Virtual Spacetime: An Environment for the Visualization of Curved Spacetimes via Geodesic Flows

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Abstract

We describe an implementation of a virtual environment for visualizing the geometry of curved spacetime by the display of interactive geodesics. This technique displays the paths of particles under the influence of gravity as described by the general theory of relativity, and is useful in the investigation of solutions to the field equations of that theory. A boom-mounted six degree of freedom head position sensitive stereo CRT system is used for display. A hand position sensitive glove controller is used to control the initial positions and directions of geodesics in spacetime. A multiprocessor graphics workstation is used for computation and rendering. We describe and illustrate with examples several techniques for visualizing the geometry of spacetime using geodesics. Though this work is described exclusively in the context of physical four-dimensional spacetimes, it extends to arbitrary geometries in arbitrary dimensions. While this work is intended for researchers, it is also useful for the teaching of general relativity.

Figure 1. The virtual spacetime system in use, showing the manipulation and observation of a spray of geodesics.

1: Introduction

According to the theory of general relativity, the motion of objects under the influence of gravity can be understood in terms of objects moving along the straightest paths in a curved four-dimensional spacetime [1][2][3]. The curvature is determined by the distribution of the mass and energy in spacetime. Given the curvature, this paper is concerned with the visualization of that curvature by studying the straightest lines, or geodesics, in that spacetime. These geodesics correspond to the actual paths of particles under the influence of gravity. Geodesics can exhibit complex three-dimensional structure, and can vary widely depending on their initial locations and directions. Virtual environments provide a natural three-dimensional display and control capability for the investigation of geodesics. The use of paths for the visualization of geometry and the interactive display and control paradigms described in this paper are inspired by earlier work on the virtual windtunnel [4].

Geodesics in curved spacetime are the shortest (or longest) path between any two points on the path. The geometry of a spacetime completely determines the geodesics. We can use geodesics to study both the curvature of spacetime and the physical motion of objects in that spacetime. Thus the visualization of spacetime via display of geodesics provides both geometric and physical insight.

This paper describes an implementation of an interactive virtual environment [4] which is dedicated to the exploration of curved spacetimes through the interactive display of geodesics (figure 1). The spacetimes considered are solutions to Einstein's equations, which specify the curvature of spacetime for a given distribution of matter and energy. The curved spacetime data provided can be closed form formulas or numerical data. Geodesics in spacetime are one-dimensional paths in four-dimensional space. The ability to rapidly move the geodesics around allows the researcher to get a sense of the overall curvature of spacetime. The interactive capability is provided by a VPL Dataglove, which measures the user's hand position, orientation and gesture. The three-dimensional structure of the geodesics is

(See color plates, p. CP-32.)

displayed via the Fake Space BOOM2C, a head-tracked wide field stereoscopic display system containing two high-resolution (1000x1000 pixels) CRT monitors. Using LCD shutters, two color channels can be displayed on the boom. The exploration capability requires the visualization software to compute and display the geodesics in response to the user inputs at a rate of about eight frames/second or better. To provide this performance, the geodesics are computed and rendered on a Silicon Graphics Iris 380GT/VGX workstation, which contains eight R3000 RISC processors for a total of 37 megaflops computational performance. This workstation also has high-performance three-dimensional graphics capability.

In section 1.1 we provide a brief survey of the role of visualization in general relativity. In section 2, the relevant mathematical framework is introduced. Section 3 describes the control and display of geodesics. Section 4 discusses the implementation of virtual spacetime, including visualization tools based on geodesics and their use in an example curved spacetime. Section 5 presents future work.

1.1 Visualization and General Relativity

Historically, visualization has played an important role in the development and presentation of general relativity. Many modern texts in general relativity (e. g. [1][3]) contain pictures which use visual arguments to motivate and argue significant technical points. This usefulness of visualization derives from the fact that general relativity is fundamentally a geometric theory. With the exception of Einstein's equations, which express the dependance of the curvature on the matter and energy distribution, the entire content of general relativity is to be found in the examination of curved space through the mathematics of differential geometry. Thus the visual display of the shapes of various aspects of spacetime geometry can provide real physical insight.

The difficulty of visualization in general relativity is that a curved spacetime is a four dimensional object. The difficulties of visually displaying the geometry of four-dimensional spaces are considerable. The are several methods in common use in general relativity to surmount these difficulties. They fall into two broad categories: embedding diagrams and physical simulations.

Embedding diagrams [1] are curved two-dimensional surfaces in flat three-dimensional space whose geometry corresponds with that of a two-dimensional slice through the curved four-dimensional spacetime. While embedding diagrams are a visually striking method of displaying curvature, they provide only indirect physical insight and only exist for simply curved spacetimes.

Physical simulations are typically based on the computation of geodesics. This is often done via direct computation of the paths in closed form [1][5]. This is a powerful method of studying the physical significance of a curved spacetime. Closed form computations are

possible only in cases of simple curvatures. Numerical computation of geodesics can be done for general spacetimes, including the results of numerical spacetime calculations [6][7]. The technique discussed in this paper uses computational geodesics and provides a real-time three-dimensional interface for both display and control of geodesics.

2: Spacetime, Geometry Data, and Geodesics

This section outlines the mathematics and concepts necessary to compute geodesics in spacetime given the geometry data for that spacetime. The necessary concepts will be given without motivation, proof, or complete context. Only information required to implement our visualization environment is given. For more detail, the reader should refer to the standard texts [1][2][3].

2.1 Spacetime and World Lines

Physics takes place in a four-dimensional space, called spacetime [8], which is the direct union of the three-dimensional space of our experience with time taken as a direction. From this point of view, objects which are bounded in space are, due to their persistence in time, extended objects in spacetime. Their extent in the time direction (in appropriate units) is much greater than their extent in the space direction. This gives the appearance of forming a line in spacetime oriented in the time direction called the object's *world line*. All objects in three-dimensional space form world lines in spacetime, so the study of the physics of objects in spacetime is the study of the behavior of the world lines of those objects. When these objects are not subject to any physical forces other than gravity, their world lines will be geodesics in curved spacetime.

Some simple examples will bring out important aspects of world lines. Consider a stationary object relative to some reference frame in three-dimensional space. In spacetime, that object is moving in the time direction. We can represent this motion in time by drawing a spacetime diagram, a two-dimensional diagram with the horizontal direction representing position in space and the vertical direction representing the position in time (figure 2). In this diagram, our motionless object has a world line that is vertical and parallel to the time axis.

An object moving from left to right will have a tilted world line. The slope of the moving object's world line will be 1/(speed of object). The velocity of an object in three-dimensional space corresponds to a tilt of that object's world line in spacetime, so specifying a velocity in three-dimensional space is the same as specifying a direction in spacetime. We will use this fact to control the initial direction of geodesics in curved spacetime. An object under some varying acceleration will have a curved world line (figure 2).
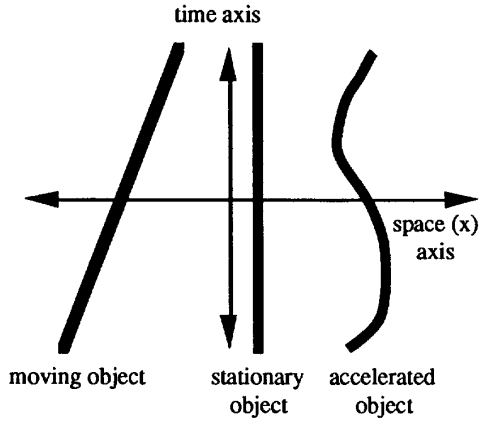
Figure 2. Example spacetime diagram showing world lines in a flat spacetime.

The spacetime diagram discussed above assumes a flat spacetime. If the spacetime is curved, the straight world lines would become curved geodesics. If these geodesics were observed without knowledge of the curvature of the spacetime, one would say that the objects corresponding to those curved world lines were accelerated by a force. In fact, they are not accelerated by a force but are following the curvature of spacetime. This is why gravity looks like a force when it is in fact due to the curvature of spacetime.

While space and time are directions in the geometric sense, the geometry in the time direction is different from that in the space directions. This is true for both flat and curved spacetime. For the moment, let us discuss the flat spacetime case. We can express the geometry of a space in terms of the formula for the distance between two points. In three-dimensional space, the distance between two points is given by the familiar Pythagorean theorem:

$$d^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

where x, y, and z are the usual Cartesian coordinates and d is the distance. In spacetime the distance formula is given by:

$$s^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

where t is the time coordinate, s is the distance in spacetime and c is the speed of light. In this formula we have chosen the 'timelike' convention, so that s^2 is positive for physical world lines which satisfy $c^2(\Delta t)^2 > (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ for any two points on that world line (called a timelike world line). A *spacelike* world line satisfies the inequality $c^2(\Delta t)^2 < (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ and corresponds to the path of a particle that moves faster than the speed of light. One could just as easily have chosen the 'spacelike' convention, reversing the signs in

the distance formula. We shall adopt the timelike convention throughout this paper.

The spacetime distance formula has the remarkable property that s can have the value zero for physically separate points. A world line of a light ray has the property $s = 0$ for any two points along that world line, and are called *lightlike* or *null* geodesics. These geodesics are important because causal influences in spacetime travel at speeds slower than or equal to light. Displaying the paths of light rays give information about the causal structure of the spacetime. Thus null geodesics play an important role in spacetime visualization.

2.2 Geometry Data: Metrics

The mathematical description of curved spacetime in general relativity is based on pseudo-Riemannian geometry, a part of differential geometry that describes the curvature of a space via distance relations between points of that space. For a general curved space of any dimension, these distance relations are expressed in a generalization of the Pythagorean theorem.

Let dx^μ denotes an infinitesimal distance in the x^μ direction in some coordinate system, where greek indices range from 0 to 3 with 0 labeling the time coordinate. The distance formula in spacetime is given by:

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}(x)dx^\mu dx^\nu$$

where each $g_{\mu\nu}(x)$ is an entry in a 4x4 matrix $g(x)$ of functions of a point x in spacetime called the *metric*. In the case of flat spacetime described in section 2.1 above, the metric is given by

$$g(x) = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In a general curved spacetime, any of the entries in the metric may be a non-constant function. On the other hand, the example of the distance formula in flat two dimensional space in polar coordinates, $d^2 = (\Delta r)^2 + r^2(\Delta\theta)^2$, shows that non-constant functions in the metric do not imply curvature. The Riemann curvature tensor is a function of the first and second derivatives of the metric which vanishes if and only if the space described by that metric is flat. This curvature tensor appears linearly as part of Einstein's field equations of general relativity, so Einstein's equations are second-order non-linear partial differential equations for the metric. Solutions to these equations for a particular distribution of matter are physical metrics that describe physical spacetime.

Metrics are computed in particular coordinate systems suited to different physical aspects of the spacetime. The

293

same physical spacetime is often described in different coordinates, implying different formulas for the metric. Since the metric is used to compute the geodesics of a spacetime, we will take as the basic geometry data for a spacetime a coordinate system and metric in those coordinates. These data can be in the form of exact closed form formulas, or as data on a computational grid.

2.3 Computing Geodesics

Given a metric $g(x)$, the geodesic starting at the point $x = (x^0, x^1, x^2, x^3)$ with initial spacetime direction $dx/ds = (dx^0/ds, dx^1/ds, dx^2/ds, dx^3/ds)$, where s is a parameter often taken to be arc length, may be obtained by iteratively solving the geodesic equations [1][2][3]:

$$\frac{d^2 x^\mu}{ds^2} = \sum_{\nu,\lambda=0}^{3} \Gamma^\mu{}_{\nu\lambda}(x) \frac{dx^\nu}{ds} \frac{dx^\lambda}{ds}.$$

The $\Gamma^\mu{}_{\nu\lambda}(x)$ are functions of $g(x)$, and are given by

$$\Gamma^\mu{}_{\nu\lambda}(x) = \sum_{\alpha=0}^{3} \frac{1}{2} g^{\mu\alpha}(x) \left(\frac{dg_{\alpha\nu}(x)}{dx^\lambda} + \frac{dg_{\alpha\lambda}(x)}{dx^\nu} - \frac{dg_{\nu\lambda}(x)}{dx^\alpha} \right)$$

These $\Gamma^\mu{}_{\nu\lambda}(x)$ are called connection coefficients or Christoffel symbols, and $g^{\mu\nu}(x)$ are the entries of the matrix $g^{-1}(x)$. Connection coefficients measure how tangent vectors in curved spacetime are turned by the curvature as they are translated in spacetime. The geodesic equations are simply the statement "acceleration relative to curved spacetime = 0". The connection coefficients are symmetric in two indices and so have 40 independent components. The connection coefficients can be numerically computed from metric data, but this induces numerical error and is time consuming. We have found it advantageous to include the connection coefficients as part of the geometry data. When the metric is available as exact formulas, the formulas for the connection coefficients are derived and included in the geometry data. For computational spacetimes, the metric and connection coefficients are directly available as data on a computational grid.

A numerical solution to the geodesic equations is obtained by selecting a Δs and integrating once to obtain, for each value of μ, a new dx^μ/ds, which is again integrated to find a new x^μ. This process is repeated, producing a path $x(s) = (x^0(s), x^1(s), x^2(s), x^3(s))$, which can then be displayed for visualization. When Δs is sufficiently small, Euler integration is sufficient. At

the nth step, $s = n\Delta s$, so using Euler integration the solution to the geodesic equations at the nth step is given by

$$\frac{dx^\mu(n\Delta s)}{ds} =$$

$$\Delta s \sum_{\nu,\lambda=0}^{3} \left(\Gamma^\mu{}_{\nu\lambda}(x((n-1)\Delta s)) \frac{dx^\nu((n-1)\Delta s)}{ds} \frac{dx^\lambda((n-1)\Delta s)}{ds} \right),$$

$$x^\mu(n\Delta s) = \Delta s \frac{dx^\mu(n\Delta s)}{ds}.$$

It is not always possible to choose a small Δs however, since one desires a sufficiently long path to display the geometry of spacetime over long distances. With a small Δs, many iterations would be required to produce a path of sufficient length. We also require, however, that the geodesics be computed and displayed sufficiently quickly that one can move the geodesics around in real time, and display them from a moving point of view. This puts a constraint on the number of points that can be computed for a path. Thus larger values of Δs can become necessary, in which case a more accurate numerical integration method than Euler integration may be required. We have found the Adams-Bashforth predictor-corrector integration [9] fast enough to be useful.

3: Control and Display of Geodesics

3.1 Mapping User Inputs to Geodesic Initial Conditions

Giving the user real-time control of geodesics amounts to giving the user control over the initial position $x = (x^0, x^1, x^2, x^3)$ and the initial direction vector $dx/ds = (dx^0/ds, dx^1/ds, dx^2/ds, dx^3/ds)$. As pointed out in section 2.1, the initial direction in four-dimensional spacetime can be thought of as a velocity vector in three-dimensional space. Using the VPL Dataglove, which provides the position and orientation of the user's hand, the user's hand position can be used to control the initial three-dimensional position of the geodesic. The initial time can be chosen as the current timestep in the case of dynamic spacetimes. The orientation of the user's hand in combination with a preset velocity magnitude can be used to control the initial direction vector. The precise mapping from the velocity vector to the initial spacetime direction will be described in sections 3.2.

We control the preset velocity magnitude "off-line" using mouse controls. In this design the user is interacting with the geodesic path of a particle with a fixed velocity and interactively controlled initial direction. This choice is motivated by ease of implementation and

the direct physical intuition it provides. Other choices of control would be interesting to explore. These could include using the degree of bend of a finger or some other interactive control.

3.2 Computation of Initial Spacetime Direction

When possible, we will be parametrizing our geodesics by arc length in spacetime. This is not possible for null geodesics, however, as they have spacetime arc length = 0. When converting the velocity vector to an initial spacetime direction vector, the timelike geodesics and null geodesics must be handled differently. We do not consider spacelike geodesics, though they can be handled in an analogous fashion. The kind of geodesic being generated is determined by the length of the initial velocity vector, which is computed by using the spacetime metric.

In this section the indices i and j refer to the three-dimensional space coordinates in the four-dimensional spacetime, and range from 1 to 3. We denote the time coordinate by t. From the definition of the metric in section 2.2,

$$ds^2 = g_{00}(x)dt dt + 2\sum_{i=1}^{3} g_{i0}(x)dx^i dt + \sum_{i,j=1}^{3} g_{ij}(x)dx^i dx^j$$

dividing by dt^2,

$$\left(\frac{ds}{dt}\right)^2 = g_{00}(x) + 2\sum_{i=1}^{3} g_{i0}(x) v^i + \sum_{i,j=1}^{3} g_{ij}(x) v^i v^j,$$

where $v = dx/dt$ is just the three-space velocity ($dx^i/dt = v^i$). In the case of timelike lines, this will be greater than zero and we can solve for $\frac{dt}{ds}$:

$$\frac{dt}{ds} = (g_{00}(x) + 2\sum_{i=1}^{3} g_{i0}(x) v^i + \sum_{i,j=1}^{3} g_{ij}(x) v^i v^j)^{-\frac{1}{2}}.$$

This gives one component of the initial spacetime direction. The other components are found via the chain rule:

$$\frac{dx^i}{ds} = \frac{dx^i}{dt}\frac{dt}{ds} = v^i \frac{dt}{ds}.$$

For null lines, the above expression for $\left(\frac{ds}{dt}\right)^2$ vanishes and we cannot simply solve for $\frac{dt}{ds}$. In this case the magnitude of the velocity is set to the speed of light and we can use an arbitrary parameterization of the geodesic. This allows us to set the $\frac{dt}{ds}$ component of the initial spacetime direction to an arbitrary value. Note that because the speed of light is, by definition, the velocity for which the spacetime distance vanishes, which is in turn dependent on the local geometry through the metric, the value of the speed of light in a fixed coordinate system will vary from point to point in space. Given a spacetime velocity vector, the magnitude of that vector may be greater than the speed of light in some regions of spacetime. We shall always consider such cases to be initial directions for null geodesics. Thus given a velocity vector v at some position x in space, when the magnitude of v is greater than the speed of light at x, we must scale v so that its magnitude is exactly the speed of light at x. Let a denote this scale factor, so $v \to av$. Substituting av into the expression for $\left(\frac{ds}{dt}\right)^2$ gives:

$$0 = g_{00}(x) + 2a\sum_{i=1}^{3} g_{i0}(x) v^i + a^2 \sum_{i,j=1}^{3} g_{ij}(x) v^i v^j.$$

This is a quadratic equation for a. Setting $A = \sum_{i,j=1}^{3} g_{ij}(x) v^i v^j$ and $B = 2\sum_{i=1}^{3} g_{i0}(x) v^i$, we find

$$a = \begin{cases} \dfrac{-B + \sqrt{B^2 - 4g_{00}(x) A}}{2A} & A \neq 0 \\[2ex] \dfrac{-g_{00}(x)}{B} & A = 0 \end{cases}$$

The three space components of the initial spacetime direction are found in the same way as in the timelike case:

$$\frac{dx^i}{ds} = \frac{dx^i}{dt}\frac{dt}{ds} = av^i \frac{dt}{ds}.$$

3.2 The Display of Geodesics

The geodesics that result from the calculations described above are paths $x(s) = (x^0(s), x^1(s), x^2(s), x^3(s))$ in four dimensions. This path must be converted into the appropriate coordinates for display

295

using the coordinate part of the geometry data. Since each point on the path has four components and our virtual environment renders these paths in a virtual three-dimensional space, there is some freedom as to how these paths are mapped to the display. There are several standard possibilities.

The first possibility is to map the spatial part of the paths $(x^1(s), x^2(s), x^3(s))$ to the three dimensional graphics space directly. This choice will be called a *spatial display*. The paths in this display are the paths in space followed by pointlike objects of infinitesimal mass under the influence of the gravity generated by the curvature of that spacetime.

The second possibility is to map the coordinates $x(s) = (x^0(s), x^1(s), x^2(s), x^3(s))$ to the three-dimensional point $(x^1(s), x^2(s), x^0(s))$ replacing the third spatial coordinate by the time coordinate, along with any necessary coordinate transformations. This choice will be called a *spacetime display*. The paths in this display are the world lines of pointlike objects as in the previous paragraph.

The exploration of other display options are of great interest. It would be interesting to explore the freedom that a virtual environment provides by developing mixed spatial and spacetime displays.

4: The Implementation of Virtual Spacetime

The implementation of the concepts described in section 3 with the interface described in section 4 will be discussed using a simple example spacetime, that of a non-rotating perfectly spherical mass of infinitesimal radius. The solution of Einstein's equations in this case is known in terms of closed functions and is known as the Schwarzschild solution. We will explicitly give the geometry data and discuss various tools and display options for visualization.

4.1 The Schwarzschild Solution

The Schwarzschild solution [1][2][3] is a solution to Einstein's equations corresponding to the exterior of a perfectly spherical, non-rotating body of mass m. This solution gives a good description of the spacetime around a planet or star and geodesics correspond to the paths of objects under the mass's gravitational influence. We shall choose time and mass units, called natural units, so that the speed of light c and Newton's gravitational constant G both have the values c = G = 1. In these units both time and mass have units of length. We shall work in Schwarzschild coordinates (t, r, θ, ϕ), which correspond to spherical coordinates at infinity and have a simple physical meaning. These coordinates are related to Cartesian coordinates (t, x, y, z) by

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi,$$

$$z = r\cos\theta.$$
$$t = t$$

In these coordinates, the metric is

$$g(t, r, \theta, \phi) = \begin{pmatrix} 1-\dfrac{2m}{r} & 0 & 0 & 0 \\ 0 & -\left(1-\dfrac{2m}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

The non-vanishing connection coefficients are:

$$\Gamma^t_{rt} = \Gamma^t_{tr} = \frac{m}{r(r-2m)}$$

$$\Gamma^r_{rr} = -\frac{m}{r(r-2m)}, \quad \Gamma^r_{\theta\theta} = -(r-2m),$$

$$\Gamma^r_{\phi\phi} = -\sin^2\theta(r-2m), \quad \Gamma^r_{tt} = \frac{m}{r^2}\left(1-\frac{2m}{r}\right),$$

$$\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\,\theta} = \frac{1}{r}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\,\phi} = \frac{1}{r}, \quad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta$$

There are two singularities in this metric, at r=0 and r=2m. The singularity at r=0 is a true singularity of the geometry in the sense that the curvature diverges there. The singularity at r=2m is the famous Schwarzschild radius and is an artefact of the coordinates which does not occur in other coordinate systems. This singularity is associated with an *event horizon*, however, since the region r<2m cannot send causal influences to the region r>2m. This is called a black hole, as anything that falls into the r<2m region can never come out again. Exploration of these event horizons with interactive null geodesics is an expected use of the system described in this paper.

Using single geodesics, various features of the Schwarzschild solution may be discovered with the system described in this paper. Geodesics with initial direction perpendicular to the radius vector describe orbits of the spacetime. When these orbits are not circular, they show a characteristic rosette pattern that corresponds to the phenomena of precession of perihelia in planetary orbits around the sun [1][2]. Null geodesics, which correspond to the paths of light rays, are seen to be bent and exhibit circular orbits when their initial directions are perpendicular to the radius vector at r=3m [1][2].

4.2 Visualization tools

Visualization tools in virtual spacetime are single geodesics or groups of geodesics analogous to rakes in the visualization of simulated fluid flow [4]. The number of geodesics in a tool are constrained by performance requirements. We currently use at most eight geodesics in a tool distributed across the eight processors in the

workstation. The position and orientation of the tools are manipulated with the VPL Dataglove. Tools that have been implemented besides single geodesics are sprays, rakes and tubes.

Sprays are groups of geodesics all of which have the same initial positions and velocity magnitudes but different initial directions (figures 1 and 3). Currently these directions all lay in the same plane which is oriented via the Dataglove. Sprays are useful for exploring the general geometry of a curved spacetime.
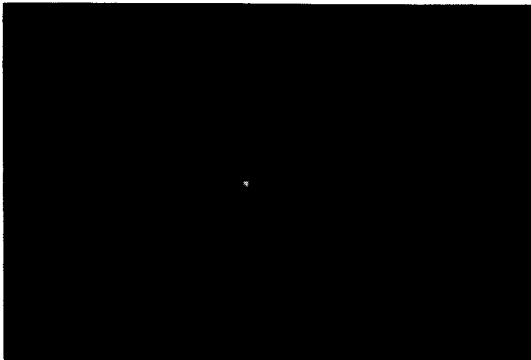

Figure 3. A spray of geodesics.

Rakes are lines of initial positions of geodesics all of which have the same initial velocity vectors. The geodesics are initially parallel and diverge or converge due to the curvature of spacetime. This divergence and convergence correspond to tidal forces and are a direct visualization of the components of the Riemann curvature tensor [1].

Tubes are analogous to rakes, with the initial positions distributed in a circle perpendicular to the initial velocities (figure 4). This allow the twisting tidal forces to be observed, along with other components of the curvature tensor.
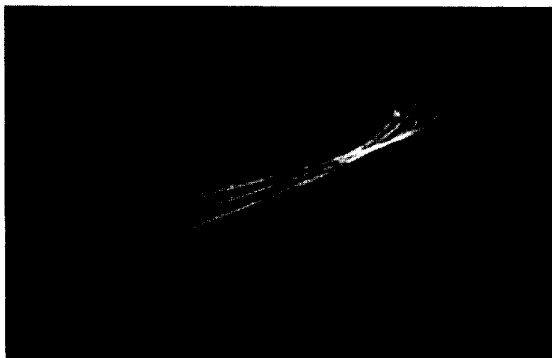

Figure 4. A tube of geodesics

A *lightsphere* is a set of short null geodesics with the same initial position whose initial directions are equidistributed over a sphere (figure 5). Lightspheres are useful for the detection of places of large curvature. By using very short geodesics, a full sphere in three-dimensional space may be displayed.
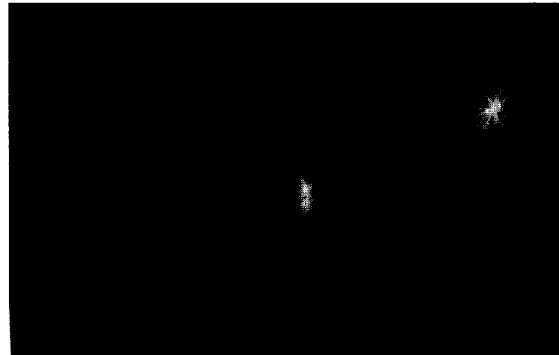

Figure 5. Lightspheres.

A *lightcone* is a spacetime display of the initial spacetime directions of a spray of null geodesics (figure 6). Lightcones are a standard visualization technique in general relativity, and are useful for probing the local causal structure of spacetime [1][2][3].
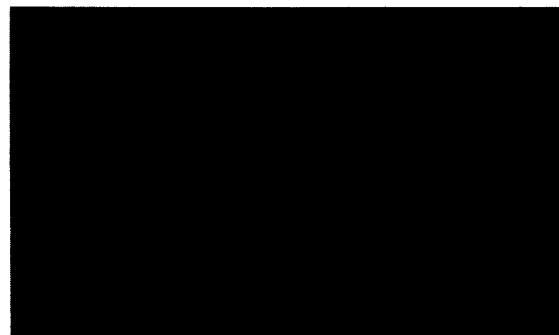

Figure 6. Lightcones

4.3 Performance

The real-time computation of geodesics requires fairly high computational power. The system described in this paper is capable of controlling, computing and displaying one to eight geodesics with lengths of about 300 points at about eight frames/second. This is sufficient for many purposes. A less expensive system capable of this performance with single geodesics is the Iris Indigo, which has a single MIPS R3000 processor. The computation of geodesics has also been implemented for a three-dimensional spacetime (time plus two space dimensions) on a Macintosh IIci, which can compute and display a geodesic with 200 points at about 8 frames/second.

(See color plates, p. CP-32.)

5: Further Work

The system described in this paper has demonstrated the feasibility of real-time interactive virtual environment techniques for the visualization of curved spacetimes using geodesics. Currently the system has been implemented only for spacetimes whose geometry data are available in closed form formulas and for spacetimes whose data are on simple static computational grids.

There are interesting spacetimes whose metrics are available as exact formulas. The Kerr [2][3][5] solution for a rotating black hole has been implemented. Incorporation of others into virtual spacetime should be straightforward. Examples include the classical Godel solution [3], Bianchi type IX cosmological solutions [10], and a metric describing collapsing dust or photons [11].

A more difficult problem is including the results of computational spacetime simulations such as those describing colliding black holes, collapsing stars, and gravitational waves [12]. There has been considerable interest expressed by the computational spacetime group at the National Center for Supercomputing Applications (NCSA) in Urbana, Illinois in using virtual spacetime to view the results of their computations. We are currently developing a collaboration for this purpose with Ed Seidel and Larry Smarr. Problems addressed by this collaboration include: the meaning of the coordinates used for the numerical simulation and how these coordinates should be mapped into the virtual space, developing an interpolation scheme to compute geometry data from the computational grid that is fast enough to allow real-time computation of geodesics; and managing the very large amounts of data that are the products of these unsteady spacetime computations. The data and speed problems are similar to those that arise in the virtual windtunnel [4]. Distributing the computation to a supercomputer over a high-speed network [13] may be necessary.

6: Acknowledgements

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