Torus and Hypertorus in 4D: A Synthetic Approach

Paolo Freguglia*

Dipartimento di Scienze, Università di Chieti-Pescara, Viale Pindaro, 42
I - 65127 PESCARA, Italy

Abstract. This paper would like to give a contribution to the analysis of the geometry of 4D phase space, which is very relevant for the study of betatron motion in beam physics. We will show fundamental surfaces and hypersurface (in particular the torus and the hypertorus) in 4D and some of their properties. Afterwards we will present the axiomatic bases of the 4D geometry.

1. Introduction

It is well-known that the betatronic motion is studied by a 4D phase space (for example see [7], [8]) and it is very important for this study the analysis of the geometrical manifolds. Usually the methods of analytical geometry are utilized for the study of the dynamical systems and in particular for the study of the hamiltonian systems. The torus and the geometry on the toral surface are analytically well described. If we want, for example, to determine *a specific* curve on the toral surface it is necessary the use of the coordinates, but if we want to consider *a generic* curve on this surface it is more convenient to utilise the methods of the syntetic geometry because these methods are less complicated than the analytical methods. We also think that it is necessary for a proper application of geometrical concepts a more exact knowledge of them and of their authentic meaning like the synthetic geometry suggests. This geometry does not need to explain its fundamental concepts (i.e. point, line, surface etc.) of other mathematical notions. For example a point is a primitive concept and it is not indispensable that this coincides with a pair of real numbers. The synthetic geometry is the "true" geometry. But the analitical methods are an essential tool especially for the geometrical interpretation of the algebraic expressions which derive from the techniques of mathematical analysis.

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Beyond theoretical reasons, the synthetical approach to geometry is useful for the techniques of the ***computer graphics***.
The bases of the syntetic geometry of four dimensions will be given in 3rd paragraph. In the following 2nd paragraph, we shall examine some hypersurfaces in S^4 (euclidean syntetic space of four dimensions, while we shall denote by $\mathbf{R}^4$ the correspondent analytical space) and so the torus and the hypertorus.

2. Torus and Hypertorus in 4D

Now we shall present some "classic" hypersurfaces of S^4 also useful for the study of phenomena. Considering in particular "round hypersurfaces" because these have also a more direct topological meaning.
A *hyperconical hypersurface* (of *first kind*) consists of the straight lines (*elements*) determined by the points of a hyperplane surface (that is a part of a S^3 of S^4, see par.3) (*directing-surface*) and a point (*vertex*) which does not belong to the hyperplane (a S^3 of S^4) of this surface. The hyperconical hypersurface has two nappes. A *hypercone of first kind* consists of the points of a hyperconical hypersurface together with the interior points to the portion of hyperspace delimitated by this hypersurface and with the interior points to the directing-surface. These points together with those of directing-surface form the *base* of the hypercone. The directing-surface can be a plane, a sphere, a circular conical surface or portions of these. We obtain a hyperplane in the case of the plane. A *plano-conical hypersurface* (or *hyperconical hypersurface of second kind*) consists of the planes (*elements*) determined by the points of a plane curve (*directing-curve*) and a line (*vertex-edge*) which does not belong to hyperplane of this curve. It is trivial the definition of *hypercone of second kind.*

Prop. 1: "A hyperplane which contains the directing-curve of a plano-conical hypersurface and a point of the vertex-edge, intersects the hypersurface in a conical surface" (see [2] p.71).

A *hypercylindrical hypersurface* (or *of first kind*) consists of the parallel straight lines (*elements*) passing through the points of a hyperplane surface (*directing-surface*) not lying in the hyperplane of the surface. This surface can be a plane, or a sphere, or a conical or cylindrical surface with directing-circle or a combination of parts of such surfaces. A *hypercilinder of first kind* is the hypersolid formed by the points of the hypercylindrical hypersurface together with the interior points of the portion of hyperspace delimited by this hypersurface and with the interior points of the directing-surface. A *plano-cylindrical hypersurface* or *hypercylindrical hypersurface of second kind* is formed by the parallel planes (*elements*) passing through the points of a plane curve (*directing-curve*) and intrsecting the plane of the curve only in these points. In particular we shall consider the case where the directing-curve is a circle. It is trivial the definition of *hypercylinder of second kind.*

Prop. 2: If a right triangle takes all possible positions with a fixed side, then the vertices and the points of the other two sides of the triangle make up a hypercone of first kind which has a sphere as a base and the straight line from the vertex to the centre of the sphere is perpendicular to the hyperplane to which belongs the sphere (*right spherical hypercone*). The fixed side is the axis, the hypothenuse is an element and the other side is a radius of the base (see [2] p.204).

Proof: In S^3 (see Fig.1) if the cathetus-base of a right triangle takes all possible positions, with the other cathetus fixed, these all possible positions are taken in a plane S^2 and being fixed the intersecion point between two cathetus, the cathetus-base makes a circle. Analogously in S^4 the cathetus-base can take all possible positions in a S^3, therefore this cathetus-base make a sphere.

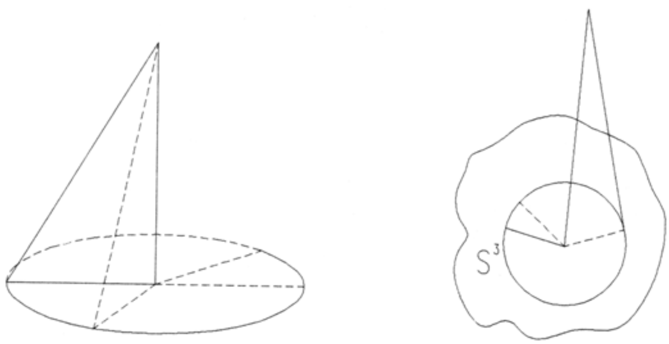


Fig.1

Prop. 3: If a rectangle takes all possible positions with one fixed side, the vertices and the points of the other three sides of the rectangle make up a hypercilinder of first kind which has two spheres as bases (top and bottom) and the elements of this hypercilinder are perpendicular to the hyperplanes of the bases (*right spherical hypercilinder*). "The fixed side is the axis, the opposite side is an element, and the other two sides are radi of the bases". (see [2] p.255).
(Analogous proof to that of the Prop.2)

Considering the hypercilindrical hypersurface of second kind, in the case where the directing-curve is a circle (*directing-circle*) the plane which passes through the centre of the circle can turn around itself and this plane is each time parallel to a element (plane) of the hypersurface. This plane which passes through the centre of the circle is an axis-plane of the hypersurface and every point of it is a centre of symmetry. Therefore always in this case (directing-circle), the hypersurface can be generated by the rotation of one of the elements (planes) around the axis-plane, that is by the rotation of one of two parallel planes around the other. We can call a *plano-cylindrical hypersurface* (or a *hypercilindrical hypersurface of second kind*) *of revolution*. The same considerations are valid for the hyperconical hypersurfce of second kind.

We call *layer* that portion of a hyperplane which lies between two parallel planes (*faces of the layer*). A *plano-prismatic hypersurface* consists of a finite number of parallel planes taken in a definite (cyclical) order, and of the layers which lie between consecutive planes. A cube belonging to a hyperplane S^3_0 of S^4 can be obtained if we intersect appropriately (in S^3_0 but no in S^4) three equal layers (see Fig.2 a). Moreover a layer can be decomposed into further layers.

Prop. 4: "When the elements of a plano-prismatic hypersurface intersect the elements of a hypercylindrical hypersurface of second kind (only in points), the intersection of the two hypersurfaces consists of the lateral surfaces of a set of cylinders lying in the *cells* of the prismatic hypersurface and joined together by their bases, together with the curves whose interiors are these bases" (see [2] p.258 and Fig.2 b)

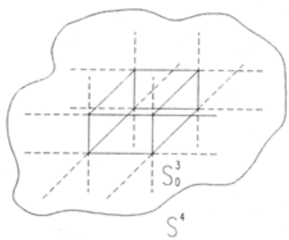


Fig.2a

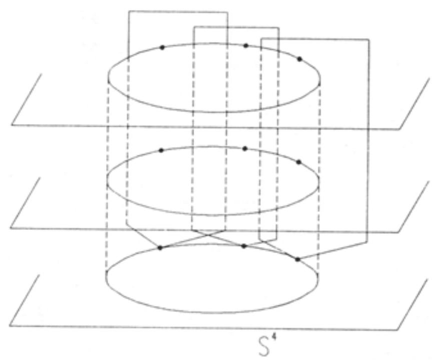


Fig.2b

The definitions of *hypersphere* (which consists of the points at a given distance from a given point) and of *hypercube* are elementary and well-known. These two hypersurfaces are particularly important for the geometry in S^4 like the circle and the square in S^2 and the sphere and the cube in S^3. For example the hypersphere has these following properties:

Prop. 5: A hyperplane which intersects a hypersphere gives a sphere having for centre the projection of the centre of the hypersphere upon the hyperplane (see [2] p.207).

"When a hyperplane passes through the centre of hypersphere the section is called *great sphere*. Other spheres of the hypersphere are called *small spheres*." (see [2] p.207)

Prop. 6: A 4-simplex of points (see the paragraph 3, that is four non-complanar points) of a hypersphere determines a sphere of the hypersphere, and a 3-simplex of points (that is three points) not complanar with the centre of the hypersphere determines a great sphere.

There are hence a lot of properties about the hypersphere and the hypercube (see [2], [10] and [1]).

An interesting surface (and solid) and a interesting hypersurface (and hypersolid) are the *torus* and the *hypertorus* (or *three-torus*) in S^4. In S^3 geometrically a torus T^2 is generated by a revolution of a circle around an axis . This rotation can be seen as a geometrical product $C^1 \times C^1$, that is $T^2 = C^1 \times C^1$, where (see [10] p.90):

1. All the vertical circles generated remain of the same size
2. All the horizontal circles generated are perpendicular to every vertical circle.

It is well-known (see [1] and [10]) that in S^3 a plane square surface (see Fig.3) can have one possibility each time of folding in such a way as to make an identification between two of its opposite edges. And we can make a torus if we deform the cylindrical surface (or the cylinder) (which we get by the identification of two opposite sides of the square) and we identify the two circles bases. Therefore a square - flat torus has the same topology of the doughnut surface.

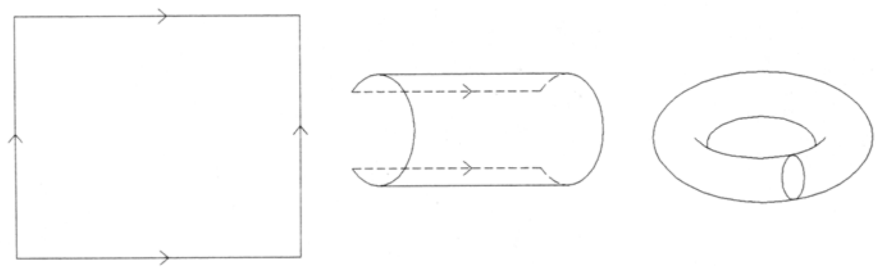


Fig.3

Now we consider a cube in S^4 (see [10] pp. 91 - 92), this cube can be seen like a set of layers delimited by horizontal squares (see Fig.4a). And we can imagine the same cube to be filled with intervals standing on end (see Fig. 4b).

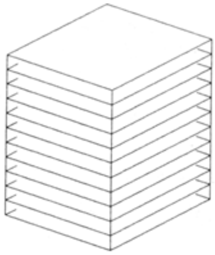


Fig.4a

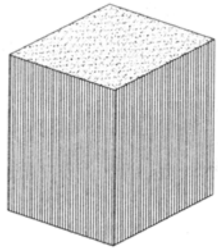


Fig.4b

The hypertorus $T^3 = C^1 \times C^1 \times C^1$ is given hence by the geometrical product $T^3 = T^2 \times C^1$. In fact when in S^4 the cube's sides (top and bottom) are glued we have that the cube of layers becomes a circle of tori and the cube of intervals becomes a torus of circles. This product is geometrical because all the horizontal tori are the same size and all the vertical circles are the same size and each torus is perpendicular to each circle. In this way we can consider T^3 as a hypersolid of rotation.

For T^2 in S^4 it is interesting and fundamental the following proposition 7, but in the mean time we remember that in S^3 a torus, surface of revolution (see Fig.5), is represented in $\mathbf{R}^3$ by the following vectorial equation:

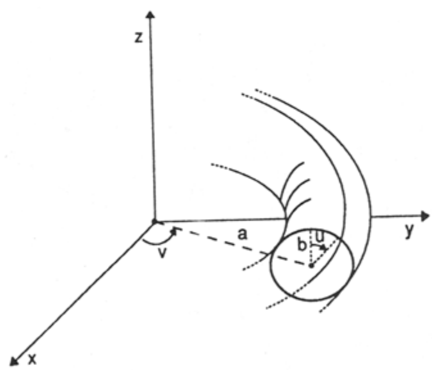


Fig.5

r(u,v) = (a + b sin u)cos v **i** + (a + b sin u)sin v **j** + b cos u **k**

with $0 < b < a$ and $(u, v) \in [0, 2\pi] \times [0, 2\pi]$

or rhe following Cartesian equation:

$$(\sqrt{(x^2 + y^2)} - a)^2 + z^2 = b^2$$

Prop. 7: The intersection of two hypercylinder hypersurfaces of second kind, perpendicular between them, gives a torus T^2 .

Proof: The elements (planes) of the first hypersurface intersects the elements (planes) of the second only in points. Each element of one hypersurface intersects the other hypersurface in a directing-circle, and the surface of intersection consists of the circles of either one of these sets. The interiors of the circles of each set lie in one of the hypersurfaces and in the interior of the other (see [2] p.262 and Fig. 6).

The analytical expressions of the two hypercylinder hypersurfaces in Cartesian space $\mathbf{R}^4$ associated to S^4 are:

$$(!) \quad \begin{cases} x^2 + y^2 = k & \forall z \, \forall w \\ z^2 + w^2 = h & \forall x \, \forall y \end{cases}$$

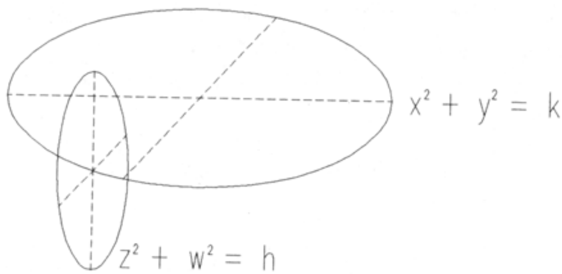
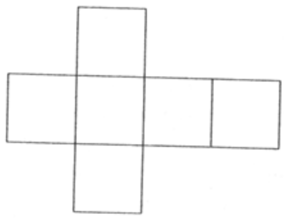


Fig.6

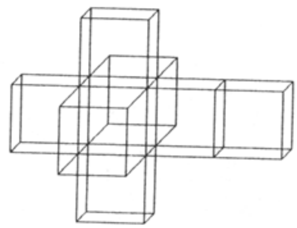
Hence the system (!) is the Cartesian representation of a torus T^2 in $\mathbf{R}^4$.

We shall now study a property which topologically tie the hypersphere, the hypercube and the torus. First we must consider the unfolding in S^3 of a hypercubic hypersurface analogous to the plane unfolding in S^2 of a cubic surface (see Fig. 7). And still we must keep in mind the procedure of identification of sides (which happens by particular rotations) which rebuilds the hypercubic hypersurface and the cubic surface.



Cube

Fig.7a



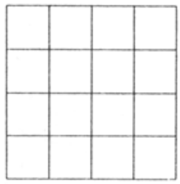
Hypercube

Fig.7b

The hypercube has 24 square faces (see Fig.7b) and each edge is common to three square faces. But (see [1]) we can take 16 squares only in such a way as each edge of the hypercube is common to two square faces. These squares can constitute a two-dimensional square surface in four-dimensional space S^4 (see Fig. 8a), which we denote by the name *square-polyhedrical torus*. In fact in S^4 it is geometrically possible to fold this square in such a way as to have identifications between the pairs of opposite edges and hence we obtain a *doughnut-polyhedrical torus*. It is trivial the

Prop. 8: The doughnut-polyhedrical torus coincides with the central projection in S^3 of the hypercube.

The vertices of a hypercube lie upon the hyperspherical hypersurface which circumscribes it as the vertices of a cube lie upon the spherical surface which circumscibes it. The 16 vertices of polyhedral torus also lie upon a hyperspherical hypersurface because these vertices are the vertices of a hypercube. If we divide ulteriorly the square polyhedrical torus in S^4 (for istance we can have 8×8 or 16×16 etc. little squares) in such a way as all vertices of this figure lie upon a hypersphere, we can use the central projection to obtain the image of this new polyhedrical torus in three-dimensional space. Therefore we have geometrically:



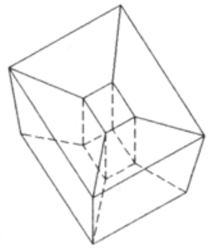


Fig.8a Fig.8b

Prop. 9: If we increase the subdividing by little squares of the square-polyhedrical torus in S^4, the approximation of the doughnut-polyhedrical tori, originated from this subdivision, to the doughnut torus of revolution T^2 in S^4 increases also (see Fig. 8).

Now we go on to some topological properties. We must premise the following important:

Prop. 10: A hypercubic hypersurface is topologically homeomorphic to hyperspherical hypersurface (like a cubic surface is topologically homeomorphic to spherical surface).

We have then (see [1 bis]):

Prop. 11: The hyperspherical hypersurface is obtained topologically by the union of two solid tori, identifying the edges in such a way that a parallel circle of the first is glued with a meridian circle of the second and vice versa.

Proof: For the Prop.10 we take a hypercubic hypersurface which we untie in S^3. We consider the two parts of the unfolding (see Fig.9):

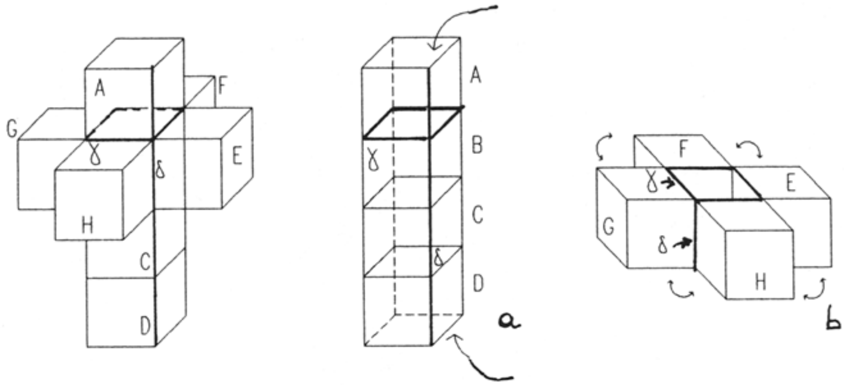


Fig.9

By identification in S^4 the part a) of Fig.9 becomes the part a) of Fig.10 and the part b) of Fig.9 becomes the part b) of Fig.10.

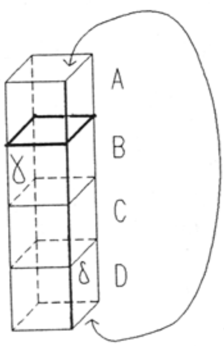


Fig.10a

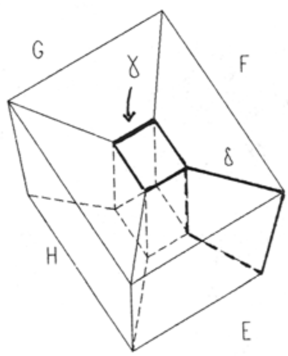


Fig.10b

We can transform c) and d) in two solid tori T^2. Now we join together again the two parts. We shall have so the following figure (Fig.11), which is hence topologically homeomorphic to the hyperspherical hypersurface.

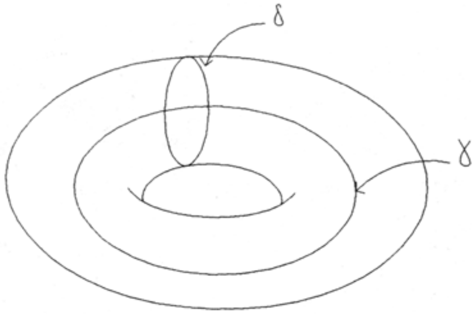


Fig.11

3. The bases of the geometry of four dimensions

A geometry is synthetically characterized by axioms. The first axioms are those of incidence. We shall start from the euclidean space S^3 and we will propose the axioms bearing those of Veronese (see [9] and [4]) in mind.

Ax.1: Given the (euclidean) space S^3, must exist at least a point P which does not belong to S^3.

Def.1: A 5-simplex is a geometrical figure determinated by five points not contained in the space S^3.

Prop.12: A 5-simplex exists at least
(*Proof.*: from Ax.1 and Def.1)

A 5-simplex is also called 'four dimensional simplex' and a 4- simplex is called 'three dimensional simplex' etc. A 2-simplex or an 'one dimensional simplex' is a segment and a 1-simplex is a point. A k-simplex (with $k \leq 5$) determines also a simplex of k - 1 straight lines (or k - 1 vectors) which have their origin at a common point (Fig.12).

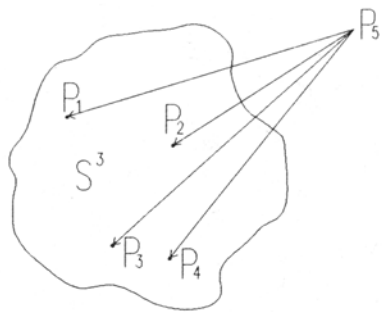


Fig.12

Ax.2: A 5-simplex determines the space S^4 and a k-simplex (with $k \leq 4$) determines a subspace of k - 1 dimensions. A simplex of k straight lines determine a space of k dimensions.

A S^3 in S^4 is called *hyperplane*

Ax.3: Let be S^p and S^q two subspaces of S^4 (that is: $0 \leq p, q \leq 4$), it is:

$$p + q = n + k \qquad (\textit{Grassmann's equation})$$

where $n \leq 4$ and $k \geq 0$ and k is the dimension of the intersection space S^k between S^p and S^q. If $p + q < 4$ (that is $k < 0$) then the two spaces S^p an S^q are *skew*.

There are also the axioms of belongings, order, continuity and congruence appropriately adapted and concerning in particular the straight lines and the planes. For example:

Ax.a : Two distinct points determine a unique straight line (It is a axiom of belongings)

Ax.b : If A and C are two points of a straight line, must exist at least a point between A and C and a point D such that C is between A and D (It's a axiom of order)

Prop. 13: In S^4 there are infinite points.

Proof: Given the space S^3 and a point P which does not belongs to S^3 (for Ax.1). We have a straight line determinated by P and a point $Q \in S^3$ (for Ax.a). There is a point R not belonging to S^3 and distinct from P (for Ax.b). Then we can iterate the reasoning (see Fig.13).

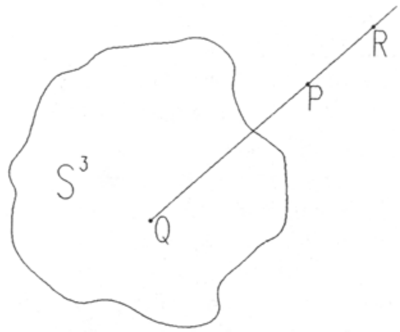


Fig. 13

By the Archimedes-Hilbert axiom of continuity it is possible to prove this infinity of points is a continuous infinity.
By S^0, S^1, S^2 we denote respectively a point, a straight line, a plane. We have now the following

Prop. 14 : *Generally* in S^4, if the relatives intersections exist, then:

a) $S^1, S^2 \in S^4 \rightarrow S^1 \cap S^2 = \varnothing$ that is S^1 and S^2 skew there exist.
In fact for Ax.3 from the Grassmann's equation we have: $1 + 2 = 4 - 1 < 4$.
But if S^1 , $S^2 \in S^3 \in S^4$ then it results from Grassmann's equation: $1 + 2 = 3 + 0$ that is:
$S^1 \cap S^2 = S^0$

b) S^1 , $S^3 \in S^4 \rightarrow S^1 \cap S^3 = S^0$, in fact $1 + 3 = 4 + 0$
But if $S^1 \in S^3 \in S^4 \rightarrow S^1 \cap S^3 = S^1$, in fact $1 + 3 = 3 + 1$

c) S^2_1 , $S^2_2 \in S^4 \rightarrow S^2_1 \cap S^2_2 = S^0$, in fact : $2 + 2 = 4 + 0$, that is pairs of planes which have a single point in common there exist. These plans are called *semiskew*.
But if $S^2_1 \cap S^2_2 = S^1 \rightarrow S^2_1$, $S^2_2 \in S^3 \in S^4$, in fact : $2 + 2 = 3 + 1$

The concepts of parallelism and perpendicularity are extended to S^4, for istance:

Def.2: Let S^p and S^q be two subspaces of S^4 (that is $p, q \leq 3$) and $p < q$. If the space at infinity of S^p belongs to the space at infinity of S^q then these two spaces are called *absolutely parallel.*

S^p and S^q, with $p \leq q$, have their respective spaces at infinity determined by a p-simplex and a q-simplex . Instead S^p and S^q are determined by a (p + 1)-simplex and a (q + 1)-simplex, for Ax.2, but we must remember that for example a plane at infinity, which has dimension 2, is the greatest space at infinity of a hyperplane, which has dimension 3. We denote with $\mathfrak{J}^p$ and $\mathfrak{J}^q$ the greatest space at infinity respectively of S^p and of S^q, then we suppose $\mathfrak{J}^p \cap \mathfrak{J}^q$ equal to a subspace of dimension r - 1. This last space is determinated by a r-simplex. If $r = p$ we have the absolute parallelism, otherwise S^p and S^q are called *partly parallel* with a *degree of parallelism* expressed by the ratio r/p.

The approach to concept of perpendicularity is analogous to that of parallelism (see [4]). But we can follow an other way, more specific for a four dimensional geometry (see [11]). We have the following theorems:

Prop. 15: The straight lines which are perpendicular to a given straight line r at the given point $P \in r$ do not all lie in the same plane, but they lie in the same hyperplane.

Proof: (see [11] pp.192 - 195) .This theorem establishes a differentiation in comparison with S^3.

Prop. 16: Let d_1, d_2, d_3 be a simplex of three straight lines with their common point O. This simplex determines a space S^3. If a straight line g, which passes through O, is perpendicular to every line d_j (j = 1, 2, 3), then so it is for every other lines which passes through O.
This theorem is valid for a simplex of m straight lines.

Proof: When we have two lines d_1 and d_2 the proof is immediate. If g is perpendicular to both d_1 and d_2 then g is perpendicular to every straight line belonging to the plane (d_1, d_2) and passing through O (see Fig.14).

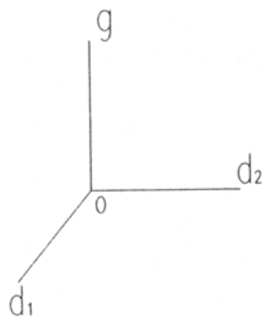


Fig.14

In the case of a simplex of three lines d_1, d_2, d_3 , if g is perpendicular to these three lines then g will be perpendicular in particular for d_1 and d_2. Therefore g will be perpendicular to every line of the plane (d_1, d_2) according to the foregoing case of two lines. Let d be any straight line belonging to the space S^3 determined by the simplex made by d_1, d_2, d_3. Then d_1 and d_2 form a plane which intersects the plane (d_1, d_2) at the straight line d'. Because g is perpendicular to every line of the plane (d_1, d_2) g will be perpendicular for d' which belongs to (d_1, d_2). But $g \notin$ (d, d') therefore if $g \perp d'$ then also $g \perp d$.

Def. 3: A hyperplane and a straight line *r* are called *perpendicular* if their intersection is a point and every straight line in the hyperplane, which passes through this point, is perpendicular to the line *r*.

Def. 4: Two planes which intersect at a single point are *absolutely perpendicular* if every straight line of one of them which passes through their common point P is perpendicular to every line of the other which passes through P.

Def. 5: If a plane α is semiskew to an other plane β at a given point P and contains one and only one line perpendicular to the plane β at the point P, then we say the plane α is *semiperpendicular* to the plane β.

We can prove that if a plane α is semiperpendicular to a plane β then β is semiperpendicular to α (see [11] p. 198 - 199).

Lastly we shall say something about the *rotation* in S^4 (see Fig.15). We consider a $S^2{}_0 \in S^4$ (see Fig. 4), from any point $M \notin S^2{}_0$ we pull down a plane $S^2{}_1$ absolutely perpendicular to plane $S^2{}_0$ which intersect the $S^2{}_1$ at a point P. In $S^2{}_1$ we make an other point M' such that the angle MPM' is equal to given value φ. We say the point M' is obtained by a rotation of angle φ from M around the *axis-plane* $S^2{}_0$. Let N be a point different from M. Then the points M, N, P and M' determine a space S^3 which intersects $S^2{}_0$ at the straight line *a* which is perpendicular to the plane MPM'. In this S^3 the point M so has had the rotation of angle φ around the axis *a* and also the point N and all the points of S^4 (see [4] p.305).

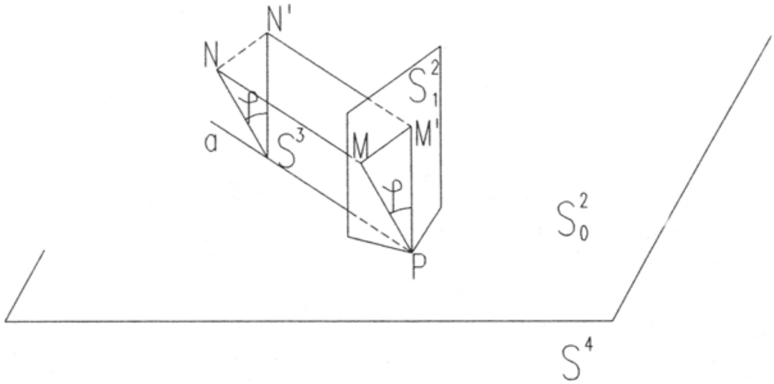


Fig.15

It is very important that any figure in hyperspace S^4 can rotate around a plane (see [2] pp.141-145).
We could illustrate other fundamental notions of geometry 4D, but we prefer to refer to [2], [4], [9], [11].

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