Annals of Mathematics
Vol. 45, No. 2, April, 1944

THE INTRINSIC MEASURE THEORY OF RIEMANNIAN AND EUCLIDEAN METRIC SPACES

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(Received March 29, 1943,
Revised January 21, 1944)

1. Introduction

Euclidean spaces are boundedly compact and have the metric property that any two spheres are similar. In particular, if a closed sphere of radius r can be covered by n open spheres of radius x, then, for any positive a, every closed sphere of radius ar can be covered by n open spheres of radius ax. We shall show that this combinatorial similarity property is sufficient in any boundedly compact metric space for the development of ordinary Lebesgue measure theory. One result is the validity of the usual formula for the volume of a sphere. That is, apart from a multiplicative constant there is one and only one measure which is a volume in the sense that spheres of equal radii have equal measures, and there is an α such that the volume of a sphere of radius r is r^α. The existence proof will be presented in a general enough form to include the development of the intrinsic measure theories of Riemannian metric spaces and of metric spaces like the Cantor sets for which the dimension α is non-integral.

2. The existence theorem

Let M be a boundedly compact metric space for which the following combinatorial congruence axiom is satisfied:

Postulate 1. *There is a constant K ($K \geqq 1$) such that if some closed sphere of radius r can be covered by n open spheres of radius x, then, for any positive a, every closed sphere of radius ar can be covered by n open spheres of radius Kax.*

Let $h(r, x)$ be the smallest number such that every closed sphere of radius r can be covered by $h(r, x)$ open spheres of radius x.[1] By Postulate 1, $h(r, x)$ is finite and satisfies the inequality:

$$h(ar, Kax) \leqq h(r, x). \tag{1}$$

It is also evident from the definition that

$$h(r, x) \leqq h(r, y)h(y, x). \tag{2}$$

Then $h(ar, ax) \leqq h(ar, Kax)h(Kax, ax) \leqq h(r, x)h(K^2, 1)$, so that

$$h(ar, ax) \leqq Ah(r, x), \tag{3}$$

where $A = h(K^2, 1)$.

[1] This function is thus similar to the Haar covering function. See Saks, *Theory of the Integral* (1937), p. 315.

LEMMA 1. *There is a positive constant B such that*

$$h(r, y)h(y, x) \leqq Bh(r, x) \tag{4}$$

whenever $x < y < r$.

If we define $f(r, x)$ to be the largest number such that every open sphere of radius r contains $f(r, x)$ disjoint closed spheres of radius x, then we clearly have $f(r, x) \geqq f(r, y)f(y, x)$. Some open sphere U with radius r contains $f(r, x)$ disjoint closed spheres S_i with radius x and contains no larger such set. But then (if $x < r/2$) the open spheres concentric with the S_i and with radius $3x$ cover the closed sphere concentric with U and with radius $r/2$. For if p were an uncovered point the closed sphere about p with radius x would lie in U and touch no S_i, contradicting the choice of the S_i as a maximal set. We thus infer that $f(r, x) \geqq h(r/2, 3Kx)$ whenever $x < r$, the inequality being obvious if $r/2 \leqq x < r$. Finally, $f(r, x) \leqq h(r, x)$. For if a set of open spheres of radius x covers a set of disjoint closed spheres of radius x then no open sphere can contain the center of more than one closed sphere.

These three inequalities on $f(r, x)$ imply that

$$h(r, x) \geqq f(r, x) \geqq f(r, y)f(y, x) \geqq h(r/2, 3Ky)h(y/2, 3Kx).$$

But by (1) and (2)

$$h(r, y) \leqq h(r, r/2)h(r/2, 3Ky)h(3Ky, y) \leqq [h(3K^2, 1)]^2 h(r/2, 3Ky)$$

and similarly for $h(y, x)$ so that the lemma follows with $B = [h(3K^2, 1)]^4$.

THEOREM 1. *There is a positive constant M and a unique positive*[2] *exponent* α *such that*

$$M^{-1}\left(\frac{r}{x}\right)^\alpha \leqq h(r, x) \leqq M\left(\frac{r}{x}\right)^\alpha \tag{5}$$

whenever $x \leqq r$.

The inequalities (2), (3) and (4) show that there is a positive constant C ($C > 1$) such that

$$C^{-1}h(1, a)h(1, b) < h(1, ab) < Ch(1, a)h(1, b)$$

where $a, b < 1$. If we abbreviate $h(1, a)$ as $h(a)$ and apply this inequality repeatedly to $h(a^m)$, we obtain the inequality:

$$C[C^{-1}h(a^m)]^n < h(a^{mn}) < [Ch(a^m)]^n C^{-1} \tag{6}$$

Let (6′) be the result of interchanging m and n in (6). Suppose that $m \geqq n$ so that $C^{1/m} \leqq C^{1/n}$. We then obtain from (6) and (6′), by dividing and extracting mn-th roots, that

$$C^{-2/n} < \frac{h(a^n)^{1/n}}{h(a^m)^{1/m}} < C^{2/n}$$

[2] We shall assume that the space M contains more than 1 point. Then by Postulate 1 the function $h(r, x)$ is unbounded and α must be positive.

whenever $m \geqq n$. Thus if a is fixed ($a < 1$), $h(a^n)^{1/n}$ converges to a limit l as $n \to \infty$ and $C^{-2}l^n \leqq h(a^n) \leqq C^2l^n$. The monotonicity of $h(1, x)$ implies that for any positive y

$$(C^2l)^{-1}l^y \leqq h(1, a^y) \leqq (C^2l)l^y.$$

We obtain Theorem 1 from this inequality (with $M = AC^2l$) by replacing a^y by x/r, taking α as $-\log l / \log a$ so that $l^y = (r/x)^\alpha$, and applying (3).

THEOREM 2. *If N disjoint closed spheres S_i with radii r_i lie in a sphere S with radius r, then*

$$\sum_{i=1}^{N} r_i^\alpha \leqq K^{2\alpha}r^\alpha.$$

If S is covered by $h(r, x)$ open spheres of radius x, where x is less than one-half the distance between any two S_i, then no covering sphere can touch more than one S_i. By the definition of $h(r, x)$ and Postulate 1, the number touching S_i is at least $h(r_i, Kx)$ so that $\sum^N h(r_i, Kx) \leqq h(r, x)$, or by (1),

$$\sum_{i=1}^{N} h(r, rK^2x/r_i) \leqq \sum_{i=1}^{N} h(r_i, Kx) \leqq h(r, x). \tag{7}$$

By Theorem 1, $h(r, x)$ can be written as $M(r, x)(r/x)^\alpha$ where $M^{-1} \leqq M(r, x) \leqq M$. Then (7) implies at once that

$$\sum_{i=1}^{N} \frac{M(r, rK^2x/r_i)}{M(r, x)} r_i^\alpha \leqq K^{2\alpha}r^\alpha.$$

If x_n ($x_n \to 0$) is chosen so that $M(r, x_n) \to \underline{\lim}_{x\to 0} M(r, x)$ as $n \to \infty$, the theorem follows.

THEOREM 3. *If the closed sphere S with radius r is covered by N open spheres S_i with radii r_i, then*

$$r^\alpha \leqq K^{2\alpha} \sum_{i=1}^{N} r_i^\alpha .$$

The proof is similar to that of Theorem 2.

We now introduce Hausdorff α-dimensional volume. The outer measure $\mu(A)$ of an arbitrary set A is defined to be the limit as $\epsilon \to 0$ of the greatest lower bound of sums $\sum^\infty r_n^\alpha$ such that 1) A is covered by a countable family S_n of open spheres with radii r_n, and 2) $r_n < \epsilon$ for all n. It follows[3] that $\mu(A)$ is a Caratheodory outer measure and the usual development of measure theory is valid. Moreover, $\mu(A)$ is a non-trivial measure, for by Theorems 1 and 3 the measure of any closed sphere S with radius r is finite and positive, satisfying in fact the inequality:

$$K^{-2\alpha}r^\alpha \leqq \mu(S) \leqq Mr^\alpha. \tag{8}$$

[3] See, for example, Saks, loc. cit., pp. 53, 54.

THEOREM 4. *Postulate* 1 *is a necessary and sufficient condition that a boundedly compact metric space have a measure* $\mu(A)$ *such that for suitable positive constants* α, C_1 and C_2,

$$C_1 r^\alpha < \mu(S) < C_2 r^\alpha, \tag{9}$$

where S *is any sphere with radius* r.

The sufficiency was remarked above. Now suppose that $\mu(A)$ satisfies (9) and that a closed sphere of radius r is covered by n open spheres of radius x. Then $C_1 r^\alpha \leqq nC_2 x^\alpha$. Now let S be any closed sphere with radius ar. If m is the smallest number of open spheres with radius Kax required to cover S, then, as was observed in the proof of Lemma 1, the open sphere concentric with S and with twice the radius contains a set of m disjoint closed spheres with radius $Kax/3$. Hence $mC_1(Kax/3)^\alpha < C_2(2ar)^\alpha$. It follows from these two inequalities that if K is taken as $6(C_2/C_1)^{2/\alpha}$ then $m \leqq n$ and Postulate 1 holds.

It is obvious from these remarks that if Postulate 1 is satisfied and if a measure $m(A)$ has the property that there are positive constants A_1 and A_2 such that for any two closed spheres S_1 and S_2 of the same radius

$$A_1 \leqq \frac{m(S_1)}{m(S_2)} \leqq A_2,$$

then again (9) is satisfied.

To get more exact information concerning the measures of spheres we need a simple form of the Vitali covering theorem. A proof will be included for completeness though it can be read almost word for word from any standard account.[4] We suppose given a bounded set A, every point of which lies in arbitrarily small spheres of a family F of closed spheres.

THEOREM 5. *There is a sequence* S_n *of disjoint spheres of* F *such that, if* T_n *is the open sphere concentric with* S_n *and with five times the radius, then for every* N

$$A - \sum_{n=1}^{\infty} S_n \subset \sum_{n=N}^{\infty} T_n .$$

We restrict ourselves to the spheres of F lying in some bounded open set O which contains A. A finite number of the S_n can be chosen arbitrarily (but disjoint!) and beyond that S_m can be taken as any sphere not touching $\sum^{m-1} S_i$ whose radius is at least one-half the least upper bound δ_m of the radii of the spheres of F (in O) not touching $\sum^{m-1} S_i$.

Any point of A not in $\sum^\infty S_i$ is at a positive distance from $\sum^N S_i$, and so lies in some sphere S of F not touching $\sum^N S_i$. Since $\delta_i \to 0$, S must touch a first S_m ($m > N$), so that $r(S) < \delta_m \leqq 2r(S_m)$, and S lies in T_m, concluding the proof of the theorem.

Let $m(A)$ be any measure satisfying (9). Then there is a constant B such that $m(T_n) \leqq Bm(S_n)$, and

$$m\left(A - \sum_{n=1}^{\infty} S_n\right) \leqq \sum_{n=1}^{\infty} m(T_n) \leqq B \sum_{n=N}^{\infty} m(S_n).$$

[4] See Saks, loc. cit., p. 109.

Since $\sum^{\infty} m(S_n) \leqq m(0) < \infty$, the last member approaches O as $N \rightarrow \infty$. Thus the sequence S_n covers A except for a set of measure O. If, in particular, A is open and the spheres of F lie in A, then $m(A) = \sum^{\infty} m(S_n)$.

THEOREM 6. *If S is an open sphere with radius r, then*

$$K^{-2\alpha} r^{\alpha} \leqq \mu(S) \leqq K^{2\alpha} r^{\alpha}.$$

Let S_n with radii r_n be a Vitali sequence for S chosen from the closed spheres in S. Theorem 2 applied to the partial sums implies that $\sum^{\infty} r_n^{\alpha} \leqq K^{2\alpha} r^{\alpha}$. But the sequence S_n can be enlarged to an open covering of S (by replacing S_n by T_n for $n > N$ and enlarging slightly the first N spheres) in such a way that the increase in $\sum^{\infty} r_n^{\alpha}$ is less than an arbitrary δ. Since the spheres of S_n can be taken with all radii less than any given ϵ, it follows that $\mu(S) \leqq K^{2\alpha} r^{\alpha} + \delta$, which proves the right hand inequality of Theorem 6.

Since any covering of S covers every closed sphere in S, Theorem 3 implies that $\mu(S) \geqq K^{-2\alpha} t^{\alpha}$ for every t less than r, and this completes the proof of the theorem.

3. Euclidean metric spaces

By a *Euclidean metric space* we shall mean any boundedly compact metric space in which Postulate 1 holds with $K = 1$, that is, in which the smallest number of open spheres of radius x required to cover a closed sphere of radius r depends only on the ratio x/r. Ordinary Euclidean spaces are special cases. We restate Theorem 6.

THEOREM 7. *In a Euclidean metric space of dimension α the Hausdorff α-dimensional volume of a sphere of radius r is r^{α}.*

We have thus established the ordinary formula for the volume of a sphere. We now show that, to within a multiplicative constant, this is the only possible volume.

THEOREM 8. *If $m(A)$ is a measure in M which is a volume in the sense that closed spheres of equal radius have equal measures, then for some positive k, $m(A) \equiv k\mu(A)$.*

Let O be any bounded open set. We norm the measure $m(A)$ so that $m(0) = \mu(0)$. The closed spheres in O for which $m(S) > \mu(S)$ cannot form a Vitali set for O, for otherwise we could choose a Vitali sequence S_n from them and obtain the contradiction:

$$m(0) = \mu(0) = \sum_{n=1}^{\infty} \mu(S_n) < \sum_{n=1}^{\infty} m(S_n) = m(0).$$

Hence there is a point p in O and a positive ϵ such that $m(S) \leqq \mu(S)$ for every closed sphere touching p with radius less than ϵ. But then $m(S) \leqq \mu(S)$ for every closed sphere with radius less than ϵ. Similarly $m(S) \geqq \mu(S)$ for every closed sphere in O with radius less than a suitable ϵ', so that $m(S) = \mu(S)$ for all sufficiently small closed spheres. Hence, by the Vitali theorem, $m = \mu$ for all bounded open sets and so for all measurable sets.

4. Riemannian metric spaces

The spaces of Riemannian geometry have the property that given any point p and any number d greater than 1 there is an open sphere S about p such that any two open (or closed) spheres in S are similar to within a multiplicative error of d. That is, if S_1 and S_2 with radii r_1 and r_2 are two such spheres, then there is a one-to-one mapping f such that $S_2 = f(S_1)$ and

$$d^{-1}\left(\frac{r_1}{r_2}\right) \leqq \frac{\rho(p, q)}{\rho(f(p), f(q))} \leqq d\left(\frac{r_1}{r_2}\right)$$

for every pair of points p and q in S_1. Thus ordinary Riemannian spaces are included in the following definition. A *Riemannian metric space* is a connected locally compact metric space in which Postulate 1 holds with any value of K greater than 1 for the closed spheres contained in a sufficiently small fixed sphere about any given point.[5] A Riemannian metric space is *locally Euclidean* if Postulate 1 holds locally with $K = 1$.

The existence theory of section 2 can now be applied only locally. Let U be a sphere (radius r_0) in which Postulate 1 holds and let V be the concentric closed sphere with radius $r_0/3$. We define $h(r, x)$ with $x \leqq r \leqq r_0/3$ in terms of coverings of $S \frown V$ where S is the general closed sphere with radius r, and we define $f(r, x)$ in terms of packings of spheres contained in V. Then (1) and (2) hold and the rest of section 2 applies automatically if only spheres lying interior to V are considered. We assume the space to be connected in order to insure that the resulting dimension α is the same over the whole space. This effect can also be obtained, without postulating connectedness, by assuming that Postulate 1 holds for any K greater than 1 in any compact set, provided that only spheres are considered having radii less than a suitable ϵ. With this common value for α the Hausdorff α-dimensional volume is defined for all separable Borel sets and is finite and non-zero for every open set with compact closure. The following theorem is evident from the definition of Riemannian metric spaces and Theorem 6.

THEOREM 9. *If F is a compact set in a Riemannian metric space of dimension α, then the Hausdorff α-dimensional volumes of spheres in F with radius r are uniformly asymptotic to r^α as $r \to 0$.*

Similarly, the formula for the volume of a sphere holds in the small in a locally Euclidean space.

THEOREM 10. *If $m(A)$ is a measure for which closed spheres of equal radius in a compact set have asymptotically equal measures as $r \to 0$, then there is a positive constant k such that $m(A) \equiv k\mu(A)$.*

The proof of Theorem 8 is general and shows that there is some point p in O such that $m(S) \leqq \mu(S)$ for all sufficiently small closed spheres touching p and a second point q for which the inequality is reversed. It follows that $m(S)$ is

[5] Compare this definition with the properties considered by Busemann, *Metric Methods in Finsler Spaces and in the Foundations of Geometry*, p. 40, (1), (1a) and (1b).

asymptotically equal to $\mu(S)$ as $r(S) \to 0$, and by the Vitali theorem that $m = \mu$ in O.

Thus section 2 contains a development of the intrinsic measure theory of Riemannian metric spaces, an application being a coordinate-free development of measure theory in ordinary Riemannian spaces.

5. Cantor spaces

In this section we shall assume Postulate 1 in the Euclidean form for a restricted set of spheres. Let M be a boundedly compact metric space and let O be a family of open spheres in M with radii of the form ba^n for fixed a and b $(0 < a < 1)$. We suppose that i) there is a positive constant C such that any sphere in M with radius Ca^n $(n \geqq 1)$ lies in a sphere of O with radius ba^n, and that ii) the smallest number of spheres of O with radius x required to cover the closure of a sphere of O with radius r depends only on the ratio x/r. The theory of section 2 can be applied to the spheres of O. In the proof of Lemma 1 the concentric spheres of radius $3x$ may not be in O, but by property i) above, each such sphere lies in a sphere of O with radius at most $(b/Ca)3x$. The outer measure μ is now defined using coverings by spheres from O. Thus if S is the closure of a sphere in O with radius ba^n, then (inequality (8))

$$(ba^n)^\alpha \leqq \mu(S) \leqq M(ba^n)^\alpha.$$

Now let S be any sphere in M whose radius x is at most Ca. For some n, $2ba^n < x \leqq 2ba^{n-1}$ so that if S_1 is the closure of any sphere of O which contains the center of S and has radius ba^n, then S_1 lies in S. Also there is a sphere S_2 of O which contains S and has radius at most bx/Ca. Hence

$$\left(\frac{a}{2}\right)^\alpha x^\alpha \leqq (ba^n)^\alpha \leqq \mu(S_1) \leqq \mu(S) \leqq \mu(S_2) \leqq M\left(\frac{b}{Ca}\right)^\alpha x^\alpha.$$

Thus (9) holds, and μ is a measure for which the Vitali theorem is valid.

The ordinary Cantor sets are spaces of this type. Let M_0 be the closed unit interval $[0, 1]$. We suppose inductively that M_n consists of 2^n closed intervals of length $((1 - l)/2)^n$ and we form M_{n+1} from M_n by removing the open interval of length $l((1 - l)/2)^n$ from the center of each closed interval of M_n. Then $M = \Pi^\infty M_i$ is the ordinary Cantor set. The 2^{n+1} endpoints of the 2^n open intervals removed from M_n make up those endpoints of M_{n+1} which are not endpoints of M_n. Let O_{n+1} be the set of 2^{n+1} open spheres in M with these points as centers and with radius $((1 - l)/2)^{n+1} + kl((1 - l)/2)^n$ $(= D((1 - l)/2)^n)$ where $1 < k < ((1 - l)/2)^{-1}$. Each interval of M_n is covered by either of the two spheres of O_{n+1} having endpoints in it, and M (and M_n) can be covered by 2^n of these spheres but by no fewer. Any sphere of radius less than $kl((1 - l)/2)^n$ lies in a sphere of O_{n+1}. Finally any two spheres of $O = \sum^\infty O_n$ are similar. Thus properties i) and ii) are satisfied, and it remains only to compute the dimension α of the space.

By Theorem 1, $M^{-1}x^{-\alpha} \leqq h(1, x) \leqq Mx^{-\alpha}$ and we have seen that $h(1, x) = 2^n$ if $x = D((1 - l)/2)^n$. Hence

$$M^{-1}D^{-\alpha}\left(\frac{1-l}{2}\right)^{-n\alpha} \leqq 2^n \leqq MD^{-\alpha}\left(\frac{1-l}{2}\right)^{-n\alpha}.$$

Taking nth roots and letting $n \to \infty$ we find that $((1 - l)/2)^{-\alpha} = 2$, i.e., that $\alpha = \log 2 / \log (2/(1 - l))$. If M is formed by removing middle thirds ($l = \frac{1}{3}$) then $\alpha = \log 2 / \log 3$.

There seems to be no way in which the general measure of this section can be considered a unique volume. Let us now replace ii) by the following stronger property: iii) if the closure of some sphere of O with radius r_1 can be covered by N spheres of O with radii x_i, $i = 1, \cdots, N$, then the closure of any sphere of O with radius r_2 can be covered by N spheres of O with radii $(r_2/r_1)x_i$. It follows from iii) that if S_1 and S_2 are the closures of spheres in O with radii ba^n and ba^m then $\mu(S_2) = a^{(m-n)\alpha}\mu(S_1)$. Therefore if μ is multiplied by a suitable positive constant, $\mu(S) = r^\alpha$ where S is the closure of any sphere in O and r is its radius. Thus μ can be considered a volume with respect to these spheres. Finally, since the closures of spheres in O form a Vitali set for the space, Theorem 8 can be applied and shows that μ is essentially (to within a multiplicative constant) the only such volume.

The replacement of ii) by iii) can certainly be made whenever, as in the case of the Cantor sets or Euclidean spaces, the spheres of O are all similar.

6. Cartesian products

Let M_1 and M_2 with dimensions α and β be boundedly compact metric spaces satisfying Postulate 1 and let M be their Cartesian product with distance defined by the usual Euclidean formula. Then M satisfies Postulate 1 and its dimension is $\alpha + \beta$. This follows readily from the fact that any sphere in M with radius r contains the "square" whose sides are spheres in M_1 and M_2 with radius $r/2^{\frac{1}{2}}$, and is contained in the square whose sides have radius r. Hence

$$h(r, \sqrt{2}x) \leqq h_1(r, x)h_2(r, x)$$

$$f(r, x) \geqq f_1(r/\sqrt{2}, x)f_2(r/\sqrt{2}, x),$$

where f is the packing function of Lemma 1, and the result is immediate.

If M_1 and M_2 are spaces in which closed spheres are similar then M has the same property. Thus $K_1 = K_2 = K = 1$ and all three spaces are Euclidean metric spaces.

But in general it does not follow, at least from the above argument, that $K = 1$ whenever $K_1 = K_2 = 1$. Nevertheless it is easily verified by integration that the volume of a sphere in M with radius r is proportional to $r^{\alpha+\beta}$. Thus is raised the problem of determining combinatorial conditions in a boundedly compact metric space which are necessary and sufficient for the existence of a measure which has the value r^α on a sphere of radius r.

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