see [18] for a complete account). Let $r\geq 3$ be an integer and $t\doteq e^{\frac{2\pi i}{r}}\in\mathbb{C}$; for each $n\in\mathbb{N}$ let:

$$[n]=\frac{t^n-t^{-n}}{t-t^{-1}},\ [0]=[1]=1$$

$$[n]!=\prod_{0\leq i\leq n}[i]$$

Let us define complex-valued functions on $\frac{\mathbb{N}}{2}$ as follows:

$$w_j=(\sqrt{-1})^{2j}\sqrt{[2j+1]}$$

We say that a triple (i, j, k) of elements of $\frac{\mathbb{N}}{2}$ is *admissible* if the following conditions are satisfied:

$$i + j \geq k,\ \ i + k \geq j,\ \ j + k \geq i$$

$$i + j + k \leq r - 2,\ \ i + j + k \in \mathbb{N}$$

For any triple of elements of $\frac{\mathbb{N}}{2}$ we define

$$\Delta(i, j, k) = \sqrt{\frac{[i + j - k]![i + k - j]![j + k - i]!}{[i + j + k + 1]}}$$

if the triple is admissible and zero otherwise. Finally, for any 6-tuple of elements of $\frac{\mathbb{N}}{2}$ we define its $6j$*-symbol* as follows:

$$\begin{pmatrix} i\ j\ k \\ l\ m\ n \end{pmatrix} = \sum_{z\in\mathbb{Z}} \frac{(\sqrt{-1})^{-2(i+j+k+l+m+n)}\Delta(i,j,k)\Delta(i,m,n)\Delta(j,l,n)\Delta(k,l,m)(-1)^z[z+1]!}{[z-i-j-k]![z-i-m-n]![z-j-l-n]![z-k-l-m]![i+j+l+m-z]![i+k+l+m-z]![j+k+m+n-z]!}$$

where the sum is taken over all z such that all the arguments of the "quantum factorials" in the denominator of the r.h.s. are non negative. Let furthermore:

$$W \doteq \frac{\sqrt{2r}}{t - t^{-1}} \qquad\qquad S \doteq W^{-1} \sum_{0\leq i\leq \frac{r-2}{2}} (w_i)^4 e^{2\pi\sqrt{-1}(i - \frac{j(j+1)}{r})}$$

We define a *coloring* on a special polyhedron (P, gl) as an assignment of an element of $\frac{\mathbb{N}}{2}$ to each region of P. Given a coloring on P, for each region R let $w[R] \doteq w_j e^{2\pi\sqrt{-1}gl(R)(i-\frac{i(i+1)}{r})}$ where j is the color of R_i; similarly, to each vertex we associate its $6j$-symbol where (i, j, k, l, m, n) are the colors of the regions around the vertex and (i, l) (j, m) and (k, n) are the pairs of colors corresponding to regions which, near the vertex, intersect only in the vertex itself. Finally, let $sign(P, gl)$ be the signature of the self intersection form of $H_2(M_{(P,gl)}; \mathbb{Z})$, and $nul(P, gl)$ be the dimension of the maximal real subspace of $H_2(M_{(P,gl)}; \mathbb{R})$ contained in the annihilator of the form. The *state sum* of (P, gl) is:

$$|(P, gl)|_r = W^{1-\chi(P)-nul(P,gl)} S^{-sign(P,gl)} \sum_{colorings} \prod_{regions} w[R] \prod_{vertices} \begin{pmatrix} i & j & k \\ l & m & n \end{pmatrix}$$

Theorem 3.18 (Turaev [18])**.** *Let N be a 3-manifold and (P, gl) be such that $N = \partial M_{(P,gl)}$. Then $|(P, gl)|_r$ is an invariant of N denoted $|N|_r$: if (P', gl') is another polyhedron such that $N = \partial M_{(P',gl')}$ then $|(P', gl')|_r = |(P, gl)|_r$. Moreover, if $N = S^3\#_k S^2 \times S^1$ for some $k \geq 0$ then $|N|_r = 1,\ \forall r \geq 3$.*

Remark 3.19.
1. The normalization we used, slightly differs from Turaev's original one to better suit our need of identifying polyhedra with gleams describing "closable" 4-manifolds.
2. The gleam of P is irrelevant for the selection of the admissible colorings so that the explicit form of the state sum $|(P, gl)|$ does not change if one changes gl. This allowed us to perform extensive computer based calculations of Reshetikhin-Turaev invariants of $\partial M_{(P,gl)}$ for a fixed polyhedron with varying gleams.

3.2. **Classification of low-complexity 4-manifolds.**

3.2.1. 0-*complexity* 4-*manifolds.* In this subsection we prove the first part of Theorem 3.10 by means of a case by case analysis. More precisely, for each polyhedron P of Figure 7 we will list all the possible gleams such that $\partial M_{(P,gl)} = S^3\#_k S^2\times S^1$. Then for each of these gleams we identify the 4-manifold obtained by closing $M_{(P,gl)}$.

Case 0.1. In that case $\pi_1(P)=\mathbb{Z}_3$ and $H_2(P)=0$, so, by Proposition 3.12, $\partial M_{(P,gl)}$ cannot have the form $S^3\#_k S^2\times S^1$ for any gleam on P.

Case 0.2. In that case P has two regions: let R_1 be the one passing once over $Sing(P)$ and R_2 the other one; let moreover P' be a regular neighborhood of $Sing(P)$ in P and P'_i the polyhedra obtained by gluing the regions R_i to P'. It can be checked that if R_1 is equipped with gleam gl_1 (necessarily an half integer) then the pair $(\partial M_{(P'_1,gl_1)},\partial P'_1)$ is $(S^3,T(2gl_1,2))$, where $T(p,q)$ is the (p,q)-torus knot. In particular, $\partial P'_1$ is a trivial knot in S^3 only if $gl_1=\pm\frac{1}{2}$ and so by Theorem 3.13, if (P,gl) produces a "closable" 4-manifold, then $gl(R_1)=\pm\frac{1}{2}$. Hence let us now suppose that $gl(R_1)=\frac{1}{2}$ (up to reversing the orientation of $M_{(P,gl)}$ we can do that); notice that $H_2(P;\mathbb{Z})=\mathbb{Z}$ and is generated by the cycle represented by $2R_1+R_2$ whose self intersection is $gl(R_2)+4gl(R_1)$ (see Proposition 3.12). Hence by Fact 6 of Proposition 3.12 it must hold: $2+gl(R_2)=\pm1$ or $2+gl(R_2)=0$, and so $gl(R_2)$ is in $\{-1,-2,-3\}$. It is not difficult to check that in these cases, using the tricks of Figure 6, P can be simplified to a sphere with gleam respectively $1,0,-1$. Such spheres are shadows respectively of $\mathbb{CP}^2$, S^4 and $\overline{\mathbb{CP}}^2$. Hence P with gleam $(\frac{1}{2},-1)$ is a shadow of $\mathbb{CP}^2$, with gleam $(\frac{1}{2},-2)$ of S^4 and with gleam $(\frac{1}{2},-3)$ of $\overline{\mathbb{CP}}^2$.

Case 0.3. Let R_1, R_2 and R_3 be the regions of P oriented so that R_1+R_3 and R_2+R_3 are cycles and gl_i, $i=1,2,3$ be their (integer) gleams. It can be checked that $\partial M_{(P,gl)}$ is the Seifert manifold $S^2(gl_1,1)(gl_2,1)(gl_3,1)$, which, according to the classification of Seifert 3-manifolds, can be S^3 or $S^2\times S^1$ only if $|gl_i|\le 3\ \forall i$. Moreover, in the chosen basis of $H_2(M_{(P,gl)})$ the self intersection matrix of $M_{(P,gl)}$ is (see Proposition 3.12):

$$\begin{pmatrix} gl_1+gl_3 & gl_3 \\ gl_3 & gl_2+gl_3 \end{pmatrix}$$

Hence, by Fact 4 of Proposition 3.12, it must hold $(gl_1+gl_3)(gl_2+gl_3)-{gl_3}^2=\pm1,0$. In particular it turns out that, up to symmetries of P and multiplication by -1 of gl (which changes the orientation of $M_{(P,gl)}$), the only cases are: $(k,0,0)$, $(1,\pm1,k)$, with $k\in\{-3,-2,-1,0,1,2,3\}$. A case by case study shows that in the first family, k has to be in $\{-1,0,1\}$ producing respectively $\overline{\mathbb{CP}}^2, S^4, \mathbb{CP}^2$; the only interesting cases of the second family turn out to be $(1,-1,1)$ and $(1,-1,3)$ which give $S^2\times S^2$, $(1,-1,0)$ and $(1,-1,2)$ which give $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, $(1,1,0)$ and $(-1,-1,0)$ which give $\mathbb{CP}^2\#\mathbb{CP}^2$ and $\overline{\mathbb{CP}}^2\#\overline{\mathbb{CP}}^2$ respectively.

3.2.2. *Complexity* 1 *four manifolds.* Let us first clarify the general strategy we follow for each polyhedron P of Figure 8. Let $R_1,\dots R_n$ be the regions of P, P' the regular neighborhood of $Sing(P)$ in P and, for each subset s of $\{1,\dots n\}$, let P_s be the polyhedron obtained by gluing to P' each region R_i with $i\in s$. By Theorem 3.14 the manifold $\partial M_{(P',\emptyset)}$ is hyperbolic with n cusps and gluing back a region R_i to P' corresponds to performing an integer Dehn filling along the i-th cusp. We want to list all the possible gleams on P such that $\partial M_{(P,gl)}$ is $S^3\#_k S^2\times S^1$ for some $k\ge 0$: this brings us produce a finite list of non-hyperbolic (possibly partial) Dehn fillings of P' by recursively applying Theorem 3.16; our main tool is Jeff Week's Snappea [20]. So, starting from the hyperbolic manifold $\partial M_{(P',\emptyset)}$ $(s=\emptyset)$, we iterate the following algorithm:

(1) Choose sections $C_i,\ i \notin s$ of the given hyperbolic manifold, let $v_i,\ i \notin s$ be the volumes they cut out of the manifold and let $g_i,\ i \notin s$ be the gleams on the regions R_i corresponding to integer Dehn fillings along shortest geodesics in C_i.
(2) For each cusp $C_i,\ i \notin s$, perform the following steps. Let $s_i = s \cup \{i\}$ and for each integer j such that $-\frac{18}{v_i} \leq j \leq \frac{18}{v_i}$, let $gl(R_i) = g_i + j$ and gl_{s_i} be the set of gleams on the regions with indices in $s \cup \{i\}$:
   - If $\partial M_{(P_{s_i}, gl_{s_i})}$ is not hyperbolic, add (P_{s_i}, gl_{s_i}) to the list of non-hyperbolic fillings of $M_{(P', \emptyset)}$.
   - If it is hyperbolic and has non empty boundary, apply Step 1 to $\partial M_{(P_{s_i}, gl_{s_i})}$ otherwise, if $j < \frac{18}{v_i}$ increase j, otherwise choose another cusp C_k with $k \in \{1, \ldots, n\} - i$ and follow Step 2.

The result of the above algorithm will be in general a finite list of non hyperbolic 3-manifolds possibly with boundary. If all the manifolds in the list are closed, one has a finite number of cases to check: in particular, we did it using Theorem 3.18. In what follows, a clever use of the tools of Subsection 3.1 allowed us to treat the cases when some element of the list has non-empty boundary and show that in fact if a 4-manifold has a complexity 1 special shadow, then it also has a complexity 0 one. It is worth to note that the above general algorithm was necessary only in few cases since most of the polyhedra of Figure 8 can be studied "by hand": let us then start from the easiest cases.

Cases 1.1-. . . -1.5. In all these cases $H_1(P)$ is a finite, non-trivial group an so by Proposition 3.12 there is no gleam on P such that $\partial M_{(P,gl)} = S^3 \#_k S^2 \times S^1$ for some $k \geq 0$.

Cases 1.6-1.7. In these cases P has only one region whose valency is 6. By Corollary 3.17 there is no gleam on P such that $\partial M_{(P,gl)} = S^3 \#_k S^2 \times S^1$.

Case 1.8. Let R_1 be the region whose valency is 5, R_2 the other one and $gl_i,\ i = 1, 2$ their gleams. Since the $\mathbb{Z}_2$-gleams of R_1 and R_2 are respectively 1 and 0, then $gl_1 \in \frac{\mathbb{Z}}{2}$ and $gl_2 \in \mathbb{Z}$. Following the general algorithm, we obtain a finite list of pairs (gl_1, gl_2) such that $M_{(P,gl)}$ is closed and non-hyperbolic and only one non-closed case: $(P'_2, gl_2 = 0)$. Using our state-sum formulation using Reshetikhin-Turaev invariants with $r = 5, 7, 9$ (see Theorem 3.18) we excluded all the closed cases. The non-closed case corresponds to the infinite family of gleams on P of the form $gl = (gl_1, 0),\ gl_1 \in \frac{\mathbb{Z}}{2}$ all of which can be simplified by the upper trick of Figure 6 obtaining a contractible shadow of S^4.

Case 1.9. This case is similar to the preceding one. Let R_1 and R_2 be respectively the valency 5 and 1 regions of P, and $gl_i,\ i = 1, 2$ be their gleams (note that in this case $gl_i \in \mathbb{Z},\ i = 1, 2$). The general algorithm gives a list containing only two non-closed non-hyperbolic manifolds and a finite number of closed ones. As in the preceding case, using Theorem 3.18 we excluded all the closed ones. The first non closed non-hyperbolic manifold is $\partial M_{(P'_2, 0)}$ whose Dehn fillings correspond to pairs (P, gl) with $gl = (gl_1, 0),\ gl_1 \in \mathbb{Z}$ which can be simplified to a contractible shadow of S^4 using the upper trick of Figure 6. The second non-hyperbolic manifold is $\partial M_{(P'_1, 0)}$; using S. Matveev's Recognizer [15], we checked that this manifold has JSJ-decomposition $D^2(2,1)(3,-2) \cup N^2 \cup D^2(2,1)(3,-2)$, and contains two incompressible tori which can be compressed only if the Dehn-filling on the boundary corresponds to the 0 gleam on R_2. But since $S^3 \#_k S^2 \times S^1$ are atoroidal, $gl_2 = 0$ which falls in the preceding case.

Case 1.10. Let R_1 be the valency 4 region, R_2 and R_3 the remaining two (they are exchangeable through a symmetry of P). It is easy to check that $H_1(P; \mathbb{Z}) = 0$ and $H_2(P; \mathbb{Z}) = \mathbb{Z}$ with generator represented by R_1. By Proposition 3.12, it must hold $gl_1 = 0$ or $gl_1 = \pm 1$, then, up to multiplying gl by -1, we reduce to study two manifolds: $\partial M_{(P'_1, 0)}$ and $\partial M_{(P'_1, 1)}$. Using S. Matveev's Recognizer, one sees that the first one has JSJ decomposition $N^2 \cup N^2$ (where N^2 is the product of a thrice punctured sphere with S^1), and contains two incompressible tori, which can be compressed only if at least one of gl_2 and gl_3 is zero, in which case (P, gl) can be simplified using the upper trick of Figure 6 obtaining a shadow which is a sphere equipped with a gleam equal to gl_1. On the contrary, $\partial M_{(P'_1,1)}$ is hyperbolic and can be treated using the general algorithm. The result is a finite list of closed non-hyperbolic manifolds which can be excluded using Theorem 3.18, and two non-closed non-hyperbolic manifolds corresponding respectively to $(P'_{1,2},(1,0,\emptyset))$ and $(P'_{1,3},(1,\emptyset,0))$ which can be simplified using the tricks of Figure 6.

Case 1.11. Let R_i $i = 1, 2, 3$ be the region of valency i in P. The 4-manifold $M_{(P,gl)}$ has a handle decomposition induced by P such that the two 1-handles induced by the edges of P are annihilated by the two 2-handles corresponding to R_1 and R_2. Hence $\partial M_{(P'_{1,2},(gl_1,gl_2,\emptyset))}$ is the complement of a knot k in S^3, so, by Theorem 3.13, we search for the cases when k is the trivial knot. To do this, we calculated the Alexander polynomial of k using gl_1 and gl_2 as parameters and Turaev's surgery formulas for Reidemeister torsion ([19]). It holds:

$$\Delta(k) = \frac{t + t^2 + t^{3c_1} + t^{(3c_1+2c_2+1)} + t^{(6c_1+2c_2-1)} + t^{(6c_1+2c_2)}}{(1+t)(1+t+t^2)}$$

$$c_1 \doteq gl_1 + \frac{1}{2},\ c_2 \doteq gl_2 - \frac{1}{2}$$

It is easy to check that the above fraction defines an element of $\mathbb{Z}[t, t^{-1}]$ well defined up to products by $t^{\pm 1}$. Then, to find the cases when k is an unknot we study when $\Delta(k) = t^r$ for some r. To do that, we associate to $\Delta(k)$ its span, that is, the (well defined) difference between the highest and the lowest degree in any of its expressions as an element of $\mathbb{Z}[t, t^{-1}]$. It is simple to see that this span depends on gl_1 and gl_2 as a piecewise affine function; a careful analysis of all the possible combinations of (gl_1, gl_2) shows that $span(\Delta(k)) = 0$ only in four cases: $(-\frac{1}{2}, \frac{3}{2}), (-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, -\frac{3}{2})$. But in each of these cases $|gl_1| = \frac{1}{2}$, hence, using the lower-left trick of Figure 6, the polyhedron can be simplified obtaining the polyhedron 0.2 of Figure 7.

Case 1.11. Let R_1 and R_2 be the two valency 2 regions of P and R_3 and R_4 the remaining two: it is easy to see that there are symmetries of P exchanging them in pairs. If either $gl_3 = \pm\frac{1}{2}$ or $gl_4 = \pm\frac{1}{2}$ then P can be simplified obtaining the special complexity 0 polyhedron, hence, we exclude from now on all the quadruples of gleams satisfying one of the above equalities. The application of the general algorithm produces again a finite list of closed non-hyperbolic Dehn fillings of $\partial M_{(P',\emptyset)}$ and of non-closed ones. The former can be shown to be different from $S^3 \#_k S^2 \times S^1$ by means of Theorem 3.18. To exclude that the latter have integer Dehn fillings of that form, we use the following facts:

(1) If S is a Seifert 3-manifold having the homology of S^3 ($S^2 \times S^1$), then $S = S^3$ ($S = S^2 \times S^1$) iff its base orbifold is S^2, its singular fibers are not more than 3 and their degree of singularity is at most 3.
(2) The determinant of the self-intersection matrix of $M_{(P,gl)}$ is $(gl_1 + gl_3 + gl_4)(gl_2 + gl_3 + gl_4) - (gl_3 - gl_4)^2$ and, by Proposition 3.12, it has to be either 0 or ± 1.
(3) $gl_1, gl_2 \in \mathbb{Z}$ and $gl_3, gl_4 \in \frac{\mathbb{Z}}{2}$.

Up to symmetries of P and multiplication of gl by -1, the list of (partial) non-closed non-hyperbolic Dehn-fillings is encoded by the following quadruples of gleams on P: $(0, \emptyset, \emptyset, \emptyset)$, $(2, -2, \emptyset, \emptyset)$, $(1, \emptyset, \emptyset, \emptyset)$, $(2, \emptyset, -\frac{3}{2}, \emptyset)$, $(2, -3, -\frac{5}{2}, \emptyset)$, $(2, -3, \frac{3}{2}, \emptyset)$, $(3, 3, -\frac{3}{2}, \emptyset)$, where we denoted by $\emptyset$ the non-filled regions. Using S. Matveev's Recognizer, one sees that the integer Dehn fillings of the first two quadruples are $S^2(gl_2, -1)(2gl_3, -gl_3 + \frac{1}{2})(2gl_4, -gl_4 + \frac{1}{2})$ and $S^2(2, 1)(2gl_3, 2)(2gl_4, 2)$ respectively. Using the above three facts, one can show that these Dehn fillings are "closable" if and only if either $gl_3 = \pm\frac{1}{2}$ or $gl_4 = \pm\frac{1}{2}$ which we excluded from the beginning. Similarly, a Dehn filling corresponding to $(1, gl_2, gl_3, gl_4)$ gives $D^2(gl_3 + \frac{1}{2}, -1)(gl_4 + \frac{1}{2}, -1) \cup D^2(2,1)(gl_2 + 1, 1)$ which contains an incompressible torus unless either gl_3 or gl_4 are $\pm\frac{1}{2}$. The quadruples $(2, gl_2, -\frac{3}{2}, gl_4)$, $(2, -3, -\frac{5}{2}, gl_4)$, $(2, -3, \frac{3}{2}, gl_4)$ satisfy the equation of Fact 2 above, only in a finite number of cases, all of which can be excluded by means of Theorem 3.18. The last quadruple $(3, 3, -\frac{3}{2}, gl_4)$ satisfies the determinant equation for all gl_4, but it can be checked that $H_1(\partial M_{(P,gl)}; \mathbb{Z})$ has always torsion unless $gl_4 = -\frac{3}{2}$ or $gl_4 = -\frac{5}{2}$; these two cases can then be excluded by means of Theorem 3.18.

3.3. **Higher complexity manifolds and exotic pairs.** Let us provide some examples of 4-manifolds having higher special complexity. Some "trivial" examples can be obtained by applying Theorem 3.9: each connected sum of a pair of special complexity 0 manifolds has special complexity at most 4; more in general, the special complexity of $k\mathbb{CP}^2 \# h\overline{\mathbb{CP}}^2$ is bounded above by $2k + 2h$. A first non-trivial example is $\mathbb{RP}^2 \tilde{\times} S^2$: his special shadow with 2-vertices can be constructed by applying Theorem 3.9 to its non special shadow whose underlying polyhedron is obtained from 0.3 of Figure 7 by gluing two discs with gleams ± 1 and one Möbius strip. More in general, if F is a genus g-surface, the manifold $F \times S^2$ has special complexity bounded above by $4g$ if F is orientable and by $4g + 2$ otherwise. Defining the complexity of a pair of manifolds as the maximum between the complexities of the two manifolds, the following natural question arises:

Question 3.20 (Complexity of exotic pairs)**.** *Which is the pair of homeomophic but non diffeomorphic closed/non-closed* 4*-manifolds with the lowest complexity/special complexity?*

We now produce upper estimates to the answer of the above question in the case of non-special complexity for closed manifolds and in the case of special complexity for non-closed ones.

It is not difficult to provide upper estimates for the non-special complexity of a class of notable closed 4-manifolds: the elliptic surfaces $E(n)$. Using a Kirby-calculus presentation of these manifolds (see [9], Theorem 8.3.2), one can check that $c(E(n)) \leq 6n + 2$. Then, an example of "exotic" 4-manifold with non-special complexity ≤ 14 is $E(2)\#\overline{\mathbb{CP}}^2$ which is homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#20\overline{\mathbb{CP}}^2$ (whose complexity is 0). Hence an upper estimate for the answer of the above question in the closed case for the non-special complexity is 14; we expect such an estimate to be non-optimal and that lower complexity examples will be found.

In sharp contrast with the empty boundary case, examples of homeomorphic but non diffeomorphic 4-manifolds with boundary are much easier to provide: the 4-thickening of the polyhedron 1.10 of Figure 8 equipped with gleams $(-1, 1, 2)$ (using the notation of Subsection 3.2.2) admits a non-diffeomorphic model having a special shadow with 3-vertices (see [9], Theorem 11.4.8). Hence, in particular an upper estimate for the answer of the above question is 3 even in the case of special complexity.

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