Branched Spines of 3-Manifolds and Reidemeister Torsion of Euler Structures

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To the memory of our colleague and friend Marco Reni

Abstract: We consider homotopy classes of non-singular vector fields on three-manifolds with boundary and we define for these classes torsion invariants of Reidemeister type. We show that torsion is well-defined and equivariant under the action of the appropriate homology group using an elementary and self-contained technique. Namely, we use the theory of branched standard spines to express the difference between two homotopy classes as a combination of well-understood elementary catastrophes. As a special case we are able to reproduce Turaev's theory of Reidemeister torsion for Euler structures in the special case of closed manifolds of dimension three.

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Introduction

The theory of Euler structures and their Reidemeister torsion was founded by Turaev in [15] for compact manifolds of arbitrary dimension, and the special case of dimension three was later discovered to have deep connection with the three-dimensional version of the Seiberg-Witten invariants [11], [16], [17]. In [2], restricting to dimension three, we have generalized Turaev's theory by considering boundary configurations more general than those he allows, and as a main application we have shown how to lift the classical Alexander invariant of knots to (pseudo-)Legendrian knots. This generalization strongly relies on sophisticated results of Turaev [15].

In the present paper we develop a more elementary and purely combinatorial approach to a certain variation of the theory of Euler structures with boundary and their torsion.

The key technique we will employ below is that of *branched standard spines*, originally developed in [1]. This technique was already used in [2] to give an explicit and geometric description of the inverse of Turaev's *reconstruction map* Ψ from the space of *combinatorial* Euler structures (which are represented by suitable integral 1-chains, called *Euler chains*) to the space of *smooth* Euler structures (which are represented by non-singular vector fields having a prescribed configuration on the boundary). We have shown that if a branched standard spine P of a manifold M *carries* (in the sense of [1]) a field v which represents a smooth Euler structure ξ on M, then P entirely determines an Euler chain s_P (called the *spider* of P) which represents $\Psi^{-1}(\xi)$. As a matter of fact s_P splits as a sum $i(s_P) + b(s_P)$, where $i(s_P)$ is explicitly constructed as a certain union of half-orbits of the flow of v, whereas $b(s_P)$ lies on ∂M and actually depends only on the boundary configuration, not on v. So it is really $i(s_P)$ which carries the key information about ξ. The basic idea of the present paper is to remove all prescriptions to v on ∂M and use $i(s_P)$ only instead of s_P.

The idea just explained leads to a variation of the theory of Euler structures and torsion. Namely, since we only consider compact manifolds M with non-empty boundary and we allow non-singular vector fields with arbitrary behavior on ∂M, it turns out that an Euler structure is naturally defined just as a homotopy class of vector fields. In addition, since we only use the internal part of the spider, to define torsion we cannot use M directly, instead we consider the space X obtained from M by collapsing ∂M to a point.

As mentioned above, to show the crucial facts that torsion is well-defined and H_1-equivariant, we only use below the technology of branched standard spines [1], together with two results from [3] and [7]. The key point is to describe elementary moves combining which one can obtain from each other any two branched standard spines of the same M, and to show that equivariance holds for all these moves. Our approach is certainly less conceptual than Turaev's, but it has some technical advantages. For instance, the general notion of *subdivision rule* is a rather demanding step of Turaev's approach (which, we recall, works in any dimension). On the contrary, having only a small number of well-determined elementary catastrophes to deal with, we can treat all the subdivisions we need by hand.

As a final point we note that, when M is obtained from a closed manifold $\widehat{M}$ by removing a ball, our theory of Euler structures for M is equivalent to Turaev's theory for $\widehat{M}$, and by [2] torsion coincides (because $i(s_P) = s_P$ in this case). So, in the three-dimensional closed case, the present paper gives a faithful, elementary, and self-contained alternative approach to Turaev's original theory.

1 Euler structures

From now on, M will be a compact, connected, oriented 3-manifold with non-empty boundary. Using the *Hauptvermutung*, we will freely intermingle the differentiable, piecewise linear and topological viewpoints. Homeomorphisms will always respect orientations. All vector fields used in this paper will be non-singular unless the contrary is explicitly stated, and they will be termed just *fields* for the sake of brevity.

We will denote by $\mathrm{Vect}(M)$ the set of (non-singular) vector fields on M up to homotopy through (non-singular) fields. Using the fact that $\partial M \neq \emptyset$, one can show that $\mathrm{Vect}(M)$ has the structure of an affine space over $H_1(M, \partial M; \mathbb{Z})$. We will denote by $\alpha : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to H_1(M, \partial M; \mathbb{Z})$ the map giving this structure. For $\xi = [v]$ and $\eta = [w]$ in $\mathrm{Vect}(M)$, we will call $\alpha(\xi, \eta)$ the *comparison* class between ξ and η, and to compute it we will use the following geometric construction. Up to homotopy we can assume that v and $-w$ are unitary with respect to some metric and in general position with respect to each other and to ∂M. Then $\{x \in M : v(x) = -w(x)\}$ is a properly embedded curve. Looking at v and $-w$ as sections of the unit tangent bundle to M we can now see this curve as the projection of the intersection of two oriented 3-manifolds in an oriented 5-manifold, and hence we can give the curve a canonical orientation. Now $\alpha(\xi, \eta)$ is precisely the homology class of this oriented curve. We also mention that for $\xi = [v] \in \mathrm{Vect}(M)$ and $h \in H_1(M, \partial M; \mathbb{Z})$, the element $\xi + h$ of $\mathrm{Vect}(M)$ such that $\alpha(\xi + h, \xi) = h$ can also be defined by an explicit geometric procedure (see [15, §. 5.2] and [1, §. 6.2]).

Recall now from [15] that, for a closed manifold $\widehat{M}$, Turaev's refined Reidemeister torsion is a function of $\widehat{M}$ and an *Euler class* on $\widehat{M}$. The set $\mathrm{Eul}(\widehat{M})$ of Euler classes can be defined as $\mathrm{Vect}(\widehat{M})/_\sim$, where $\xi \sim \eta$ if there exist representatives v and w of ξ and η such that $v(x) = w(x)$ for all x in $\widehat{M} \setminus B$, where $B \subset \widehat{M}$ is an embedded 3-ball. Of course this definition makes sense also for $\partial M \neq \emptyset$, but in this case one canonically has $\mathrm{Eul}(M) \cong \mathrm{Vect}(M)$. As Turaev shows, $\mathrm{Eul}(\widehat{M})$ is an affine space over $H_1(\widehat{M}; \mathbb{Z})$. We can now show that the closed situation is actually contained in the bounded situation described above. Namely, take M to be $\widehat{M} \setminus B$, so that $\partial M \cong S^2$ and $\widehat{M}$ can be identified with $M/\partial M$. Denote by $x_0 \in \widehat{M}$ the image of ∂M under the projection $p : M \to \widehat{M}$.

Proposition 1.1. *The projection* $p : M \to \widehat{M}$ *induces a well-defined and canonical bijection* $\mathrm{Vect}(M) \to \mathrm{Eul}(\widehat{M})$. *This bijection is an isomorphism*

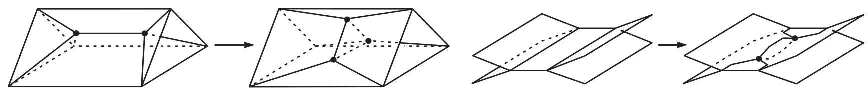

Figure 1: The MP-move (on the left) and the lune move (on the right).

of affine spaces under the isomorphism of $H_1(M, \partial M; \mathbb{Z})$ *with* $H_1(\widehat{M}; \mathbb{Z})$ *induced by* p *and the inclusion* $(\widehat{M}, \emptyset) \hookrightarrow (\widehat{M}, \{x_0\})$.

Proof of 1.1. We confine ourselves to a sketch. Of course p, viewed as a smooth map, is singular on ∂M, but, if we consider a ball B in $\widehat{M}$ centred at x_0 and we take a field v on M, we see that $p_*(v)$ is non-singular on $\widehat{M} \setminus B$. A computation of Euler characteristic now shows that $p_*(v)$ extends to a non-singular field $\widehat{v}$ on $\widehat{M}$, and by definition of $\mathrm{Eul}(\widehat{M})$ we can set $p_*([v]) = [\widehat{v}]$. Equivariance under the actions of homology is now easy, and it implies bijectivity. 1.1

2 Branched spines

In this section we recall the notion of (branched) spine and ideal triangulation of a 3-manifold M as above.

Standard spines A *quasi-standard* polyhedron P is a finite, connected, and purely 2-dimensional polyhedron with singularities of stable nature (triple lines and points where 6 non-singular components meet). Such a P is called *standard* if all the components of the natural stratification given by singularity are open cells. Depending on dimension, we call the components *vertices*, *edges*, and *regions*.

A quasi-standard polyhedron P is called a *spine* of M if it embeds in M so that M collapses onto P. It is by now well-known, after the work of Casler [4], Matveev [8] and Piergallini [14], that a standard spine determines M up to homeomorphism, that every M has standard spines (because $\partial M \neq \emptyset$), and that any two standard spines of the same M can be transformed into each other by certain well-understood moves. More precisely, assuming both spines have at least two vertices, the *MP-move* shown in Fig. 1-left and its inverse are already enough. We will often call an MP-move *positive* to emphasize that we are considering the move in the direction which increases (by one) the number of vertices.

Embedded spines In the sequel it will be more convenient to switch to the embedded viewpoint of the theory of standard spines, described with care in [18, Chapter IX, §. 2] and [1, §. 4.1]. The idea is just to fix M and consider the set $\mathrm{StSpin}(M)$ of all standard spines P embedded in the interior of M, where the embedding is regarded up to isotopy in M. With a slight abuse, we will write just P also for the isotopy class of P.

It is quite easy to see that the MP-move mentioned above can be defined within $\mathrm{StSpin}(M)$, and the embedded version of the Matveev-Piergallini theorem can now be stated (and proved) just as the non-embedded version: *two elements of* $\mathrm{StSpin}(M)$ *both having at least two vertices are related by embedded MP-moves and their inverses.*

Another move in $\mathrm{StSpin}(M)$ used in the sequel is the *lune move*, shown in Fig. 1-right. Since this move is non-local, it must be described with some care. This move, which increases by two the number of vertices, is determined by an arc α properly embedded in a region R of P. The move acts on P as in Fig. 1-right, but to define its effect non-ambiguously we must specify which pairs of regions, out of the four regions incident to R at the ends of α, will become adjacent to each other after the move. This is achieved by noting that R is a disc, so its regular neighborhood in M is a product, and we can choose for R a transverse orientation. Using it, at each end of α we can tell from each other the two regions incident to R as being a positive and a negative one, and we can stipulate that the two positive regions will become incident after the move, and similarly for the negative ones.

In the rest of the paper we will always regard M to be fixed, and we will only consider spines and moves embedded in M, without explicit mention.

Branched spines and fields A *branching* on $P \in \mathrm{StSpin}(M)$ is a collection b of one orientation for each region of P, such that no edge is induced the same orientation three times by the three regions incident to it. As mentioned in [1, §. 3.1], b can be used to consistently smoothen the singularity of P so to turn it into a branched surface, see [19]. Namely, the embedding of P can be isotoped so that an oriented tangent plane is defined at each point, and all the regions are smoothly immersed in P in an orientation-preserving way. A pair (P, b) will be called a *branched spine* of M, and b will often be dropped from the notation. The set of branched spines of M will be denoted by $\mathrm{BrSpin}(M)$. Note that not every $P \in \mathrm{StSpin}(M)$ carries a branching. The following is taken from [1, §. 4.1].

Proposition 2.1. *Every* $P \in \mathrm{BrSpin}(M)$ *defines up to isotopy a field* $v(P)$ *on* M *such that* $v(P)$ *is positively transversal to* P.

The topological construction which underlies this proposition is illustrated in a cross-section in Fig. 2. From the proposition we get a well-defined map $\Phi : \mathrm{BrSpin}(M) \ni P \mapsto [v(P)] \in \mathrm{Vect}(M)$ which we call

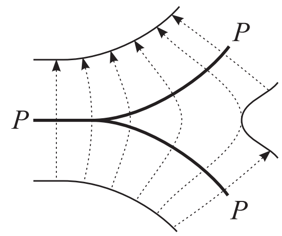

Figure 2: Field associated to a branched spine.

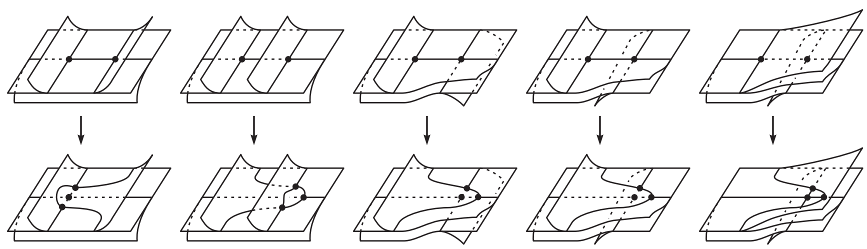

Figure 3: Sliding MP-moves.

reconstruction map. The following was essentially shown in [1, §. 4.6], see below for details.

Theorem 2.2. *The reconstruction map* $\Phi : \mathrm{BrSpin}(M) \to \mathrm{Vect}(M)$ *is onto.*

MP-move on branched spines When $P \in \mathrm{BrSpin}(M)$ and a positive MP-move is applied to P to get some $P' \in \mathrm{StSpin}(M)$, all the regions of P survive through the move, and a new (triangular) one is created. It is a fact (see [1, §. 3.5]) that the new region can always be oriented so to turn P' into a branched spine. Any move within $\mathrm{BrSpin}(M)$ arising as just described will be called a *branched* MP-move.

Two quite different types of branched MP-move exist. In the first type of move, the field carried by the spine is unchanged up to homotopy and the move itself can be viewed as a continuous evolution of a branched surface within M, with one instance of non-generic singularity along the evolution. The moves of this type are pictured in Fig. 3 and called *sliding* MP-moves. The second type of move is called *bumping* MP-move and shown in Fig. 4. As opposed to the case of sliding MP-moves, one can show (see [1, §. 4.6]) that under a bumping MP-move the homotopy class of the field carried by the spine gets modified (at least locally), see Proposition 4.5.

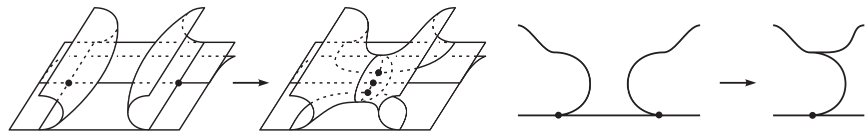

Figure 4: Bumping MP-move: complete picture and a vertical cross-section.

Remark 2.3. In a non-branched context, a positive MP-move is completely determined by the edge which disappears during the move. On the contrary, when branchings are present, to determine the move intrinsically one must specify the orientations of all the regions involved. As a result, there are 16 different sliding MP-moves and 4 bumping MP-moves. In Figg. 3 and 4 we are actually showing fewer moves, because they allow to understand all the essential physical modifications.

We can now explain better how Theorem 2.2 is deduced from [1]. First recall that, by Proposition 3.4.7 of [1], $\mathrm{BrSpin}(M)$ is non-empty. Then the theorem follows from Theorem 4.6.4 of [1], which we now recall:

Theorem 2.4. *Given $P \in \mathrm{BrSpin}(M)$ and $\xi \in \mathrm{Vect}(M)$, there exists a sequence of sliding MP-moves and bumping MP-moves which transforms P into an embedded branched spine P' such that $[v(P')] = \xi$.*

To be completely precise, another move between branched spines besides the MP-moves was considered in Theorem 4.6.4 of [1], but this move was later shown in [3] to be implied by the branched MP-moves.

Lune move on branched spines When considering the lune move in the context of branched spines, the same phenomena mentioned above for the MP-move appear again. Namely, if $P \in \mathrm{BrSpin}(M)$ and P is transformed into P' by a positive lune move, then P' can be turned into a branched spine so that all the regions which survive through the move, including the region split into two new regions, keep their orientation. The resulting transformation of $P \in \mathrm{BrSpin}(M)$ into $P' \in \mathrm{BrSpin}(M)$ will be called a *branched* lune move. As for the MP-move, there are two sorts of branched lune move, called *sliding* and *bumping* respectively and pictured in Fig. 5.

Remark 2.5. As for MP-moves (see Remark 2.3), there are several branched versions of the lune move. More precisely, there are 4 sliding lune moves and 2 bumping lune moves, but the essential phenomena are those described in Fig. 5.

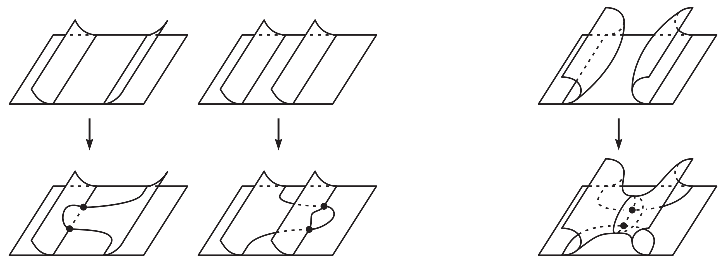

Figure 5: Branched versions of the lune move.

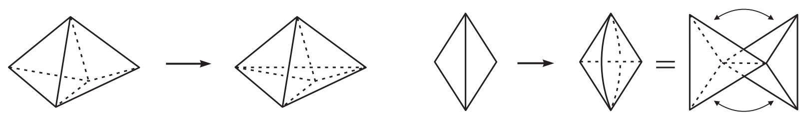

Figure 6: The dual version of the MP and lune moves.

Ideal triangulations An *ideal triangulation* of a manifold M with non-empty boundary is a partition $\mathcal{I}$ of $\mathrm{Int}(M)$ into open cells of dimensions 1, 2, and 3, induced by a triangulation $\mathcal{I}'$ of the space $Q(M)$, where:

1. $Q(M)$ is obtained from M by collapsing to a point each component of ∂M, so $Q(M)$ minus its vertices can be identified with $\mathrm{Int}(M)$;
2. $\mathcal{I}'$ is a triangulation only in a loose sense, namely self-adjacencies and multiple adjacencies of tetrahedra are allowed;
3. The vertices of $\mathcal{I}'$ are precisely the points of $Q(M)$ which correspond to the components of ∂M.

Duality It turns out ([10], [13], [9]) that there exists a natural bijection between standard spines and ideal triangulations of a 3-manifold. Given an ideal triangulation, the corresponding standard spine is just the 2-skeleton of the dual cellularization, as illustrated in Fig. 7-left. The inverse of this correspondence is denoted by $P \mapsto \mathcal{I}(P)$, and $\mathcal{I}(P)$ is called the ideal triangulation *dual* to P. It will be convenient in the sequel to call *centre* of a cell c of P the only point at which c meets the simplex of $\mathcal{I}(P)$ dual to c.

Before turning to a branched context, we show in Fig. 6 the MP and lune moves on a spine in terms of the dual ideal triangulations.

Now assume $P \in \mathrm{BrSpin}(M)$. First of all, we can realize $\mathcal{I}(P)$ in such a way that each edge is an orbit of the restriction of $v(P)$ to $\mathrm{Int}(M)$, and each 2-face is a union of such orbits. Since the edges of $\mathcal{I}(P)$ are orbits,

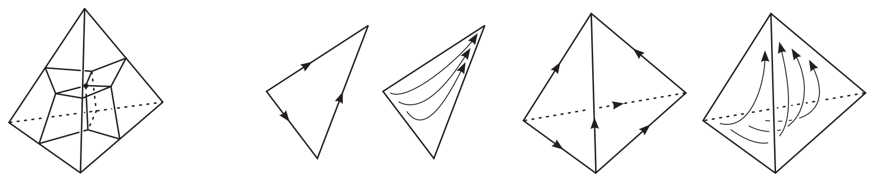

Figure 7: Left: portion of spine dual to a tetrahedron of an ideal triangulation. Right: how to deduce the field from the orientation of the edges.

they have a natural orientation, and the branching condition means that if we look at a (triangular) face of a tetrahedron of $\mathcal{I}(P)$, we never see the boundary of the triangle oriented as a closed circuit. So the branching can be encoded by an assignment of edge-orientations on $\mathcal{I}(P)$. Moreover, edge-orientations allow us to describe the orbits of the field also on the interior of the triangles and the tetrahedra of the triangulation, as shown in Fig. 7-right.

It is worth remarking that not only the edges, but also the faces and the tetrahedra of $\mathcal{I}(P)$ have natural orientations. For tetrahedra, we just restrict the orientation of M. For faces, we first note that the edges of P have a natural orientation (the prevailing orientation induced by the incident regions). Now, we orient a face of $\mathcal{I}(P)$ so that the algebraic intersection in M with the dual edge is positive.

3 Basic definition of torsion

Fixing M as above, we define in this section torsion functions on $\mathrm{Vect}(M)$, using suitable representations of the fundamental group of the space $X = M/\partial M$. We will denote by $p : M \to X$ the projection, and by x_0 be the image of ∂M. Using p, we will often tacitly identify $\mathrm{Int}(M)$ with $X \setminus \{x_0\}$.

Spider carried by a branched spine Let $P \in \mathrm{BrSpin}(M)$, and consider the field $v(P)$ and the ideal triangulation $\mathcal{I}(P)$ defined by P on M. Since $X \setminus \{x_0\} \cong \mathrm{Int}(M)$, we can consider $v(P)$ to be defined on $X \setminus \{x_0\}$, and note that the α-limit and the ω-limit in X of any orbit of $v(P)$ are both given by $\{x_0\}$ only. When we project $\mathcal{I}(P)$ to X we obtain a triangulation (in a loose sense) of X, which we will denote by $\mathcal{T}(P)$, with only one vertex x_0 and open positive-dimensional simplices which correspond to those of $\mathcal{I}(P)$ and are unions of orbits of $v(P)$.

Remark 3.1. Stipulating x_0 to be positive and using the facts already remarked, we see that also in $\mathcal{T}(P)$ all the simplices have a natural orientation.

Definition 3.2 (spider associated to P). We define $s(P)$ as the singular 1-chain in X obtained as $\sum_p \beta_p$, where p runs over the centres of cells of P and β_p is the closure of the positive orbit of $v(P)$ which starts at p. Note that the final endpoint of each β_p is x_0.

Remark 3.3. Let $\epsilon(p) = (-1)^d$ if p is the centre of a d-cell of P. We note right here that $\epsilon(p) = (-1)^{d'+1}$ when p is viewed as the centre of the dual d'-cell of $\mathcal{T}(P)$, because $d' = 3 - d$. When $\partial M \cong S^2$, so X is a closed manifold $\widehat{M}$, the chain $\sum_p \epsilon(p)\beta_p$ is easily recognized to be an Euler chain on $\widehat{M}$ with respect to the cellularization $\mathcal{T}(P)$, according to Turaev's terminology [15]. For arbitrary ∂M we have

$$\partial\left(\sum\nolimits_p \epsilon(p)\beta_p\right) = (1 - \chi(X)) \cdot x_0 - \sum\nolimits_p \epsilon(p)p,$$

where the Euler characteristic of X is given by $1 + \chi(M) - \chi(\partial M) = 1 - \chi(M)$.

For the sake of brevity, in the sequel we will denote $\pi_1(X, x_0)$ just by π. We denote now by $(\widetilde{X}, \widetilde{x}_0)$ a universal cover of (X, x_0). The reason for considering pointed spaces is that any two such covers are *canonically* isomorphic and all our constructions will obviously be equivariant under such isomorphisms. On $\widetilde{X}$ we consider the action of π defined using the basepoint $\widetilde{x}_0$. We denote by $\widetilde{\mathcal{T}}(P)$ the π-invariant lifting of $\mathcal{T}(P)$ to $\widetilde{X}$. We will consider in the sequel the complex $C_*^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ of integer chains in $\widetilde{X}$ which are cellular with respect to $\widetilde{\mathcal{T}}(P)$. In a natural way, $C_*^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ is a complex of $\mathbb{Z}[\pi]$-modules. Moreover, each $C_i^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ is a free $\mathbb{Z}[\pi]$-module: a free basis is determined by the choice of an ordering for the i-simplices in $\mathcal{T}(P)$ and one lifting for each of them (as remarked, orientations are canonical).

Lifted spider and free bases We define $\widetilde{s}(P)$ as the singular 1-chain $\sum_p \widetilde{\beta}_p$ in $\widetilde{X}$, where $\widetilde{\beta}_p$ is the only lifting of β_p with final endpoint $\widetilde{x}_0$. We choose $\widetilde{x}_0$ as preferred lifting of x_0. For a positive-dimensional simplex of $\mathcal{T}(P)$ dual to a cell with centre p, we choose as preferred lifting the one which contains the initial endpoint of $\widetilde{\beta}_p$. If σ is an ordering of the simplices in $\mathcal{T}(P)$, we denote by $\mathfrak{g}_i(P, \sigma)$ the free $\mathbb{Z}[\pi]$-basis of $C_i^{\text{cell}}(\widetilde{X}; \mathbb{Z})$ obtained from σ and from these preferred liftings.

We briefly review now the general algebraic machinery used to define torsions [12]. We consider a ring Λ with unit, with the property that if n

and m are distinct positive integers then Λ^n and Λ^m are not isomorphic as Λ-modules. We recall that the Whitehead group $K_1(\Lambda)$ is defined as the Abelianization of $\mathrm{GL}_\infty(\Lambda)$. Moreover, $\overline{K}_1(\Lambda)$ is the quotient of $K_1(\Lambda)$ under the action of $-1 \in \mathrm{GL}_1(\Lambda) = \Lambda_* \subset \Lambda$.

Given a free Λ-module M and two finite bases $\mathfrak{b} = (b_k)$ and $\mathfrak{b}' = (b'_k)$ of M, the assumption on Λ guarantees that $\mathfrak{b}$ and $\mathfrak{b}'$ have the same number of elements, so there exists an invertible square matrix (λ_k^h) such that $b'_k = \sum_h \lambda_k^h b_h$. We denote by $[\mathfrak{b}'/\mathfrak{b}]$ the image of (λ_k^h) in $K_1(\Lambda)$.

Twisted homology and chain bases Going back to the topological situation, let us consider now a group homomorphism $\varphi : \pi \to \Lambda_*$ and its natural extension to a ring homomorphism $\widetilde{\varphi} : \mathbb{Z}[\pi] \to \Lambda$. We can define now the twisted chain complex $C_*^\varphi(P)$, where $C_i^\varphi(P)$ is defined as $\Lambda \otimes_{\widetilde{\varphi}} C_i^{\mathrm{cell}}(\widetilde{X};\mathbb{Z})$ and the boundary operator is induced from the ordinary boundary. Note that $C_i^\varphi(P)$ is a free Λ-module and each $\mathbb{Z}[\pi]$-basis of $C_i^{\mathrm{cell}}(\widetilde{X};\mathbb{Z})$ determines a Λ-basis of $C_i^\varphi(P)$. We denote by $\mathfrak{g}_i^\varphi(P,\sigma)$ the Λ-basis of $C_i^\varphi(P)$ corresponding to $\mathfrak{g}_i(P,\sigma)$ and by $H_i^\varphi(P)$ the i-th homology group of the complex $C_*^\varphi(P)$. The canonical isomorphism which exists between two pointed universal covers of (X,x_0) induces an isomorphism of the corresponding homology groups, so $H_*^\varphi(P)$ is intrinsically defined.

Remark 3.4.

1. It readily follows from the excision and dimension axioms of homology that $H_1(M,\partial M;\mathbb{Z})$ is canonically isomorphic to $H_1(X;\mathbb{Z})$. (A special case of this fact was tacitly used in Proposition 1.1 above.)

2. Given $\varphi : \pi \to \Lambda_*$ as above, if we compose φ with the natural projection $\Lambda_* \to \overline{K}_1(\Lambda)$, we get a new homomorphism $\varphi' : \pi \to \overline{K}_1(\Lambda)$. Now, noting that $\overline{K}_1(\Lambda)$ is Abelian and using the isomorphism just mentioned, we obtain another homomorphism
$$\varphi'' : H_1(M,\partial M;\mathbb{Z}) \to \overline{K}_1(\Lambda)$$
which will be crucial below.

3. The isomorphism $H_1(M,\partial M;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is most easily understood using a spine P, because the 1-cells of $\mathcal{I}(P)$ are precisely the same as those of $\mathcal{T}(P)$, and they are all cycles. This will allow us below to compute φ'' directly on X, using $\mathcal{T}(P)$.

Torsion - Acyclic case Assume that $H_*^\varphi(P)$ is equal to 0. Then we can apply the general definition of torsion of an acyclic chain complex of free Λ-modules with assigned bases. We briefly review this definition, confining ourselves to the case where the boundary modules are free (in general, stable bases should be used). So, let $\mathfrak{b}_i$ be a finite subset of $C_i^\varphi(P)$ such that $\partial\mathfrak{b}_i$ is a basis of $\partial C_i^\varphi(P)$. The complex being acyclic, $(\partial\mathfrak{b}_{i+1})\cdot\mathfrak{b}_i$ is now a basis of $C_i^\varphi(P)$, so we can compare it with $\mathfrak{g}_i^\varphi(P,\sigma)$.

Definition 3.5. We set

$$\tau_0^\varphi(P,\sigma)=\prod_{i=0}^{3}[(\partial\mathfrak{b}_{i+1})\cdot\mathfrak{b}_i/\mathfrak{g}_i^\varphi(P,\sigma)]^{(-1)^{i+1}}\in K_1(\Lambda).$$

Independence of the $\mathfrak{b}_i$'s and invariance under isomorphism of pointed universal covers is readily checked. Of course σ is responsible of at most a sign change, so $\tau^\varphi(P)=\pm\tau_0^\varphi(P,\sigma)\in\overline{K}_1(\Lambda)$ is well-defined.

Torsion - General case Often $C_*^\varphi(P)$ is not acyclic. It is a general fact that a torsion $\tau^\varphi(P,\mathfrak{h})\in\overline{K}_1(\Lambda)$ can be defined also in this case, provided the homology Λ-modules are free and have assigned bases $\mathfrak{h}_*$. Namely, if $\mathfrak{h}_i$ is a Λ-basis of $H_i^\varphi(P)$, we replace $(\partial\mathfrak{b}_{i+1})\cdot\mathfrak{b}_i$ in the above formula by $(\partial\mathfrak{b}_{i+1})\cdot\widetilde{\mathfrak{h}}_i\cdot\mathfrak{b}_i$, where $\widetilde{\mathfrak{h}}_i$ is a lifting of $\mathfrak{h}_i$ to $C_i^\varphi(P)$.

It is maybe appropriate here to remark that the choice of bases $\mathfrak{h}_*$ of $H_*^\varphi(P)$ and the definition of $\tau^\varphi(P,\mathfrak{h})$ implicitly assume a description of the universal cover of X, which is typically undoable in practical cases. However, if one starts from a representation of π into the units of a *commutative* ring Λ, one can use from the very beginning the maximal Abelian rather than the universal cover, which makes computations more feasible.

4 Torsion of vector fields

This section is devoted to the (long) proof of the following:

Theorem 4.1. *If $P_i\in\mathrm{BrSpin}(M)$ and $\xi_i=\Phi(P_i)\in\mathrm{Vect}(M)$ for $i=1,2$, then the following equality holds in $\overline{K}_1(\Lambda)$:*

$$\tau^\varphi(P_2,\mathfrak{h})=\varphi''(\alpha(\xi_2,\xi_1))\cdot\tau^\varphi(P_1,\mathfrak{h}).\qquad(\star)$$

Before plunging into the proof, we state the main result of the present paper, which follows directly from Theorems 2.2 and 4.1:

Corollary 4.2. *If we set $\tau^\varphi(\xi,\mathfrak{h})=\tau^\varphi(P,\mathfrak{h})$ for $P\in\Phi^{-1}(\xi)$, we get a well-defined map*

$$\tau^\varphi(\,\cdot\,,\mathfrak{h}):\mathrm{Vect}(M)\to\overline{K}_1(\Lambda)$$

such that

$$\tau^\varphi(\xi+h,\mathfrak{h})=\varphi''(h)\cdot\tau^\varphi(\xi,\mathfrak{h})$$

for all $\xi\in\mathrm{Vect}(M)$ and $h\in H_1(M,\partial M;\mathbb{Z})$.

The proof of Theorem 4.1 is organized as follows. We first show that the equivariance formula $(\star)$ holds when P_2 is obtained from P_1 either by a positive branched MP-move, or by a positive branched lune move, or by a change of branching. Then we conclude using a recent result of A. Makovetskii [7] on the existence of a spine which dominates, as far as the positive MP and lune moves are concerned, any two given spines of M.

4.1 Equivariance under MP-moves

To show that the equivariance formula $(\star)$ holds when P_2 is obtained from P_1 by a positive branched MP-move, namely by a move either as in Fig. 3 or as in Fig. 4, we first discuss how to compute torsion using a subdivision of a triangulation carried by a spine.

Spider of a subdivision Given a triangulation $\mathcal{T}$ and an Euler chain s for $\mathcal{T}$, a general technology of Turaev [15] explains how to construct an Euler chain s' for a subdivided triangulation $\mathcal{T}'$, and shows that torsion is unchanged. In the special context of triangulations coming from branched spines we can give a simplified version of this technology, which does not use any of Turaev's results.

Consider $P \in \mathrm{BrSpin}(M)$ and the corresponding triangulation $\mathcal{T}(P)$ of the complex X, and let $\mathcal{D}$ be a subdivision of $\mathcal{T}(P)$. We are mainly interested in the case where also $\mathcal{D}$ is a triangulation (possibly with multiple and self-adjacencies). A subdivided spider $s_{\mathcal{D}}(P)$ can be defined as $\sum_p \beta_p$, where $\{p\}$ is a collection of one interior point for each simplex of $\mathcal{D}$ and β_p is (the closure of) the orbit of $v(P)$ which starts at p and reaches x_0. The reader can easily check that the choice of $\{p\}$ is inessential.

Now consider the data Λ, φ, $\mathfrak{h}$ which allow to define a torsion $\tau^{\varphi}(P,\mathfrak{h})$. We can define the Λ-modules $C_*^{\varphi,\mathcal{D}}(P)$ using cellular chains with respect to the lifting of $\mathcal{D}$ to $\widetilde{X}$ and we can construct preferred Λ-bases of these modules, using the spider $s_{\mathcal{D}}(P)$. Recall that there exists a canonical isomorphism $H_*^{\varphi,\mathcal{D}}(P) \cong H_*^{\varphi}(P)$, so we can use the same symbol $\mathfrak{h}$ for a Λ-basis of $H_*^{\varphi,\mathcal{D}}(P)$, and define a torsion $\tau^{\varphi,\mathcal{D}}(P,\mathfrak{h})$, exactly as we have done in Section 3.

Invariance of torsion under subdivision In the above setting, it is a general fact that $\tau^{\varphi,\mathcal{D}}(P,\mathfrak{h}) = \tau^{\varphi}(P,\mathfrak{h})$, but we will use this fact only in two special cases, so we concentrate on these cases, giving an elementary combinatorial proof. Namely, we refer to the situation under consideration of a positive branched MP-move which transforms a certain $P_1 \in \mathrm{BrSpin}(M)$ into another $P_2 \in \mathrm{BrSpin}(M)$. In this case there is an obvious easiest sub-

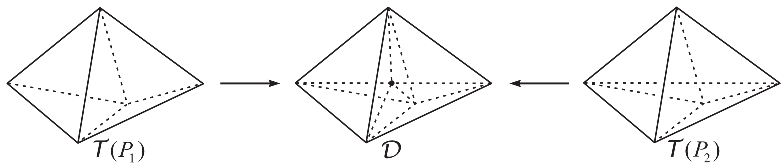

Figure 8: Common subdivision of dual triangulations for the MP-move.

division $\mathcal{D}$ of $\mathcal{T}(P_1)$ and $\mathcal{T}(P_2)$, shown in Fig. 8. And the following holds:

Proposition 4.3. *For $i = 1, 2$, we have $\tau^{\varphi,\mathcal{D}}(P_i, \mathfrak{h}) = \tau^{\varphi}(P_i, \mathfrak{h})$.*

Note that of course $\tau^{\varphi,\mathcal{D}}(P_1, \mathfrak{h})$ and $\tau^{\varphi,\mathcal{D}}(P_2, \mathfrak{h})$ can differ from each other, because the spiders $s_{\mathcal{D}}(P_1)$ and $s_{\mathcal{D}}(P_2)$ may provide different instructions for the liftings of the cells of $\mathcal{D}$.

Proof of 4.3. Note first that the definition of subdivided spider $s_{\mathcal{D}}(P)$ leads to the following very natural rule: a simplex of $\mathcal{D}$ lying in a simplex S of $\mathcal{T}(P)$ is lifted in $\widetilde{X}$ to the only preimage which lies in the lifting of S determined by $s(P)$. Now, this rule makes sense also for more general subdivisions than for triangulations, in particular for cell complexes, so we will use them. One easily sees that the subdivisions of $\mathcal{T}(P_1)$ and $\mathcal{T}(P_2)$ into $\mathcal{D}$ can be expressed as combinations of the following elementary transformations (including inverses):

1. The subdivision of an edge into two edges by insertion of a vertex;
2. The subdivision of a square into two triangles by insertion of a diagonal;
3. The transformation which inserts one triangle in a polyhedron with 5 vertices, 9 edges and 6 triangular faces, thus splitting the polyhedron into two tetrahedra.

We are left to prove that torsion is invariant under these transformations. In all three cases, the proof goes as follows:

i. We consider data $\mathfrak{g}_i$, $\mathfrak{b}_i$, $\widetilde{\mathfrak{h}}_i$, with $i = 0, \dots, 3$, which allow to compute τ before subdivision;

ii. We describe new data $\mathfrak{g}'_i$, $\mathfrak{b}'_i$, $\widetilde{\mathfrak{h}}'_i$ for the subdivided complex;

iii. We analyze the matrices $((\partial\mathfrak{b}_{i+1}) \cdot \widetilde{\mathfrak{h}}_i \cdot \mathfrak{b}_i)/\mathfrak{g}_i$ and $((\partial\mathfrak{b}'_{i+1}) \cdot \widetilde{\mathfrak{h}}'_i \cdot \mathfrak{b}'_i)/\mathfrak{g}'_i$ to show that they have the same image in $\overline{K}_1(\Lambda)$.

Note that this proves that torsion is unchanged "term by term", not only globally. We only make the proof explicit for the third type of subdivision, leaving the other two (easier) cases to the reader. Denote by Q the polyhedron which is split into tetrahedra T_1 and T_2 by the insertion of a triangle Δ. Then, in a natural way, we have that $\mathfrak{g}'_0$ is equal to $\mathfrak{g}_0$, that $\mathfrak{g}'_1$ is equal to $\mathfrak{g}_1$, and that $\mathfrak{g}'_2$ is obtained from $\mathfrak{g}_2$ by inserting the lifting of Δ which lies in the lifting $\widetilde{Q}$ of Q. To get $\mathfrak{g}'_3$ from $\mathfrak{g}_3$, we need to remove $\widetilde{Q}$ and insert the liftings $\widetilde{T}_1$ and $\widetilde{T}_2$ of T_1 and T_2 which lie in $\widetilde{Q}$. For the lifted homology bases, we have $\widetilde{\mathfrak{h}}'_i = \widetilde{\mathfrak{h}}_i$, for $i = 0, 1, 2$, whereas $\widetilde{\mathfrak{h}}'_3$ is obtained from $\widetilde{\mathfrak{h}}_3$ by replacing each occurrence of $\widetilde{Q}$ with $\widetilde{T}_1 + \widetilde{T}_2$. Similarly, we have $\mathfrak{b}'_i = \mathfrak{b}_i$ except for $i = 3$, and again $\mathfrak{b}'_3$ is obtained from $\mathfrak{b}_3$ by replacing each occurrence of $\widetilde{Q}$ with $\widetilde{T}_1 + \widetilde{T}_2$ and then inserting $\widetilde{T}_2$. The transition matrices are unchanged in dimensions 0 and 1, while in dimension 2 and 3, with obvious meaning of symbols, we have:

$$((\partial\mathfrak{b}'_3)\cdot\widetilde{\mathfrak{h}}'_2\cdot\mathfrak{b}'_2)/\mathfrak{g}'_2 = \left(\begin{array}{c|c|c|c} \partial\mathfrak{b}_3/\mathfrak{g}_2 & \begin{array}{c} * \\ \vdots \\ * \end{array} & \widetilde{\mathfrak{h}}_2/\mathfrak{g}_2 & \mathfrak{b}_2/\mathfrak{g}_2 \\ \hline 0\cdots 0 & 1 & 0\cdots 0 & 0\cdots 0 \end{array}\right)$$

$$(\widetilde{\mathfrak{h}}'_3\cdot\mathfrak{b}'_3)/\mathfrak{g}'_3 = \left(\begin{array}{c|c|c} \widetilde{\mathfrak{h}}_3/(\mathfrak{g}_3\setminus\{\widetilde{Q}\}) & \mathfrak{b}_3/(\mathfrak{g}_3\setminus\{\widetilde{Q}\}) & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline \widetilde{\mathfrak{h}}_3/\widetilde{Q} & \mathfrak{b}_3/\widetilde{Q} & 0 \\ \hline \widetilde{\mathfrak{h}}_3/\widetilde{Q} & \mathfrak{b}_3/\widetilde{Q} & 1 \end{array}\right).$$

When Λ is a field, one immediately gets the conclusion by taking determinants. For the general case, one needs to recall the definition of $\overline{K}_1(\Lambda)$, but the conclusion follows anyway. $\boxed{4.3}$

Equivariance of torsion We prove now formula ($\star$) in the case under consideration of a branched MP-move transforming P_1 into P_2. The proof is split in two steps. We first show that, after the move, torsion gets multiplied by $\varphi''(h)$ for a certain $h \in H_1(M, \partial M; \mathbb{Z})$ which we describe explicitly. Later we show that h is precisely the comparison class of the fields carried by P_2 and P_1 respectively.

Proceeding with the same notation as above, consider the subdivided spiders $s_{\mathcal{D}}(P_i) = \sum_p \beta_p^{(i)}$. Recall from Remark 3.3 that we have defined $\epsilon(p)$ as $(-1)^{d+1}$ when p is the centre of a d-cell of $\mathcal{T}(P)$. This definition of course extends to the centre of any cell of $\mathcal{D}$. It is now easy to see that

$\sum_p \epsilon(p)(\beta_p^{(2)} - \beta_p^{(1)})$ is a cycle, so we can define $h(P_2, P_1)$ to be its class in $H_1(M, \partial M; \mathbb{Z})$.

Proposition 4.4. *$\tau^\varphi(P_2, \mathfrak{h}) = \tau^\varphi(P_1, \mathfrak{h}) \cdot \varphi''(h(P_2, P_1))$.*

Proof of 4.4. By Proposition 4.3, it is enough to show that

$$\tau^{\varphi,\mathcal{D}}(P_2, \mathfrak{h}) = \tau^{\varphi,\mathcal{D}}(P_1, \mathfrak{h}) \cdot \varphi''(h(P_2, P_1)).$$

With obvious meaning of symbols, we have

$$\begin{aligned} \tau^{\varphi,\mathcal{D}}(P_2, \mathfrak{h}) &= \pm \prod_{i=0}^{3} \left[\left((\partial \mathfrak{b}_{i+1}) \cdot \widetilde{\mathfrak{h}}_i \cdot \mathfrak{b}_i \right) / \mathfrak{g}_i^{\varphi,\mathcal{D}}(P_2) \right]^{(-1)^{i+1}} \\ &= \pm \prod_{i=0}^{3} \left[\left((\partial \mathfrak{b}_{i+1}) \cdot \widetilde{\mathfrak{h}}_i \cdot \mathfrak{b}_i \right) / \mathfrak{g}_i^{\varphi,\mathcal{D}}(P_1) \right]^{(-1)^{i+1}} \cdot \left[\mathfrak{g}_i^{\varphi,\mathcal{D}}(P_1) / \mathfrak{g}_i^{\varphi,\mathcal{D}}(P_2) \right]^{(-1)^{i+1}} \\ &= \tau^{\varphi,\mathcal{D}}(P_1, \mathfrak{h}) \cdot \Big(\prod_{i=0}^{3} \left[\mathfrak{g}_i^{\varphi,\mathcal{D}}(P_1) / \mathfrak{g}_i^{\varphi,\mathcal{D}}(P_2) \right]^{(-1)^{i+1}} \Big). \end{aligned}$$

To compute the last correction factor, for $k = 1, 2$ let us denote by $\widetilde{e}_p^{(k)}$ the lifting to $\widetilde{X}$ determined by $s_{\mathcal{D}}(P_k)$ of the cell of $\mathcal{D}$ centered at p. Note that we have $\widetilde{e}_p^{(1)} = \gamma_p \cdot \widetilde{e}_p^{(2)}$, where γ_p is the loop $[(\beta_p^{(1)})^{-1} \cdot \beta_p^{(2)}]$ in π. This implies that $[\mathfrak{g}_i^{\varphi,\mathcal{D}}(P_1) / \mathfrak{g}_i^{\varphi,\mathcal{D}}(P_2)]$ is the image in $\overline{K}_1(\Lambda)$ of a diagonal matrix with entries $\varphi(\gamma_p)$, as p varies in the centres of the i-cells of $\mathcal{D}$. It easily follows that

$$\prod_{i=0}^{3} \left[\mathfrak{g}_i^{\varphi,\mathcal{D}}(P_1) / \mathfrak{g}_i^{\varphi,\mathcal{D}}(P_2) \right]^{(-1)^{i+1}} = \varphi''(h(P_2, P_1))$$

whence the conclusion. 4.4

Proposition 4.5. *If $\xi_i = \Phi(P_i) \in \mathrm{Vect}(M)$ then $\alpha(\xi_2, \xi_1) = h(P_2, P_1)$. If moreover the move from P_1 to P_2 is a sliding one, then $\alpha(\xi_2, \xi_1) = 0$.*

Proof of 4.5. Instead of treating in detail all the moves, we confine ourselves to the description of a general framework and then we apply the method to one particular move (a bumping one), the other cases being similar. Recall that Fig. 8 describes the portion U of X where the move takes place, *i.e.* the intersection with U of the triangulations $\mathcal{T}(P_1)$ and $\mathcal{T}(P_2)$ and their common subdivision $\mathcal{D}$. More precisely, the figure shows an "unfolded version" U' of the portion U, because in U all the "external" vertices of the figure are identified together (giving the point x_0), so for example each

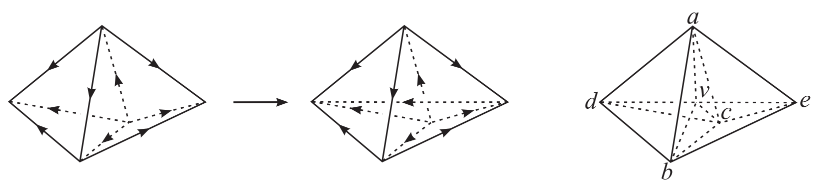

Figure 9: Left: the bumping move of Fig. 4 in terms of edge-orientations. Right: notation for the subdivided triangulation.

edge of $\mathcal{T}(P_1)$ and $\mathcal{T}(P_2)$ represents a (possibly non-trivial) element of π, and some external edges or faces could be glued together.

The basic idea of the proof is to lift the cycle $\sum_p \epsilon(p)(\beta_p^{(2)} - \beta_p^{(1)})$ to a 1-chain in U', to study this chain in U' and then show that its image in U is precisely the comparison class of the fields.

Recall that a branching on a spine is encoded by a (suitable) system of orientations for the edges of the dual triangulation. So a branched MP-move from P_1 to P_2 is described by a system of edge-orientations for $\mathcal{T}(P_1)$ and one for $\mathcal{T}(P_2)$, such that orientations coincide on common edges. The systems define fields as in Fig. 7-right, and $h(P_2, P_1)$ is constructed by integrating these fields from the centres of the cells and taking the difference. In particular, the non-zero contributions to $h(P_2, P_1)$ can only come from simplices of $\mathcal{D}$ which are not shared with both $\mathcal{T}(P_1)$ and $\mathcal{T}(P_2)$. So we only have to deal with the internal simplices of U', namely 1 vertex, 5 edges, 9 faces and 6 tetrahedra. Namely, we must compute the liftings to U' of the loops $\epsilon(p) \cdot (\beta_p^{(2)} - \beta_p^{(1)})$ for these internal simplices, and show that the projection in U gives $\alpha(\xi_2, \xi_1)$.

We will now carry out the computations for a bumping move. For a sliding move we would have that $\alpha(\xi_2, \xi_1)$ vanishes, and the computations would be very similar. The bumping move of Fig. 4 translates in terms of edge-orientations as described in Fig. 9-left. In Fig. 9-right we introduce notation for the subdivided triangulation (note that $a = b = c = d = e = x_0$ in X, because U is shown unfolded). We want to analyze the contributions of the internal simplices to the lifted chain: since U' is contractible, it is enough to determine, for both fields, the targets of the orbits starting at the centres of simplices. This is done in the next tables.

Simplex σ	v	va	vb	vc	vd	ve
$(-1)^{\dim(\sigma)+1}$	-1	$+1$	$+1$	$+1$	$+1$	$+1$
End $\widetilde{\beta}^{(1)}_{p(\sigma)}$	b	b	b	b	d	e
End $\widetilde{\beta}^{(2)}_{p(\sigma)}$	d	d	d	d	d	d
Boundary	$b-d$	$d-b$	$d-b$	$d-b$	0	$d-e$

Simplex σ	vab	vbc	vac	vae	vbe	vbd	vad	vce	vcd
$(-1)^{\dim(\sigma)+1}$	-1	-1	-1	-1	-1	-1	-1	-1	-1
End $\widetilde{\beta}^{(1)}_{p(\sigma)}$	b	b	b	e	e	d	d	e	d
End $\widetilde{\beta}^{(2)}_{p(\sigma)}$	d	d	d	d	d	d	d	d	d
Boundary	$b-d$	$b-d$	$b-d$	$e-d$	$e-d$	0	0	$e-d$	0

Simplex σ	$vabe$	$vabd$	$vace$	$vbce$	$vbcd$	$vacd$
$(-1)^{\dim(\sigma)+1}$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
End $\widetilde{\beta}^{(1)}_{p(\sigma)}$	e	d	e	e	d	d
End $\widetilde{\beta}^{(2)}_{p(\sigma)}$	d	d	d	d	d	d
Boundary	$d-e$	0	$d-e$	$d-e$	0	0

The sum of the bottom rows of the tables is equal to $b-e$, so the chain $\sum_p \epsilon(p)(\beta_p^{(2)} - \beta_p^{(1)})$ is homologous to the class of the edge eb, oriented from e to b. To conclude the proof we note now that, by Lemma 4.6.3 of [1] and the isomorphism between $H_1(M, \partial M; \mathbb{Z})$ and $H^2(M; \mathbb{Z})$, the comparison class between $v(P_2)$ and $v(P_1)$ is precisely equal to $[eb] \in H_1(M, \partial M; \mathbb{Z})$. 4.5

4.2 Equivariance under lune moves

We now show that $(\star)$ holds when P_2 is obtained from P_1 by a branched lune move (see Fig. 5).

Remark 4.6. We know from [14] that, in presence of more than one vertex, the lune move can be obtained by a finite combination of MP-moves, including inverse moves. However, this fact cannot be used directly to prove $(\star)$ for the lune move, because along the sequence of MP-moves we may find some negative ones which cannot be turned into branched moves. So we use an alternative argument.

Remark 4.7. As Fig. 6-right shows, the dual version of the lune move replaces two triangles by two tetrahedra. In particular, to find a common subdivision of $\mathcal{T}(P_1)$ and $\mathcal{T}(P_2)$ we would have to consider all the tetrahedra incident to the two triangles which get replaced, and there can be any number of such tetrahedra. Therefore, if we want to describe the transformation of $\mathcal{T}(P_1)$ into $\mathcal{T}(P_2)$ explicitly, we cannot use subdivisions only.

Square-fattening We call *square-fattening* a transformation of a triangulation which acts on a pair of adjacent triangles by enlarging them to

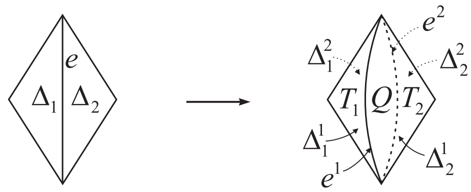

Figure 10: An unfolded version of the square-fattening.

a pillow which is then cellularized by four triangles, a bigon, and two 3-cells, as suggested in Fig. 10. Of course this transformation also requires a deformation of the surrounding simplices.

Back to the situation of a branched lune move from P_1 to P_2, let us apply the square-fattening to the triangles of $\mathcal{T}(P_1)$ which get doubled in $\mathcal{T}(P_2)$, and let us call $\mathcal{T}'$ the result. Of course $\mathcal{T}'$ and $\mathcal{T}(P_2)$ now have a common subdivision $\mathcal{D}$ which we can describe in local terms (see Fig. 11). To show $(\star)$ for the lune move we will now proceed as follows:

1. We define a specific field v' on $\mathrm{Int}(M) \cong X \setminus \{x_0\}$ and bases $\mathfrak{h}'_*$, such that a torsion $\tau^\varphi((\mathcal{T}', v'), \mathfrak{h}')$ can be defined as $\tau^\varphi(P, \mathfrak{h})$ in Section 3; moreover we show that $\tau^\varphi((\mathcal{T}', v'), \mathfrak{h}') = \tau^\varphi(P_1, \mathfrak{h})$;
2. We show that $\tau^\varphi((\mathcal{T}', v'), \mathfrak{h}')$ and $\tau^\varphi(P_2, \mathfrak{h})$ are both unchanged when passing to the common subdivision $\mathcal{D}$;
3. We conclude, as in Subsection 4.1, by first showing that $\tau^{\varphi,\mathcal{D}}(P_2, \mathfrak{h})$ differs from $\tau^{\varphi,\mathcal{D}}((\mathcal{T}', v'), \mathfrak{h}')$ by $\varphi''(h)$, where h is a specific element of $H_1(M, \partial M; \mathbb{Z})$, and then proving that $h = \alpha(\xi_2, \xi_1)$ if $\xi_i = \Phi(P_i) \in \mathrm{Vect}(M)$ for $i = 1, 2$.

Invariance under square-fattening Let us denote by Δ_1, Δ_2 and e the two triangles and their common edge before the square-fattening. After the move we denote the new triangles by Δ_i^j for $i, j = 1, 2$, the bigon by Q, the new edges by e^1 and e^2, and the new 3-cells by T_1 and T_2. Noting that there is a natural "flattening" projection $f : T_1 \cup T_2 \to \Delta_1 \cup \Delta_2$, we choose indices so that $f(T_i) = \Delta_i$, $f(\Delta_i^j) = \Delta_i$, and Δ_1^j is adjacent to Δ_2^j along e^j. See Fig. 10.

Now we define v' on $T_1 \cup T_2$ as the pull-back of $v(P_1)$ under f, and on $X \setminus (T_1 \cup T_2)$ as the pull-back under the natural diffeomorphism $f' : X \setminus (T_1 \cup T_2) \to X \setminus (\Delta_1 \cup \Delta_2)$. Since $f \sqcup f'$ is a homotopy equivalence and it fixes x_0, we see that v' is homotopic to $v(P_1)$ on $\mathrm{Int}(M)$.

Back to torsion, since every cell of $\mathcal{T}'$ is a union of orbits of v' and both the α-limit and ω-limit of any orbit of v' is $\{x_0\}$, we can define a spider

$s(\mathcal{T}', v')$ just as we did above for $\mathcal{T}(P)$ using $v(P)$. Moreover, since the canonically isomorphic modules $H_*^{\varphi,\mathcal{T}'}(P_1) \cong H_*^{\varphi}(P_1)$ are free over Λ (by hypothesis), we can select a basis $\mathfrak{h}_i'$ of $H_i^{\varphi,\mathcal{T}'}(P_1)$ corresponding to $\mathfrak{h}_i$. We can now define a torsion $\tau^{\varphi}((\mathcal{T}', v'), \mathfrak{h}') \in \overline{K}_1(\Lambda)$, exactly as we have done in Section 3 and the following lemma holds:

Lemma 4.8. $\tau^{\varphi}((\mathcal{T}', v'), \mathfrak{h}') = \tau^{\varphi}(P_1, \mathfrak{h})$.

Proof of 4.8. Let us consider $\widetilde{\mathcal{T}}(P_1)$ and $\widetilde{\mathcal{T}}'$, the π-invariant liftings respectively of $\mathcal{T}(P_1)$ and $\mathcal{T}'$ to $\widetilde{X}$, the universal cover of X. We consider the complexes $C_*^{\mathrm{cell},\mathcal{T}(P_1)}(\widetilde{X}, \mathbb{Z})$ and $C_*^{\mathrm{cell},\mathcal{T}'}(\widetilde{X}, \mathbb{Z})$ of integer chains in $\widetilde{X}$ which are cellular with respect to $\widetilde{\mathcal{T}}(P_1)$ and $\widetilde{\mathcal{T}}'$. In a natural way, each one of these complexes also has a structure of complex of $\mathbb{Z}[\pi]$-modules. The proof that the two torsions $\tau^{\varphi}(P_1, \mathfrak{h})$ and $\tau^{\varphi}((\mathcal{T}', v'), \mathfrak{h}')$ are equal goes as follows:

i. We consider data $\mathfrak{g}_i$, $\mathfrak{b}_i$, $\widetilde{\mathfrak{h}}_i$, with $i = 0, \ldots, 3$, which allow to compute $\tau^{\varphi}(P_1, \mathfrak{h})$ before the square fattening;

ii. We describe new data $\mathfrak{g}_i'$, $\mathfrak{b}_i'$, $\widetilde{\mathfrak{h}}_i'$ for the complex $C_*^{\mathrm{cell},\mathcal{T}'}(\widetilde{X}, \mathbb{Z})$;

iii. We analyze the matrices $((\partial \mathfrak{b}_{i+1}) \cdot \widetilde{\mathfrak{h}}_i \cdot \mathfrak{b}_i)/\mathfrak{g}_i$ and $((\partial \mathfrak{b}_{i+1}') \cdot \widetilde{\mathfrak{h}}_i' \cdot \mathfrak{b}_i')/\mathfrak{g}_i'$ to show that they have the same image in $\overline{K}_1(\Lambda)$.

To define $\mathfrak{g}_i$ and $\mathfrak{g}_i'$ we use respectively the spiders $s(P_1)$ and $s(\mathcal{T}', v')$: note that in the second case, due to the behavior of v', the lifting $\widetilde{T}_i$ contains in its boundary the liftings $\widetilde{\Delta}_i^1$ and $\widetilde{\Delta}_i^2$, for $i = 1, 2$. Note also that there is a natural correspondence (given by $f \sqcup f'$) between the cells of $\mathcal{T}(P_1) \setminus \{\Delta_1, \Delta_2, e\}$ and those of $\mathcal{T}' \setminus \{T_1, T_2, \Delta_1^1, \Delta_1^2, \Delta_2^1, \Delta_2^2, Q, e^1, e^2\}$. We can extend this correspondence to $\mathbb{Z}[\pi]$-module homomorphisms $F_i : C_i^{\mathrm{cell},\mathcal{T}(P_1)}(\widetilde{X}, \mathbb{Z}) \to C_i^{\mathrm{cell},\mathcal{T}'}(\widetilde{X}, \mathbb{Z})$ by defining $F_2(\widetilde{\Delta}_i) = \widetilde{\Delta}_i^1$, for $i = 1, 2$, and $F_1(\widetilde{e}) = \widetilde{e}^1$. So, we have $\mathfrak{g}_0' = F_0(\mathfrak{g}_0)$, $\mathfrak{g}_1' = F_1(\mathfrak{g}_1) \cdot \widetilde{e}^2$, $\mathfrak{g}_2' = F_2(\mathfrak{g}_2) \cdot \widetilde{\Delta}_1^2 \cdot \widetilde{\Delta}_2^2 \cdot \widetilde{Q}$ and $\mathfrak{g}_3' = F_3(\mathfrak{g}_3) \cdot \widetilde{T}_1 \cdot \widetilde{T}_2$; moreover, for the boundary modules, we have $\mathfrak{b}_0' = F_0(\mathfrak{b}_0)$, $\mathfrak{b}_1' = F_1(\mathfrak{b}_1)$, $\mathfrak{b}_2' = F_2(\mathfrak{b}_2) \cdot \widetilde{Q}$ and $\mathfrak{b}_3' = F_3(\mathfrak{b}_3) \cdot \widetilde{T}_1 \cdot \widetilde{T}_2$. It is not difficult to see that the $\mathfrak{b}_i'$'s actually are bases of the boundary modules.

Let us pass to the homology bases. If we apply F_i to a cycle α of $\widetilde{\mathfrak{h}}_i$, which represents an element of $\mathfrak{h}_i$, we do not necessarily obtain a cycle (the cycle can get "punctured" under the square-fattening); but it is not difficult to see that, adding a suitable multiple of $\widetilde{Q}$ (for the 2-cycles) or of $\widetilde{T}_1$ and $\widetilde{T}_2$ (for the 3-cycles), we obtain a cycle $F_i(\alpha)'$ which represents the class $(f \sqcup f')_*^{-1}([\alpha])$.

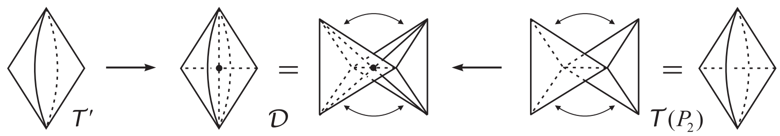

Figure 11: Common subdivision of $\mathcal{T}'$ and $\mathcal{T}(P_2)$.

The computations for step (iii) are similar to those of Proposition 4.3, so we omit them. 4.8

Subdivision In Fig. 11 we describe the common subdivision $\mathcal{D}$ of $\mathcal{T}'$ and $\mathcal{T}(P_2)$. Exactly as we have done in Subsection 4.1, we can now define torsions $\tau^{\varphi,\mathcal{D}}((\mathcal{T}',v'),\mathfrak{h}')$ and $\tau^{\varphi,\mathcal{D}}(P_2,\mathfrak{h})$ and show the following using the same arguments as in Proposition 4.3:

Proposition 4.9. *$\tau^{\varphi,\mathcal{D}}((\mathcal{T}',v'),\mathfrak{h}') = \tau^{\varphi}((\mathcal{T}',v'),\mathfrak{h}')$ and $\tau^{\varphi,\mathcal{D}}(P_2,\mathfrak{h}) = \tau^{\varphi}(P_2,\mathfrak{h})$.*

Equivariance of torsion As in Subsection 4.1, let $h(P_2,P_1)$ be the class in $H_1(M,\partial M;\mathbb{Z})$ of the cycle $\sum_p \epsilon(p)(\beta_p^{(2)} - \beta_p^{(1)})$, where p varies over centres of cells of $\mathcal{D}$ and the $\beta_p^{(i)}$'s come from the subdivided spiders $s_{\mathcal{D}}(\mathcal{T}',v') = \sum_p \beta_p^{(1)}$ and $s_{\mathcal{D}}(P_2) = \sum_p \beta_p^{(2)}$. The next two results imply $(\star)$ for the lune move. The first proof is omitted because similar to that of Proposition 4.4.

Proposition 4.10. $\tau^{\varphi}(P_2,\mathfrak{h}) = \tau^{\varphi}((\mathcal{T}',v'),\mathfrak{h}') \cdot \varphi''(h(P_2,P_1))$.

Proposition 4.11. *If $\xi_i = \Phi(P_i) \in \mathrm{Vect}(M)$ then $\alpha(\xi_2,\xi_1) = h(P_2,P_1)$. If moreover the move from P_1 to P_2 is a sliding one, then $\alpha(\xi_2,\xi_1) = 0$.*

Proof of 4.11. As already noted there are essentially 3 branched lune moves (see Fig. 5). For each of them, the calculation of $h(P_2,P_1)$ is carried out along the lines of the computation in the proof of Proposition 4.5. For the sliding lune moves, $h(P_2,P_1)$ turns out to vanish, and $\alpha(\xi_2,\xi_1)$ also vanishes, because $v(P_1)$ and $v(P_2)$ are homotopic. For the bumping lune move, $h(P_2,P_1)$ may not vanish, but it equals $\alpha([v(P_2)],[v(P_1)])$, which was computed in Lemma 4.6.3 of [1]. 4.11

4.3 Equivariance under change of branching

In this section we prove that, if $P_1,P_2 \in \mathrm{BrSpin}(M)$ are the same in $\mathrm{StSpin}(M)$, namely if they differ only for the branching, then their tor-

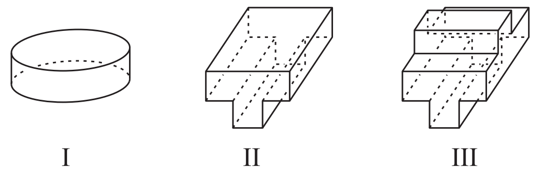

Figure 12: The three types of blocks into which the manifold M is decomposed.

sions are related as in formula $(\star)$, where $\xi_i = \Phi(P_i)$ for $i = 1, 2$.

Once again we consider the cycle $\sum_p \epsilon(p)(\beta_p^{(2)} - \beta_p^{(1)})$ and call $h(P_2, P_1)$ its class in $H_1(M, \partial M; \mathbb{Z})$. The next two results imply the desired equivariance. The proof of the first one is omitted because it is similar to that of Proposition 4.4.

Proposition 4.12. $\tau^{\varphi}(P_2, \mathfrak{h}) = \tau^{\varphi}(P_1, \mathfrak{h}) \cdot \varphi''(h(P_2, P_1))$.

Proposition 4.13. $h(P_2, P_1) = \alpha(\xi_2, \xi_1)$.

Proof of 4.13. By the geometric description of $\alpha(\xi_2, \xi_1)$ mentioned at the beginning of Section 1, it is sufficient to construct fields v_1' and v_2' such that:

- v_i' is homotopic to $v(P_i)$ for $i = 1, 2$;
- v_1' and v_2' are in general position with respect to each other and to ∂M;
- the properly embedded and oriented curve $\{x \in M : v_2'(x) = -v_1'(x)\}$ represents $h(P_2, P_1)$ in $H_1(M, \partial M; \mathbb{Z})$.

To construct v_1' and v_2' we start with some notation. Let $P \in \mathrm{StSpin}(M)$ be the non-branched spine underlying P_1 and P_2, so $P_i = (P, b_i)$ for $i = 1, 2$. The fact that P is a spine of M means that M can be viewed as a thickening of P, so the cellular structure of P induces a decomposition of M into solid blocks of the three types shown in Fig. 12. Up to homotopy, we can suppose that $v(P_1)$ and $v(P_2)$ are tangent to the boundary of the blocks and "vertical" on the blocks of type I. We also denote by $q : M \to P$ the projection induced by collapse.

We now define a singular field w_R on M for every region R of P, and another such field w_e for every edge e. We define w_R first on R to be tangent to R, radial and vanishing near ∂R, with one repelling singularity at the centre of R (recall that R is a disc). Then we extend w_R to $q^{-1}(R) \cong R \times [-1, 1]$ to be constant in the $[-1, 1]$-factor, and we set w_R to be zero outside $q^{-1}(R)$.

Similarly, w_e on e is tangent to e and null near ∂e, with one repelling singularity at the centre of e. On the rest of P the field w_e vanishes, except very close to e, where it is parallel to e with length rapidly decreasing to zero as one gets far from e. The extension to M is again taken to be "constant" with respect to $q : M \to P$.

Of course we can adjust things so that $\sum_R w_R$ is non-zero on each type I block (except on a segment at the centre of the block) and $\sum_e w_e$ is non-zero on each type II block (except on a hexagon which cuts the block into two halves). We define now v'_i to be $v(P_i) + \sum_R w_R + \sum_e w_e$. Since $\sum_R w_R + \sum_e w_e$ is never a negative multiple of $v(P_i)$, we have that v'_i is homotopic to $v(P_i)$. By construction v'_2 and $-v'_1$ are in general position, so $\alpha(v'_2, v'_1)$ can be constructed geometrically as the curve where v'_2 is a negative multiple of v'_1. Moreover v'_1 and v'_2 both point outside the blocks of type I and inside the blocks of type III. This implies that the contributions to $\alpha(v'_2, v'_1)$ can be analyzed block by block. We conclude the proof by analyzing the contributions to $\alpha(v'_2, v'_1)$ and showing that they are the same as those to $h(P_2, P_1)$.

Blocks of type I. Consider a block determined by a region R with centre p. If b_1 and b_2 are the same on R, the situation is quite simple: v'_1 and v'_2 are obviously homotopic on the block, and the chain $\beta_p^{(2)} - \beta_p^{(1)}$ is null-homologous. If $b_1(R)$ is opposite to $b_2(R)$, then v'_2 is a negative multiple of v'_1 precisely on the support of $\beta_p^{(2)} - \beta_p^{(1)}$, a properly embedded segment. If we project v'_1 and v'_2 to a cross-section orthogonal to the segment we see an index-1 repelling singularity in both cases, which easily implies that the right orientation for the segment is the same as that given by $\beta_p^{(2)} - \beta_p^{(1)}$.

Blocks of type II. Up to isomorphism there are 6 possible configurations for (b_1, b_2) near an edge e with centre p, and the analysis is easy in all cases. We only make one case explicit in Fig. 13. One sees from the figure that again v'_2 is a negative multiple of v'_1 only on the support of $-(\beta_p^{(2)} - \beta_p^{(1)})$, and a careful computation of indices shows that also orientations match.

Blocks of type III. Up to isomorphism we have 24 possible configurations for (b_1, b_2) near a vertex v. Analyzing them directly would be cumbersome, but we can avoid doing this by considering the local figure only and expressing the transition from b_1 to b_2 as a combination of more elementary transitions. Intermediate branchings may not be traces of global branchings on P, but looking at the local picture we do not need to worry about this: the local contribution to $\alpha(v'_2, v'_1)$ is just the sum of the local contributions of the elementary transitions. Showing that elementary transitions contribute as they do to $h(P_2, P_1)$ of course implies the conclusion.

To describe the elementary transitions, recall first that a system b of orientations for the edges of a tetrahedron Δ is dual to a branching of a

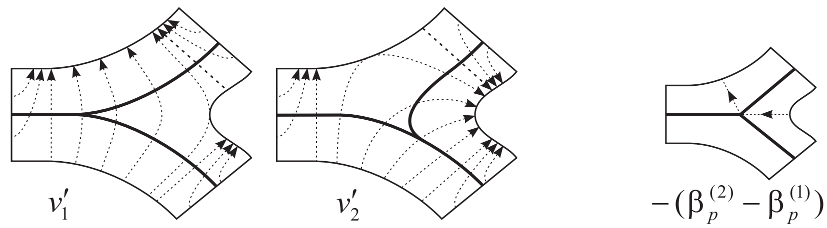

Figure 13: Left: the fields v_1' and v_2' on the hexagon where w_e vanishes. Right: the relative cycle $-(\beta_p^{(2)} - \beta_p^{(1)})$.

vertex of P if and only if one vertex of (Δ, b) is a source and one is a sink (this was first remarked in [6]). We call *little sink* the vertex of (Δ, b) with two incoming and one outgoing edge. Of course any two branchings on Δ are related by a combination of modifications of the following types:

Type A: Any of the 6 modifications of b which leave the sink fixed;

Type B: The modification of b which interchanges the sink with the little sink.

To analyze the contribution to $\alpha(v_2', v_1')$ when b_1 and b_2 are related by a type A or type B modification we need an extra construction. Inside the type-III block corresponding to the vertex under exam we consider a tetrahedron Δ as suggested in Fig. 14: note that Δ can be viewed as a shrunk copy of the tetrahedron dual to the vertex. In particular both b_1 and b_2 define orientations for the edges of Δ. Now we can further homotope v_1' and v_2' on the interior of the type III block, getting new fields v_1'' and v_2'' such that:

1. On Δ the field v_j'' coincides with the vector field determined by the edge-orientation carried by b_j as shown in Fig. 7-right;

2. On the complement of Δ both v_1'' and v_2'' point towards Δ (so all orbits asymptotically tend to some vertex of Δ).

The second condition implies that v_1'' and v_2'' can be a negative multiple of each other only within Δ. So we can consider Δ only, dismissing the rest of the type-III block. Moreover on Δ the fields are completely determined by b_1 and b_2.

If b_2 is obtained from b_1 by a type-A modification, then $\beta_p^{(2)} - \beta_p^{(1)}$ is null-homologous, and the points where v_2'' is a negative multiple of v_1'' are contained in the face of Δ opposite to the common sink. We can then get

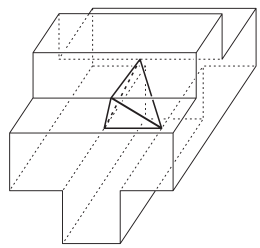

Figure 14: A shrunk version of the dual tetrahedron sits in a type-III block.

rid of these points by adding a field directed towards the common sink and supported within a neighborhood of that face.

For a type-B modification, note that b_2 only differs from b_1 on the edge e which joins the little sink to the sink of (Δ, b_1), so $[\beta_p^{(2)} - \beta_p^{(1)}] = [-e]$ in $H_1(M, \partial M; \mathbb{Z})$. The construction of v_1'' and v_2'' now readily implies that $\alpha(v_2'', v_1'')$ is precisely $[-e]$. $\boxed{4.13}$

We conclude the subsection by mentioning an alternative approach to Proposition 4.13 which has independent interest and provides a complete proof when $H_1(M, \partial M; \mathbb{Z})$ has no 2-torsion. Consider again P_1 and P_2 in $\mathrm{BrSpin}(M)$ which differ only for the branching, and the corresponding chain $\sum_p \epsilon(p)(\beta_p^{(2)} - \beta_p^{(1)})$ which represents $h(P_2, P_1)$, see above for notation. Proposition 4.13 implies that

$$2\mathrm{PD}(h(P_2, P_1)) = \mathcal{E}(v_2^\perp) - \mathcal{E}(v_1^\perp) \in H^2(M; \mathbb{Z}) \qquad (1)$$

where PD denotes Poincaré duality, $\mathcal{E}$ denotes the Euler class of a plane distribution, and $v_j^\perp$ is any plane distribution positively transversal to v_j. In addition, equation (1) implies Proposition 4.13 when the 2-torsion of $H^2(M; \mathbb{Z}) \cong H_1(M, \partial M; \mathbb{Z})$ vanishes.

We sketch now a direct geometric proof of equation (1). We first remind that in [1, § 7.1] we have exhibited a preferred 2-cochain $e(P)$ representing $\mathcal{E}(v(P)^\perp)$ for a branched spine P. This cochain is obtained by defining near $S(P)$ a non-singular field $\mu(P)$ (called the "maw" by Christy [5]) which is tangent to P and points from the locally twofold region to the locally onefold region. For a region R of P the value of $e(P)$ on R is then computed as the algebraic number of singularities which an extension of $\mu(P)$ to R must have.

Back to the situation where P_1 and P_2 are the same spine P except for the branching, we orient the regions of P as dictated by P_1 and note that $\mathcal{E}(v_2^\perp) - \mathcal{E}(v_1^\perp)$ has a certain representative $e(P_2) - \tilde{e}(P_1)$, where $e(P_2)$ is as just described and $\tilde{e}(P_1)$ is obtained from $e(P_1)$ by switching signs to the

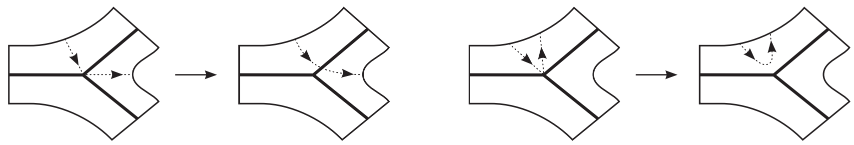

Figure 15: Modification of the 1-chain in a cross-section transversal to an edge.

duals of the regions which have different orientation in P_1 and P_2. Now we canonically modify the relative 1-chain $\sum_p \epsilon(p)(\beta_p^{(2)} - \beta_p^{(1)})$ described above to a new relative 1-chain $c(P_2, P_1)$ which is transversal to the regions of P and hence defines a 2-cochain, see Fig. 15. The modification for $\beta_p^{(2)} - \beta_p^{(1)}$ when p is the centre of an edge of P is suggested in Fig. 15. When p is the centre of a region no modification is needed, and when p is a vertex the situation is only slightly more complicated. Now one can show that

$$2c(P_2, P_1) = e(P_2) - \tilde{e}(P_1), \tag{2}$$

namely that the desired equality (1) holds at the level of preferred cochains, not only of cohomology classes. Equation (2) is most easily understood when P_2 is obtained from P_1 by reversing the orientation of all regions. In this case of course $\xi_2 = -\xi_1$ and $\tilde{e}(P_1) = -e(P_2)$, and $c(P_2, P_1)$ is easily recognized to be precisely $e(P_2)$ because it is obtained by sliding along the maw the original chain $\sum_p \epsilon(p)(\beta_p^{(2)} - \beta_p^{(1)})$.

4.4 Conclusion

The following result, stated informally above, was proved in [7]:

Theorem 4.14. *Given P_1 and P_2 in* $\mathrm{StSpin}(M)$*, there exists Q in* $\mathrm{StSpin}(M)$ *such that:*

- *Q can be obtained from P_1 using positive MP-moves only;*
- *Q can be obtained from P_2 using lune moves only.*

Let us now recall that a positive MP or lune move applied to a branched spine can always be turned into a branched move. This fact and Theorem 4.14 easily imply that any two branched spines of M can be obtained from each other by a combination of branched MP-moves, branched lune moves, and changes of branching. Having already shown that torsion is equivariant under these three elementary operations, we readily get the proof of Theorem 4.1 from the facts that α defines an affine structure on $\mathrm{Vect}(M)$ and that φ'' is a homomorphism.

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