Extrusion and boundary evaluation for multidimensional polyhedra*

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Extrusion is a basic operation allowing the generation of higher intrinsic dimension polyhedra. The paper gives closed formulas both to generate a $(d+1)$-dimensional polyhedron obtained by affine extrusion of a (d)-dimensional polyhedron, and to generate a polyhedral approximation of the curved solid generated by rotational extrusion. Algorithms for the boundary evaluation when a decompositive representation is given are also discussed.

The representation used in the paper, based on simplicial complexes, is general and simple, and allows us to represent nonconvex, unconnected, unoriented, nonmanifold and unbounded linear polyhedra. A simplicial complex triangulating the extruded polyhedron is generated by independently extruding the simplices of the input object. The approach is very efficient because no a posteriori *triangulation of the extruded polyhedron is required; furthermore, both the underlying complex and the adjacencies between cells are calculated by using closed formulas.*

solid modelling, polyhedra, multidimensional modelling

Multidimensionality is a new frontier for computer graphics. New needs for multidimensional visualization and manipulation of higher dimensional objects are in fact quickly emerging from various areas, specifically in the fields of statistical graphics, scientific visualization, modelling and simulation, and robotics. For example, time was included as a variable to extend space partition representation[1,2], and 4D solid methods were proposed for the solution of the collision detection problem[3]. Furthermore, a new method has been recently discovered[4] to construct a polyhedral approximation of the free configuration space of mobile systems by using extrusion, projection and set operations in higher dimensional spaces. In the present paper the authors aim to present a representation paradigm for high dimensional polyhedra based on well-known tools from algebraic topology and to show that with this approach efficient and powerful algorithms for managing multidimensional objects can be devised. In particular, the paper gives closed formulas to:

- generate a $(d+1)$-dimensional polyhedron obtained by linear affine extrusion of a d-dimensional polyhedron;
- generate a polyhedral approximation of the curved polyhedron generated by rotational extrusion;
- evaluate the boundary of a given decompositive representation of a d-polyhedron.

Simplicial complexes are largely used in the area of engineering design (finite element codes) but only rarely adopted in solid modelling and computer graphics. The 'winged representation' used in the paper, based on simplicial complexes, is general and simple, and allows the representation of a large class of linear polyhedra. This representation can also linearly approximate curved polyhedra, can be used with nonregular polyhedra, which have subparts of different dimensions, and is suitable for implementing nonregularized set operations. In the three dimensional case it has been shown[5] that winged representations are space optimal in representing the topology of linearized approximations of curved polyhedra.

Previous work

Basic computer graphics techniques for the wire-frame display of d-dimensional objects have been exploited by the early works of Noll[6,7]. Burton and Smith[8] have described a hidden-line algorithm for higher-dimensional scenes. Armstrong and Burton[9] presented various graphical techniques based on cues for 'hyperdimensional' objects. Banchoff[10] discussed the real-time rendering of four dimensional objects and extends basic algorithms of computer graphics (like Gouraud's shading) to 4D objects[11]. Glassner[12] gives efficient techniques for ray tracing of animated scenes working in 4-dimensional space.

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Putnam and Subrahmanyam[13] have given Boolean operators on *d*-dimensional solids. In their approach the representation of solids is done by recursively listing the boundary elements and their orientation, without explicit storage of adjacency information. Bieri and Nef[14] have described polyhedra with 'adjoined pyramids' associated to the boundary faces. Any pyramid is represented as a set of cells in the space partition induced by the set of the boundary hyperplanes. Each cell is described as a bit sequence whose length equates the number of boundary hyperplanes. Analogous representation of *d*-dimensional unconvex polyhedra is given by Günther[15] by using bit sequences. Rossignac and O'Connor[16] are developing a powerful dimension independent representation for pointsets with internal structures and incomplete boundaries by using geometrical cell complexes, where each cell is a connected manifold. The 'winged representation' of *d*-dimensional polyhedra used in this paper, proposed by Cattani and Paoluzzi[17,18] and used in its 3D boundary version in the solid modeller Minerva[5], is based on the direct use of simplicial complexes and of their maximum order adjacencies. In the present paper we detail that approach by giving closed formulas to fully compute the representation of extruded objects.

Preview

The paper is organized as follows. The second section introduces basic concepts about simplices and simplicial complexes, the definition and some properties of the 'winged' representation of a simplicial complex. It is also shown that the winged representation can be considered a paradigm for (linear) solid modelling, as it allows for a unified treatment of decompositive, boundary, sweep and hierarchical representations of (possibly unconnected, unlimited, nonconvex, non-manifold, unoriented) polyhedra. The third section illustrates the *d*-dimensional interpretation of the extrusion operation, used to build higher dimensional polyhedra. Extrusion can be regarded as a special case of the join operation, which plays a central role in algebraic topology. In the third section it is first given a closed combinatorial formula to generate a simplicial complex triangulating the $(d+1)$-polyhedral 'tube' generated by extruding a simplex. A simplicial complex triangulating an extruded polyhedron is then generated by independently extruding the various simplices of a complex associated to the input polyhedron. The approach is very efficient because no *a posteriori* triangulation of the extruded polyhedron is required. The shown formulas for affine (linear) extrusion are closed over the set of winged representations of polyhedra-with-boundary, and can be therefore applied to any decompositive representation. The fourth section introduces the polyhedral approximation of rotational extrusion by using any desired number *h* of affine (linear) steps. The fifth section discusses algorithms for the boundary evaluation and gives a formula relating the size of the boundary of an extruded polyhedron with the size of decompositive and boundary representation of the generating polyhedron.

REPRESENTATION SCHEME

In the paper the authors make use of a solid representation based on a simplicial decomposition of the interior of the object or of its boundary. A *d*-dimensional simplex is simply a *d*-dimensional triangle, which contains $d+1$ vertices. For example, a 1-dimensional simplex is a straight line segment, and a 3-dimensional simplex is a tetrahedron.

The point set generated by the convex combination of any proper subset of vertices is called a proper 'face'. So any subsets of three vertices from a tetrahedron generates a 2D face (triangle); any subset of two vertices generates a 1D face (edge); any subset of one vertex generates a 0-dimensional face (vertex). Note that the number of faces of a single tetrahedron is therefore $\binom{4}{3}+\binom{4}{2}+\binom{4}{1}=2^4-2$ (see Figure 1). A simplicial complex is a set of simplices which can be considered as a 'well-formed triangulation'. Some examples of sets of simplices which are not simplicial complexes are given in Figure 2.

In the following the authors recall some definitions[18] with a slightly different notation.

Simplices and complexes

A *d*-simplex $\sigma \subset \mathscr{R}^n$ is the convex combination of $d+1$ affinely independent points, called vertices. The set $\{v_0, v_1, \ldots, v_d\}$ of vertices is called the 0-skeleton of the simplex. The *s*-simplex generated from any subset of $s+1$ vertices of a *d*-simplex is called an *s*-face.

A (simplicial) complex is a set of simplices Σ verifying the following conditions: (a) if $\sigma \in \Sigma$, then any face of

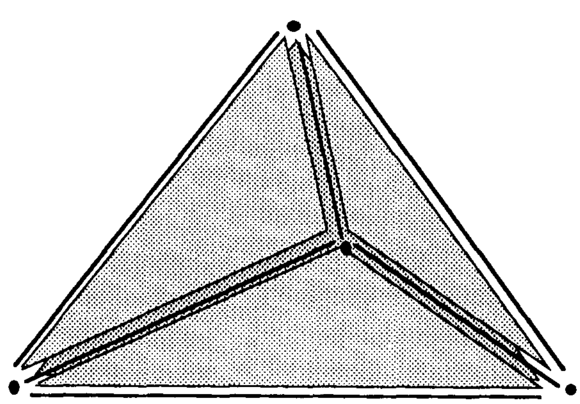


Figure 1. 3-D simplex (a tetrahedron) and its faces

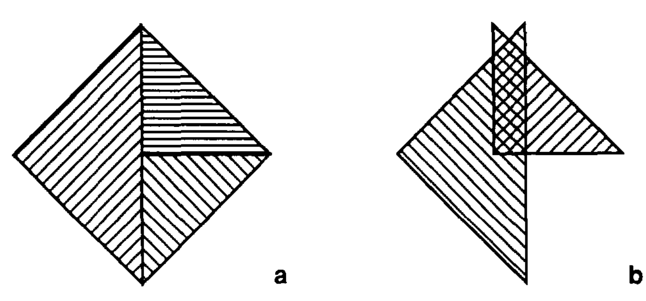


Figure 2. Two sets of simplices which are not simplicial complexes. (a) The intersection of simplices is not a common face. (b) Simplices overlap

σ belongs to Σ; (b) if $\sigma, \tau \in \Sigma$, then either $\sigma \cap \tau = \emptyset$, or $\sigma \cap \tau$ is a face of σ and of τ. The order of a complex is the maximum order of its simplices. A complex $\Sigma^{(d)}$ of order d is also called a d-complex. A d-complex is regular if each simplex is a face of a d-simplex. Two simplices σ_1 and σ_2 in a complex Σ are s-adjacent if they have a common s-face; they are s-connected if a sequence of simplices in Σ exists, beginning with σ_1 and ending with σ_2, such that any two successive terms of the sequence are s-adjacent. This sequence is called a simplicial s-chain. We denote as $K^{(s)}$, $0 \leqslant s \leqslant d$, the s-skeleton, defined as the set of the s-faces of $\Sigma^{(d)}$.

Geometric carrier $[\Sigma] = \cup_{\Sigma} \sigma$ is the point set union of simplices in a complex Σ. A linear d-polyhedron $P^{(d)} \subset \mathscr{R}^n$ coincides with the geometric carrier of a simplicial d-complex, and we write $P^{(d)} = [\Sigma^{(d)}]$. A polyhedron is regular if the associated complex is regular.

The boundary $\partial P^{(d)}$ of a regular d-polyhedron $P^{(d)} = [\Sigma^{(d)}]$ is the geometric carrier of a closed $(d-1)$-complex whose $(d-1)$-simplices are faces of exactly one d-simplex in $\Sigma^{(d)}$. Notice that $\partial\partial P^{(d)} = \emptyset$.

An ordering of the 0-skeleton implies an orientation of a simplex, according to the even or odd permutation class of the 0-skeleton. The two opposite orientations will be denoted as $+\sigma$ and $-\sigma$. Two adjacent simplices are coherently oriented when their common face* has opposite orientations. A complex is orientable when all its simplices can be coherently oriented. The oriented s-faces of the d-simplex $\sigma_k = \pm \langle v_k^0, \ldots, v_k^d \rangle$ are given by the formula:

$$\sigma_k^i = (-1)^i (\sigma_k - \langle v_k^i \rangle), \quad 0 \leqslant i \leqslant d \tag{1}$$

where σ_k^i and v_k^i denote the i-th face and the i-th vertex of σ_k, respectively, and where the minus sign denotes set subtraction.

Winged representation

In solid modelling one is usually interested in regular d-polyhedra. In this case the set $K^{(d)}$ of maximum order simplices is a complete representation of a polyhedron, since any other skeleton can be derived from $K^{(d)}$ by repeated application of the equation (1). $K^{(d)}$ alone is a complete representation, but it takes an $O(|K^{(d)}|)$ time to answer any topological query. It is useful to enrich the representation by storing the highest order adjacencies. In such a way it becomes possible to traverse efficiently the polyhedron and answer queries about the adjacency of topology elements in $O(q)$ time, where q is the size of the query output.

Definition 1 The winged representation $\mathscr{W}(P^{(d)})$ of a polyhedron $P^{(d)} \subset \mathscr{R}^n$ is a pair $(K^{(d)}, \mathscr{A})$, where $K^{(d)}$ is the d-skeleton of P, and $\mathscr{A}$: $K^{(d)} \rightarrow (K^{(d)} \cup \perp)^{d+1}$ is an adjacency function which associates each d-simplex $\sigma_k \in K^{(d)}$ with the $(d+1)$-tuple of d-simplices that are $(d-1)$-adjacent to it.

If $\sigma_k = \langle v_k^0, \ldots, v_k^d \rangle$, then $\mathscr{A}(\sigma_k) = \langle \sigma_{k_0}, \ldots, \sigma_{k_d} \rangle$, where either $\sigma_{k_i} = \perp$ or $\sigma_k \cap \sigma_{k_i} = \sigma_k^i$ and the symbol $\perp$ stands for 'undefined'. The first notation means that σ_k has no adjacent simplex along its face σ_k^i. With abuse of notation we use $\mathscr{A}(\sigma_k^i)$ to indicate σ_{k_i}, so that we can write $\mathscr{A}(\sigma_k) = \langle \mathscr{A}(\sigma_k^0), \ldots, \mathscr{A}(\sigma_k^d) \rangle$. Each d-simplex in $\mathscr{W}(P^{(d)})$ will be therefore represented by using $d+1$ pointers to its vertices and $d+1$ pointers to the adjacent d-simplices (see Figure 3). If $|K^{(d)}| = S$ and $|K^{(0)}| = V$, then the size of the winged representation $\mathscr{W}(P^{(d)})$, with $P^{(d)} \subset \mathscr{R}^n$, is $2(d+1)S + nV$.

The winged representation can be considered a paradigm, because it allows for a unified treatment of various different representation schemes (see the taxonomy of representations by Requicha[19] or Mantyla[20]). This is a direct consequence of the use of simplicial complexes. In particular the following can be shown.

Boundary representation

A winged boundary representation is a two-step mapping between the set of d-polyhedra-with-boundary, the set of $(d-1)$-polyhedra-without-boundary and the set of $(d-1)$-complexes. Both mappings are one-to-many, so that the resulting representation is complete but not unique. A d-polyhedron $P^{(d)}$ is represented as $\mathscr{W}(\partial P^{(d)})$. Some kind of 'canonical' representation can be easily defined. Non-manifold polyhedra are represented as pseudo-manifolds, where any boundary $(d-2)$-face, eventually duplicated, is contained only in two boundary $(d-1)$-faces (see Figure 3).

Decompositive representation

Winged decompositive representations are largely used in this paper. $P^{(d)}$ in $\mathscr{R}^n$ is represented as one of the complexes $\Sigma^{(d)}$ such that $[\Sigma^{(d)}] = P^{(d)}$. Triangulated decompositive representations are better than boundary triangulated ones from many points of view; the authors argue that decompositive representations have smaller size than boundary ones with the same vertices (this can be easily shown in some special cases); set operations and domain integration are simpler; decompositive representations can directly interface finite element codes; finally, many algorithms over decompositive representations can be easily parallelized.

Sweeping representation

In this paper the authors show that the winged representation is a natural way to represent the $(d+1)$-polyhedron obtained by extruding a d-polyhedron. Exact rotationally extruded polyhedra are outside of the domain of the representation, but they can be linearly approximated, as is usual in low dimensional graphics, by the polyhedron obtained by the iterative application of a suitable affine transformation.

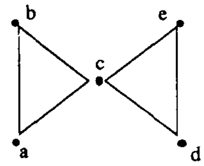
Hierarchical representation

In many geometrical computations, e.g. in computing the intersections or the visible contours of polyhedral approximations of curved surfaces, as well as in using multigrid methods for field problems over geometrical models, it is useful to use hierarchical representations, in order to improve locally the computations where it

* By now, face (without prefix) of a d-simplex stands for $(d-1)$-face.

```
? (printpol p)
Intrinsic dimension : 2
E-space dimension : 2
Simplices number : 2
Vertices number : 5

A : (6 1)
B : (1 1)
C : (7/2 9/2)
D : (6 8)
E : (1 8)
S0 : +(A C B) (0 0 0)
S1 : +(C D E) (0 0 0)
```




```
? (printpol (boundary
p))
Intrinsic dimension : 1
E-space dimension : 2
Simplices number : 6
Vertices number : 5

A : (6 1)
B : (1 1)
C : (7/2 9/2)
D : (6 8)
E : (1 8)
S0 : +(C B) (S1 S2)
S1 : -(A B) (S0 S2)
S2 : +(A C) (S0 S1)
S3 : +(D E) (S4 S5)
S4 : -(C E) (S3 S5)
S5 : +(C D) (S3 S4)
```

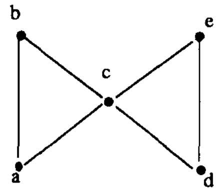


Figure 3. Winged representations of simple polyhedron $P^{(2)}$. *Above: decompositive representation; below: boundary representation. Notice that object is nonmanifold. Simplices are scaled with respect to their centroid*

```
? (printpol p)
Intrinsic dimension : 3
E-space dimension : 3
Simplices number : 6
Vertices number : 10

A0 : (6 1 0)
B0 : (1 1 0)
C0 : (7/2 9/2 0)
D0 : (6 8 0)
E0 : (1 8 0)
A01 : (6 3 2)
B01 : (1 3 2)
C01 : (7/2 13/2 2)
D01 : (6 10 2)
E01 : (1 10 2)
S0 : +(A0 C0 B0 A01) (S1 0 0 0)
S1 : +(C0 B0 A01 C01) (S2 0 0 S0)
S2 : +(B0 A01 C01 B01) (0 0 0 S1)
S3 : +(C0 D0 E0 C01) (S4 0 0 0)
S4 : +(D0 E0 C01 D01) (S5 0 0 S3)
S5 : +(E0 C01 D01 E01) (0 0 0 S4)
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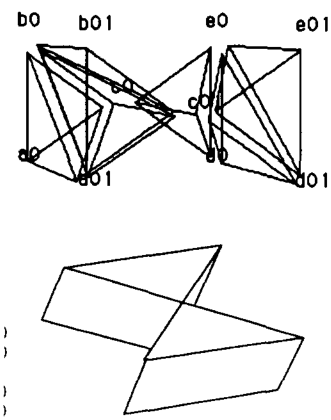

Figure 4. Winged decompositive representation of polyhedron $P^{(3)}$, *obtained as the extrusion of the object in Figure 3*

is necessary, while maintaining one or more coarser related representations. This can be easily achieved by using an ordered set of complexes, where each complex $\Sigma_i^{(d)}$ is a refinement of a subcomplex (eventually unconnected) of $\Sigma_{i-1}^{(d)}$. Such a succession of triangulations can be implemented efficiently by using a slightly modified winged representation scheme, with a few more pointers for each d-simplex. In particular, each such simplex, while maintaining topological information relative to the complex to which it belongs, will also point to the higher level simplex ('parent' simplex) containing it, as well as to one of its 'sibling' simplices, whose set constitutes a decomposition (or a finer approximation) of the parent simplex. This simplicial thread can be implemented in many ways, to improve access efficiency as desired. Such a scheme (winged representation + some more pointers) can also be used to associate geometric directories to the representation, i.e. spatial indices (see, e.g.[21]) which allow efficient access to the complex elements whose intersection with a given space region is not empty.

AFFINE EXTRUSION

In this section we analyze the extrusion of a polyhedron, used as a basic operation in order to generate higher dimensional polyhedra. It is easy to see that (a) the extrusion $P^{(d+1)}$ of a regular polyhedron $P^{(d)}$ is regular; (b) boundary points of $P^{(d)}$ are mapped onto boundary points of $P^{(d+1)}$.

Preliminaries

The point set generated by an extrusion operation over a polyhedron $P^{(d)} = [\Sigma^{(d)}] \subset \mathscr{R}^n (d < n)$ is defined as the polyhedron $P^{(d+1)} = [\Sigma^{(d+1)}]$, where the simplices in $\Sigma^{(d+1)}$ are generated by triangulating the convex combinations of corresponding simplices in $\Sigma^{(d)}$ and $T\Sigma^{(d)}$, under the conditions: (a) T is an invertible affine transformation of $\mathscr{R}^n$ and (b) $\Sigma^{(d)}$ and $T\Sigma^{(d)}$ do not lie in the same affine subspace.

Definition 2 If $P^{(d)} = [\Sigma^{(d)}] \subset \mathscr{R}^n$, T is an invertible affine transformation in $\mathscr{R}^n$ and $\langle v_0, \ldots, v_d, Tv_0 \rangle$ is a simplex for each $\langle v_0, \ldots, v_d \rangle \in K^{(d)}$, then the extrusion of $P^{(d)}$ is the set

$$P^{(d+1)} = S_T(\Sigma^{(d)}) = \{q : q = \alpha p + \beta T p\}$$

where $p \in \Sigma^{(d)}$, $\alpha, \beta \geqslant 0$, and $\alpha + \beta = 1$.

$S_T(\Sigma^{(d)})$ is a $(d+1)$-dimensional object, as any $q \in S_T(\Sigma^{(d)})$ can be uniquely expressed as $\alpha p + (1-\alpha)Tp$, and therefore is determined by the baricentric coordinates of p into the simplex to which it belongs, and by the parameter α.

Definition 3 A $(d+1)$-tube is the polyhedron $S_T(\sigma)$, where σ is a d-simplex.

There follows a combinatorial rule generating a simplicial chain $\Theta^{(d+1)}(\sigma)$ which triangulates the $(d+1)$-tube $S_T(\sigma)$. The proof is given in Paoluzzi and Cattani[18].

Theorem 1 If σ is a d-simplex, T is a suitable affine transformation of $\mathscr{R}^n (d < n)$, then a simplicial chain triangulating $S_T(\sigma)$ is

$$\Theta^{(d+1)}(\sigma) = \{ \tau_i : \tau_i = (-1)^{id} \langle v_i, \ldots, v_d, Tv_0, \ldots, Tv_i \rangle, \; i = 0, \ldots, d \} \quad (2)$$

Example 1 A 0-simplex (point) generates a 1-tube (straight line segment), where a simplicial chain triangulating the tube has one 1-simplex; a 1-simplex (straight line segment) gives a 2-tube (rectangle), where a simplicial chain triangulating the tube has two 2-simplices; a 2-simplex (triangle) produces a 3-tube (wedge), where a simplicial chain triangulating the tube has three 3-simplices; a 3-simplex (tetrahedron) generates a 4-tube, where a simplicial chain has four 4-simplices, and so on.

Example 2 The 5-tube generated by extruding a 4-simplex $\sigma = \langle v_0 v_1 v_2 v_3 v_4 \rangle$ is triangulated by the simplicial chain $\Theta^{(5)}(\sigma)$, where:

$$\Theta^{(5)}(\sigma) = \begin{cases} \tau_0 = +\langle v_0 & v_1 & v_2 & v_3 & v_4 & Tv_0 \rangle \\ \tau_1 = +\langle v_1 & v_2 & v_3 & v_4 & Tv_0 & Tv_1 \rangle \\ \tau_2 = +\langle v_2 & v_3 & v_4 & Tv_0 & Tv_1 & Tv_2 \rangle \\ \tau_3 = +\langle v_3 & v_4 & Tv_0 & Tv_1 & Tv_2 & Tv_3 \rangle \\ \tau_4 = +\langle v_4 & Tv_0 & Tv_1 & Tv_2 & Tv_3 & Tv_4 \rangle \end{cases}$$

The extrusion of a polyhedron $P^{(d)}$ is efficiently obtained (a) by generating a set of simplicial chains $\{\Theta^{(d+1)}(\sigma)\}$, where $\Theta^{(d+1)}(\sigma)$ triangulates $S_T(\sigma)$, the tube obtained by independently extruding σ, $\sigma \in K^{(d)}$, and (b) by 'gluing' the elements of such a set (i.e. resolving adjacencies between different chains), under suitable conditions on the representation $\mathscr{W}(P^{(d)})$. Since the simplicial chain $\Theta^{(d+1)}(\sigma)$ contains $2(d+1)$ vertices and $d+1$ cells ($(d+1)$-simplices), the polyhedron $P^{(d+1)} = S_T(\Sigma^{(d)})$ has $2V$ vertices and $(d+1)S$ cells.

Computing $\mathscr{W}(P^{(d+1)})$ from $\mathscr{W}(P^{(d)})$

In this section the authors show how to compute a decompositive winged representation for an extruded polyhedron $P^{(d+1)}$, starting from a decompositive winged representation of the input polyhedron $P^{(d)}$. As both the underlying complex and the explicit representation of the d-adjacencies in $\mathscr{W}(P^{(d+1)})$ are calculated by using closed formulas, the complexity is $O(d^2S)$, where $S = |K^{(d)}(P^{(d)})|$. The resulting algorithm is also $\Omega(d^2S)$. In fact, there are $(d+1)S$ $(d+1)$-simplices in the generated triangulation of the output polyhedron, and $(d+2)$ d-adjacencies must be evaluated for each $(d+1)$-simplex.

Let $P^{(d)} = [\Sigma^{(d)}]$, with $|K^{(d)}| = S$ and $P^{(d+1)} = S_T(\Sigma^{(d)})$. It is not difficult to see that:

$$P^{(d+1)} = S_T(\Sigma^{(d)}) = \left[\bigcup_{\sigma \in K^{(d)}} \Theta^{(d+1)}(\sigma) \right] \tag{3}$$

Notice that

$$\bigcup_{\sigma \in K^{(d)}} \Theta^{(d+1)}(\sigma) \tag{4}$$

is the combinatorial union of simplicial complexes, but it is not necessarily a simplicial complex. In order to guarantee that (4) is a simplicial complex, i.e. that the intersection of any pair of simplices is either empty or is a face for both simplices, we need a particular representation $\mathscr{W}(P^{(d)})$.

Definition 4 A representation $\mathscr{W}(P^{(d)}) = (K^{(d)}, \mathscr{A})$ is well-ordered if $\sigma_i^h = \sigma_i \cap \sigma_j = -\sigma_j^k$, where $\cap$ denotes combinatorial intersection of 0-skeletons, for each pair of $(d-1)$-adjacent simplices $\sigma_i, \sigma_j \in K^{(d)}$.

We note that a representation is well-ordered depending on the choice of a suitable permutation for the 0-skeleton of each d-simplex.

Example 3 Consider $P^{(2)} = [\Sigma^{(2)}]$, where $\Sigma^{(2)} = \{\sigma_0, \sigma_1\}$. The representation

$$\mathscr{W}(P^{(2)}) = \begin{cases} \sigma_0 = +\langle v_0, v_1, v_3 \rangle, \langle \sigma_1, \perp, \perp \rangle \\ \sigma_1 = +\langle v_2, v_3, v_1 \rangle, \langle \sigma_0, \perp, \perp \rangle \end{cases} \tag{5}$$

is coherently oriented but not well-ordered, because $\sigma_0^0 = +\langle v_1, v_3 \rangle$ and $\sigma_1^0 = +\langle v_3, v_1 \rangle$, and not, as required, $\sigma_1^0 = -\langle v_1, v_3 \rangle$. A coherently oriented and well-ordered representation for $P^{(2)}$ is

$$\mathscr{W}(P^{(2)}) = \begin{cases} \sigma_0 = +\langle v_0, v_1, v_3 \rangle, \langle \sigma_1, \perp, \perp \rangle \\ \sigma_1 = +\langle v_1, v_2, v_3 \rangle, \langle \perp, \sigma_0, \perp \rangle \end{cases} \tag{6}$$

Well-ordered representations play a central role because of the following theorem:

Theorem 2 The set $\Sigma^{(d+1)} = \cup_{\sigma \in K^{(d)}} \Theta^{(d+1)}(\sigma)$ is a simplicial complex if and only if $\mathscr{W}(P^{(d)})$ is well-ordered.

We can assume that representations are well-ordered, because the following closure property holds:

Theorem 3 If $\mathscr{W}(P^{(d)})$ is well-ordered, then $\mathscr{W}(P^{(d+m)})$, obtained by applying m extrusions to $P^{(d)}$, is also well-ordered.

In the following we give closed formulas for calculating the d-adjacencies in $\mathscr{W}(P^{(d+1)})$, therefore making it possible to compute a winged representation for the extruded polyhedron (see Figure 6). We suppose that simplices in $K^{(d)}$ are ordered from σ_0 to σ_{S-1}, and that simplices in $K^{(d+1)}$ range from τ_0 to $\tau_{S(d+1)-1}$. Hence the simplices in $\Theta^{(d+1)}(\sigma_p)$ are indexed as $\{\tau_{p(d+1)}, \ldots \tau_{p(d+1)+d}\}$ (see Figure 5).

A formula which generates the d-adjacencies internal to the simplicial chain $\Theta^{(d+1)}(\sigma_p)$ is easy to write. We note that any simplex in the chain (except the first and the last) is adjacent to two other simplices. So we have:

$$\begin{aligned} &\mathscr{A}(\tau_i^0) = \tau_{i+1}, && p(d+1) \leqslant i \leqslant p(d+1)+d-1 \\ &\mathscr{A}(\tau_i^{d+1}) = \tau_{i-1}, && p(d+1)+1 \leqslant i \leqslant p(d+1)+d \\ &\mathscr{A}(\tau_{p(d+1)}^{d+1}) = \mathscr{A}(\tau_{p(d+1)+d}^0) = \perp && \end{aligned} \tag{7}$$

In the following we address the problem of computing the d-adjacencies between simplices belonging to different chains. Such adjacencies are induced from the original $(d-1)$-adjacencies in $\mathscr{W}(P^{(d)})$. Since the adjacency relation is symmetric, if $\mathscr{A}(\sigma_p^h) = \sigma_q$ holds in $\mathscr{W}(P^{(d)})$, then $\mathscr{A}(\sigma_q^k) = \sigma_p$ must also hold in $\mathscr{W}(P^{(d)})$ for some k. Each of these two equations induces d adjacency values in $\mathscr{W}(P^{(d+1)})$.

In order to explicitly generate the d-adjacencies we define a function $f: \{(p, h), (q, k)\} \rightarrow \{(i, j)\}^d$ which produces an ordered sequence constituted by d index

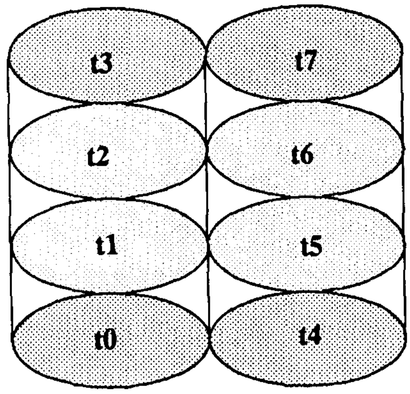

Figure 5. For d $= 3$ *and* $S = 2$, *this is the numbering of simplices in the two simplicial chains* $\Theta^{(4)}(\sigma_0)$ *and* $\Theta^{(4)}(\sigma_1)$

```
? (printpol (tsweep p))
Intrinsic dimension : 4
E-space dimension : 4
Simplices number : 24
Vertices number : 20

A00 : (6 1 0 0)
B00 : (1 1 0 0)
C00 : (7/2 9/2 0 0)
D00 : (6 8 0 0)
E00 : (1 8 0 0)
A010 : (6 3 2 0)
B010 : (1 3 2 0)
C010 : (7/2 13/2 2 0)
D010 : (6 10 2 0)
E010 : (1 10 2 0)
A001 : (6 1 0 8)
B001 : (1 1 0 8)
C001 : (7/2 9/2 0 8)
D001 : (6 8 0 8)
E001 : (1 8 0 8)
A0101 : (6 3 2 8)
B0101 : (1 3 2 8)
C0101 : (7/2 13/2 2 8)
D0101 : (6 10 2 8)
E0101 : (1 10 2 8)
S0 : +(A00 C00 B00 A010 A001) (S1 0 0 0 0)
S1 : -(C00 B00 A010 A001 C001) (S2 0 0 S4 S0)
S2 : +(B00 A010 A001 C001 B001) (S3 0 S5 0 S1)
S3 : -(A010 A001 C001 B001 A0101) (0 S6 0 0 S2)
S4 : +(C00 B00 A010 C010 C001) (S5 0 0 S1 0)
S5 : -(B00 A010 C010 C001 B001) (S6 0 S2 S8 S4)
S6 : +(A010 C010 C001 B001 A0101) (S7 S3 S9 0 S5)
S7 : -(C010 C001 B001 A0101 C0101) (0 S10 0 0 S6)
S8 : +(B00 A010 C010 B010 B001) (S9 0 0 S5 0)
S9 : -(A010 C010 B010 B001 A0101) (S10 0 S6 0 S8)
S10 : +(C010 B010 B001 A0101 C0101) (S11 S7 0 0 S9)
S11 : -(B010 B001 A0101 C0101 B0101) (0 0 0 0 S10)
S12 : +(C00 D00 E00 C010 C001) (S13 0 0 0 0)
S13 : -(D00 E00 C010 C001 D001) (S14 0 0 S16 S12)
S14 : +(E00 C010 C001 D001 E001) (S15 0 S17 0 S13)
S15 : -(C010 C001 D001 E001 C0101) (0 S18 0 0 S14)
S16 : +(D00 E00 C010 D010 D001) (S17 0 0 S13 0)
S17 : -(E00 C010 D010 D001 E001) (S18 0 S14 S20 S16)
S18 : +(C010 D010 D001 E001 C0101) (S19 S15 S21 0 S17)
S19 : -(D010 D001 E001 C0101 D0101) (0 S22 0 0 S18)
S20 : +(E00 C010 D010 E010 E001) (S21 0 0 S17 0)
S21 : -(C010 D010 E010 E001 C0101) (S22 0 S18 0 S20)
S22 : +(D010 E010 E001 C0101 D0101) (S23 S19 0 0 S21)
S23 : -(E010 E001 C0101 D0101 E0101) (0 0 0 0 S22)
```

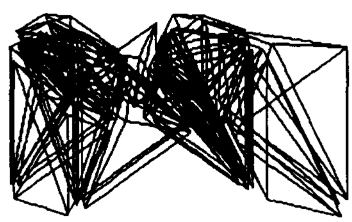

Figure 6. Four dimensional image, under extrusion operation, of the polyhedron $\mathcal{W}(\mathrm{P}^{(3)})$ *shown in Figure 4*

pairs ($(d+1)$-simplex, d-face).

$$f(p,h)=\begin{cases}(i_l,j_l)=\begin{cases}(p(d+1)+1,\ d) & h=0\\ (p(d+1),\ h) & 1\leqslant h\leqslant d\end{cases} & l=1\\[2ex] (i_l,j_l)=\begin{cases}(i_{l-1}+2,\ d) & j_{l-1}=1\\ (i_{l-1}+1,\ j_{l-1}-1) & j_{l-1}>1\end{cases} & 2\leqslant l\leqslant d\end{cases} \tag{8}$$

An analogous definition holds for $f(q, k)$, by exchanging p, h with q, k. The d adjacencies in $\mathcal{W}(P^{(d+1)})$ induced by $\mathcal{A}(\sigma_p^h) = \sigma_q$ are now given by:

$$\mathcal{A}(\tau_{i_l}^{j_l}) = \tau_{r_l}, \quad (i_l, j_l) \in f(p, h), (r_l, \cdot) \in f(q, k), 1 \leqslant l \leqslant d \tag{9}$$

For the d adjacencies induced by $\mathcal{A}(\sigma_q^k) = \sigma_p$ the same formula holds, by exchanging p, h with q, k. For a boundary face σ_p^h in $\mathcal{W}(P^{(d)})$ we have $\mathcal{A}(\sigma_p^h) = \perp$, and it is sufficient to write

$$\mathcal{A}(\tau_{i_l}^{j_l}) = \perp, \quad (i_l, j_l) \in f(p, h), 1 \leqslant l \leqslant d. \tag{10}$$

The formulas (7)(9)(10) completely solve the problem of computing the adjacency function $\mathcal{A}$ in $\mathcal{W}(P^{(d+1)})$ with time and space complexity $\Omega(d^2S)$.

The polyhedron $P^{(d+m)}$ which is obtained by means of m extrusion operations on $P^{(d)} \subset \mathcal{R}^n$ $(d + m \leqslant n)$ will have 2^mV vertices and $(d+1)\cdots(d+m)S < (d+m)^mS$ $(d+m)$-simplices, where V and S are the numbers of vertices and simplices of $P^{(d)}$, respectively. The representation $\mathcal{W}(P^{(d+m)})$ has size $O(2(d+m+1)(d+m)^mS + (n+m)2^mV)$.

Example 4 An m-dimensional hypercube (generated by independently extruding simplices) has size m!. Take a single point and apply to it a sequence of m translational extrusions. The resulting polyhedron is an m-dimensional hypercube, with $|K^{(m)}(P^{(m)})| = (0+1)\cdots(0+m)\cdot 1 = \mathrm{m}!$.

POLYHEDRAL APPROXIMATION OF ROTATIONAL EXTRUSION

In linearly approximating a rotation with angle θ by using h steps, an affine step transformation T_α depending on an angle $\alpha \doteq \theta/h$ must be applied. In order to eliminate special cases and self-crossings, we represent a rotation of $\mathcal{R}^{n-1}$ as a transformation of $\mathcal{R}^n$ composed of a rotation and a translation depending on the same parameter (see Figure 10). Without loss of generality, we can assume that the rotation is an elementary rotation in the x_1x_2 plane, i.e. is such that any point x in the coordinate plane x_1x_2 is transformed into a point Rx belonging to the same plane. More general rotations, where the rotation plane is not coordinate or does not contain the origin, can be obtained, as is usual in graphics, by composition of elementary affine transformations. The curve polyhedron $\hat{P}^{(d+1)}$, obtained as rotational extrusion of $P^{(d)} = [\Sigma^{(d)}] \subset \mathcal{R}^n$ $(n > d,\ n \geqslant 2)$, can be linearly approximated as $\hat{P}^{(d+1)} = [\cup_{i=1}^h \Sigma_i^{(d+1)}]$ by a succession $\{\Sigma_i^{(d+1)}\}$, $1 \leqslant i \leqslant h$, of suitable complexes, where $\Sigma_i^{(d+1)}$ is defined as follows.

Under the previous assumptions, the step transformation $T_\alpha : \mathcal{R}^n \to \mathcal{R}^n$ is:

$$[v_1^* \ldots v_n^*]' = T_\alpha [v_1 \ldots v_n]' \tag{11}$$

where

$$\begin{cases} v_1^* = v_1 \cos\alpha - v_2 \sin\alpha, \\ v_2^* = v_1 \sin\alpha + v_2 \cos\alpha, \\ v_i^* = v_i, & 3 \leqslant i \leqslant n-1, \\ v_n^* = v_n + \alpha. \end{cases} \tag{12}$$

The polyhedron $\hat{P}^{(d+1)}$, a linear approximation of $P^{(d+1)}$, is defined as the union of a succession of $(d+1)$-complexes obtained by iterative application of the step transformation T_α.

$$\begin{aligned} \Sigma_1^{(d+1)} &= S_{T_\alpha}(\Sigma^{(d)}) \\ \Sigma_i^{(d+1)} &= T_\alpha \Sigma_{i-1}^{(d+1)}, \quad 2 \leqslant i \leqslant h \end{aligned} \tag{13}$$

Another equivalent definition of $\{\Sigma_i^{(d+1)}\}$ is:

$$\begin{aligned} \Sigma_i^{(d+1)} &= S_{T_\alpha}(\Sigma_{i-1}^{(d)}), \qquad 1 \leqslant i \leqslant h \\ \Sigma_i^{(d)} &= T_\alpha \Sigma_{i-1}^{(d)}, \qquad 1 \leqslant i \leqslant h-1 \\ \Sigma_0^{(d)} &= \Sigma^{(d)} \end{aligned} \tag{14}$$

Notice from the previous definitions that, in building a polyhedral approximation of a curved (rotational) polyhedron, extrusion and affine transformation commute. This is a well-known property in low dimensional graphics. In other words the result can be generated either with one extrusion operation and $h-1$ affine transformations, or with h affine transformations and h extrusions. It is clear that the second construction is computationally more expensive. In the following we show how to connect the adjacent pairs of complexes in the succession $\{\Sigma_i^{(d+1)}\}$.

We remember that $S_{T_\alpha}(\Sigma^{(d)})$ is expressed as a set of simplicial chains $\Theta^{(d+1)}(\sigma)$, $\sigma \in K^{(d)}$. Their adjacencies internal to each step complex $\Sigma_i^{(d+1)}$ are known (can be determined by using the eq. (7) (9) (10)). In order to obtain a winged representation $\mathscr{W}(\hat{P}^{(d+1)})$ of the polyhedral approximation of $\hat{P}^{(d+1)}$, the problem is now to glue together the corresponding tubes in adjacent 'step' complexes $\Sigma_i^{(d+1)}$ and $\Sigma_{i+1}^{(d+1)}$.

If $|K^{(d)}| = S$, then there are S structures $\Theta^{(d+1)}$ in each $\Sigma_i^{(d+1)}$ $(1 \geqslant i \geqslant h)$, and consequently $(h-1)S$ pairs of $(d+1)$-chains must be glued together. It is sufficient to note that the final element of the $\Theta^{(d+1)}(\sigma)$ chain in the complex $\Sigma_i^{(d+1)}$ is adjacent to the starting element of the $\Theta^{(d+1)}(\sigma)$ chain in the complex $\Sigma_{i+1}^{(d+1)}$, and vice versa, for any $\sigma \in K^{(d)}$ and for any pair of adjacent step complexes.

At this point we must remember that any $(d+1)$-simplex τ_k $(0 \leqslant k \leqslant d)$ in $\Theta^{(d+1)}(\sigma)$ is d-adjacent to two elements in the chain, except the first and the last simplices (τ_0 and τ_d, respectively). These two simplices have undefined adjacencies with the exterior of the chain ($\mathscr{A}(\tau_0^{d+1})$ and $\mathscr{A}(\tau_d^0)$, respectively). If the $(d+1)$-simplices τ_k $(0 \leqslant k \leqslant d)$ in the tubes $\Theta_i(\sigma_m)$ $(0 \leqslant i \leqslant h-1)$ $(0 \leqslant m \leqslant S-1)$ are ordered sequentially, then they give an ordered set of $(d+1) \cdot h \cdot S$ elements. The simplex indexed by the triple (k, i, m) will be positioned at the address $p = k + m(d+1) + iS(d+1)$ in this ordering. In such a hypothesis the various chains can be glued together by setting the appropriate adjacencies as follows:

$$\begin{aligned} \mathscr{A}(\tau^0_{l(d+1)+d}) &= \tau_{(d+1)(l+S)} \\ \mathscr{A}(\tau^{d+1}_{(d+1)(l+S)}) &= \tau_{l(d+1)+d} \end{aligned} \qquad l = 0, \ldots, S(h-1)-1 \tag{15}$$

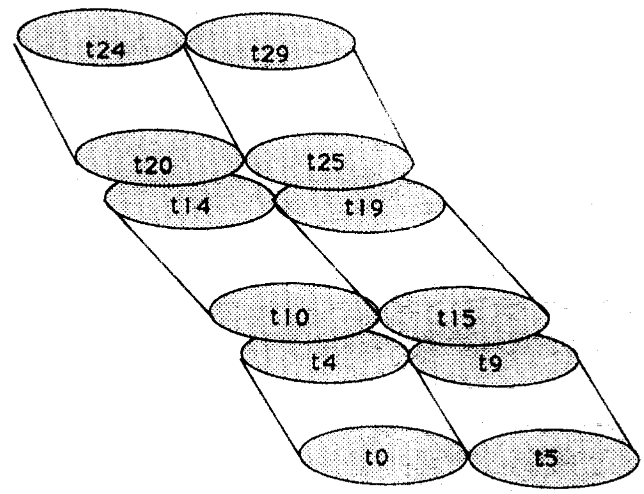

Figure 7. Structure of the simplicial (d + 1)-complex approximating in three steps (h = 3) the rotational extrusion of $P^{(4)}$ with S = 2

where $(d+1)(l+S)$ is obtained by grouping terms in the explicit address expression $(l+1)(d+1) + (S-1) \times (d+1)$.

Example 5 Let's start with $P^{(4)}$ and $S = 2$. If $h = 3$ we obtain $\hat{P}^{(5)} = [\Sigma_0^{(5)} \cup \Sigma_1^{(5)} \cup \Sigma_2^{(5)}]$. Each $\Sigma_i^{(5)}$ consists of 2 chains $\Theta^{(5)}(\sigma)$, and each $\Theta^{(5)}(\sigma)$ contains 5 simplices. Figure 7 shows such a structure and the numbering of the first and the last simplex in each chain, as explained above. Formula (15) is rewritten as

$$\begin{aligned} \mathscr{A}(\tau^0_{5l+4}) &= \tau_{5(l+2)} \\ \mathscr{A}(\tau^5_{5(l+2)}) &= \tau_{5l+4} \end{aligned} \qquad l = 0, \ldots, 3 \tag{16}$$

and d-adjacencies between corresponding pairs of simplices in different step complexes are immediately computed:

$$\begin{aligned} &\mathscr{A}(\tau^0_4) = \tau_{10} \quad \mathscr{A}(\tau^0_9) = \tau_{15} \quad \mathscr{A}(\tau^0_{14}) = \tau_{20} \\ &\mathscr{A}(\tau^0_{19}) = \tau_{25} \\ &\mathscr{A}(\tau^5_{10}) = \tau_4 \quad \mathscr{A}(\tau^5_{15}) = \tau_9 \quad \mathscr{A}(\tau^5_{20}) = \tau_{14} \\ &\mathscr{A}(\tau^5_{25}) = \tau_{19} \end{aligned} \tag{17}$$

It is easy to derive the following properties for the linear approximation $\hat{P}^{(d+1)}$ of a rotational extrusion: (a) $\hat{P}^{(d+1)}$ has exactly $(h+1)V$ vertices and $(d+1)hS$ cells; (b) $\hat{P}^{(d+m)}$ obtained with m rotational extrusions, each one approximated with h linear steps, contains $(h+1)^m V$ vertices and $(d+1)h \cdots (d+m)hS$ cells; (c) the cell number of $\hat{P}^{(d+m)}$ can be therefore expressed as $O[(d+m)^h h^m S]$. The topology representation of such a polyhedron has a size $O[2(d+m)^{m+1} h^m S]$.

BOUNDARY EVALUATION

Boundary evaluation of a winged decompositive representation of a d-polyhedron $P^{(d)}$ is very easy. First of all, notice that a direct representation of boundary faces is already explicitly embedded in the decompositive winged scheme. In fact, for each $\perp$ value in the adjacency tuples $\mathcal{A}(\sigma_k)$, where σ_k is a d-simplex, the corresponding $(d-1)$-face of σ_k is a boundary face for $P^{(d)}$, so that the skeleton $K^{(d-1)}(\partial P^{(d)})$ can be evaluated in time proportional to $(d+1)S$. Some work is necessary to compute the adjacency function between $(d-1)$-faces in $\partial P^{(d)}$. At this point two different approaches can be taken.

Simplex boundary

The boundary of a d-dimensional simplex $\sigma = +\langle v_0, \ldots, v_d \rangle$ is the set of its $(d-1)$-faces:

$$\partial\sigma = \{\sigma_i : \sigma_i = (-1)^i \cdot (\sigma - \langle v_i \rangle), \quad 0 \leq i \leq d\} \qquad (18)$$

We can immediately write a formula giving the adjacencies in the complex $\partial\sigma$:

$$\begin{aligned} \mathcal{A}(\sigma_i^j) &= \sigma_{j+1} \\ \mathcal{A}(\sigma_{j+1}^i) &= \sigma_i \end{aligned} \quad 0 \leq i \leq d,\ i \leq j \leq d-1 \qquad (19)$$

Example 6 Given the 4-simplex σ, with winged representation $\mathcal{W}(\sigma) = +\langle v_0, v_1, v_2, v_3, v_4 \rangle, \langle \perp, \perp, \perp, \perp, \perp \rangle$, then we have the boundary complex:

$$\mathcal{W}(\partial\sigma) = \begin{cases} \sigma_0 = +\langle v_1, v_2, v_3, v_4 \rangle, \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle \\ \sigma_1 = -\langle v_0, v_2, v_3, v_4 \rangle, \langle \sigma_0, \sigma_2, \sigma_3, \sigma_4 \rangle \\ \sigma_2 = +\langle v_0, v_1, v_3, v_4 \rangle, \langle \sigma_0, \sigma_1, \sigma_3, \sigma_4 \rangle \\ \sigma_3 = -\langle v_0, v_1, v_2, v_4 \rangle, \langle \sigma_0, \sigma_1, \sigma_2, \sigma_4 \rangle \\ \sigma_4 = +\langle v_0, v_1, v_2, v_3 \rangle, \langle \sigma_0, \sigma_1, \sigma_2, \sigma_3 \rangle \end{cases} \qquad (20)$$

Complex boundary

Two different methods can be given in order to evaluate the boundary $\partial P^{(d)}$ of a polyhedron $P^{(d)}$ when a decompositive representation of it is given.

Disjoint union of simplicial boundaries

The first method computes $\mathcal{W}(\partial P^{(d)})$ by using a composition operation of the representations $\mathcal{W}(\partial\sigma^{(d)})$ for each adjacent pair $\sigma_p^{(d)}, \sigma_q^{(d)} \in K^{(d)}$. In fact, both the complex $\partial\sigma^{(d)}$ and the adjacency function $\mathcal{A}^{(d-1)}: K^{(d)} \to K^d$, where $K = K^{(d-1)}(\partial\sigma^{(d)})$, are computable by using formulas (18) (19). Afterwards, the boundary complexes associated to two d-simplices σ_p, σ_q, which are adjacent in $P^{(d)}$, can be 'glued' together, by eliminating their common boundary $(d-1)$-face and by 'crossing' their $(d-1)$-adjacencies (see Figure 8). Such a 'gluing and crossing' operation, which will be called sewing and denoted with the symbol $\uplus$, can be performed as discussed in the following.

Definition 5 The sewing of the winged representations of two simplicial complexes Σ_h and Σ_k is the winged representation of the disjoint union of them:

$$\mathcal{W}(\Sigma_h) \uplus \mathcal{W}(\Sigma_k) \doteq \mathcal{W}(\Sigma_h \oplus \Sigma_k) \qquad (21)$$

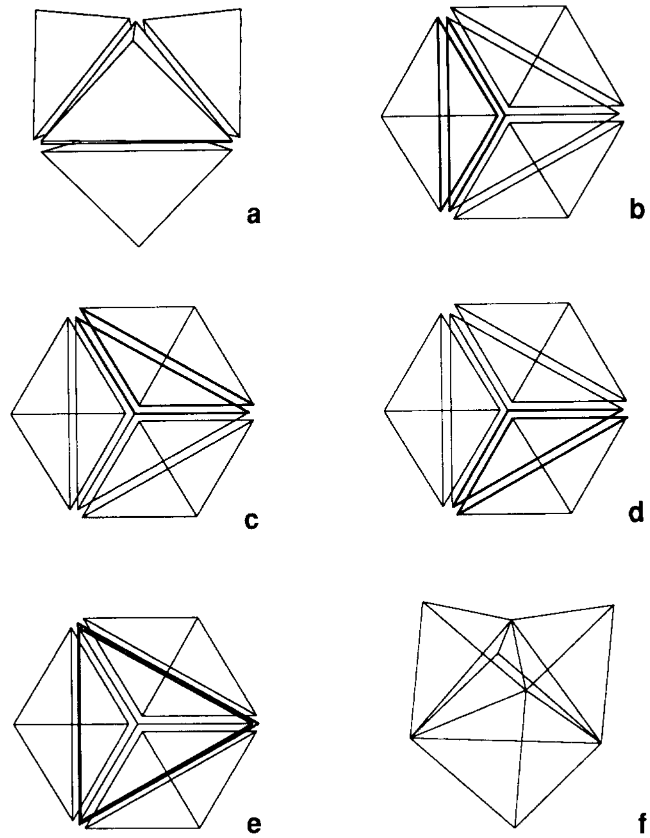

Figure 8. (a) The decomposition of simple 3-polyhedron (stellated tetrahedron). (b) (c) (d) (e) Boundary evaluation steps. (f) Boundary of the object

Let $\mathcal{W}(P^{(d)})$ be given, where $\mathcal{A}^{(d)}(\sigma_p^h) = \sigma_q$, and $\mathcal{A}^{(d)}(\sigma_q^k) = \sigma_p$, with $\sigma_p^h \equiv \sigma_q^k$. The $(d-1)$-skeleton of the disjoint union of the boundaries $\partial\sigma_p$ and $\partial\sigma_q$ is

$$K^{(d-1)}(\partial\sigma_p \oplus \partial\sigma_q) = K^{(d-1)}(\partial\sigma_p) \cup K^{(d-1)}(\partial\sigma_q) - \{\sigma_p^h\} - \{\sigma_q^k\}. \qquad (22)$$

We want now to compute $\mathcal{A}^{(d-1)}(\partial\sigma_p \oplus \partial\sigma_q)$. If the $(d-1)$-adjacencies of the common faces in $\mathcal{W}(\partial\sigma_p)$ and $\mathcal{W}(\partial\sigma_q)$ are expressed as:

$$\mathcal{A}^{(d-1)}(\sigma_p^h) = \langle \sigma_p^{h_0} \ldots \sigma_p^{h_{d-1}} \rangle \qquad (23)$$

$$\mathcal{A}^{(d-1)}(\sigma_q^k) = \langle \sigma_q^{k_0} \cdots \sigma_q^{k_{d-1}} \rangle \qquad (24)$$

then, in order to obtain $\mathcal{A}^{(d-1)}(\partial\sigma_p \oplus \partial\sigma_q)$, it is sufficient:

(a) to start with the union set of adjacencies $\mathcal{A}^{(d-1)}(\partial\sigma_p) \cup \mathcal{A}^{(d-1)}(\partial\sigma_q)$;
(b) to exchange in $\mathcal{A}^{(d-1)}(\sigma_p^{h_i})$ the occurrence of σ_p^h with $\sigma_q^{k_i} (0 \leq i \leq d-1)$ and in $\mathcal{A}^{(d-1)}(\sigma_q^{k_i})$ the occurrence of σ_q^k with $\sigma_p^{h_i} (0 \leq i \leq d-1)$;
(c) to cancel the two tuples $\mathcal{A}^{(d-1)}(\sigma_p^h)$ and $\mathcal{A}^{(d-1)}(\sigma_q^k)$.

As we have said, $\mathcal{W}(\partial P^{(d)})$ is obtained from $\mathcal{W}(P^{(d)})$ by sewing the winged representations of adjacent pairs of d-simplices. In formal terms we can write:

$$\mathcal{W}(\partial P^{(d)}) = \biguplus_{\sigma_p \in K^{(d)}} \ \biguplus_{\sigma_q \in \mathcal{A}(\sigma_p)} (\mathcal{W}(\partial\sigma_p), \mathcal{W}(\partial\sigma_q)) \qquad (25)$$

Actually, an implementation of the algorithm is instead performed by sewing the boundary of a d-simplex at a time with the complex constituted by the previously sewed boundaries:

$$\mathcal{W}(\partial\Sigma_0) = \mathcal{W}(\partial\sigma_0) \qquad (26)$$

$$\mathcal{W}(\partial\Sigma_i) = \mathcal{W}(\partial\Sigma_{i-1}) \oplus \mathcal{W}(\partial\sigma_i), \quad 1 \leqslant i \leqslant S-1 \qquad (27)$$

Executing this iterative construction we have that $\mathcal{W}(\partial\Sigma_{S-1}) \equiv \mathcal{W}(\partial P^{(d)})$.

Local evaluation

A second method of boundary evaluation starting from a decompositive representation $\mathcal{W}(P^{(d)})$ is more local, and consists of looking for the adjacent boundary face σ_q^k of a boundary face σ_p^h. This can be done by recursively visiting (with the use of the $\mathcal{A}$ function) the subcomplex Σ_{pq}^{hk} constituted by the d-simplices which contain the $(d-2)$-face in common between σ_p^h and σ_q^k. This strategy of local evaluation can be carefully mixed with that formerly described, in order to obtain an algorithm working in $O(d^2S)$.

A formula relating $|K^{(d)}(P^{(d)})|$, $|K^{(d-1)}(\partial P^{(d)})|$ and $|K^{(d+m-1)}(\partial P^{(d+m)})|$

We now give a formula for $|K^{(d+m-1)}(\partial P^{(d+m)})|$, the cardinality of the boundary of the polyhedron $P^{(d+m)}$, obtained by applying m extrusions to $P^{(d)}$. We assume that $|K^{(d-1)}(\partial P^{(d)})|$, the cardinality of the boundary of $P^{(d)}$, and $|K^{(d)}(P^{(d)})|$, the cardinality of the d-skeleton of $P^{(d)}$, are known. For $m = 1$ we can write:

$$|K^{(d)}(\partial P^{(d+1)})| = d\,|K^{(d-1)}(\partial P^{(d)})| + 2\,|K^{(d)}(P^{(d)})|. \qquad (28)$$

The expression for any m can be derived by repeatedly applying (28) as follows:

$$\begin{aligned}
&|K^{(d+m-1)}(\partial P^{(d+m)})| \\
&\quad = (d+m-1)\,|K^{(d+m-2)}(\partial P^{(d+m-1)})| \\
&\qquad + 2\,|K^{(d+m-1)}(P^{(d+m-1)})| \\
&\quad = (d+m-1)\{(d+m-2)\,|K^{(d+m-3)}(\partial P^{(d+m-2)})| \\
&\qquad + 2\,|K^{(d+m-2)}(P^{(d+m-2)})|\} \\
&\qquad + 2\,|K^{(d+m-1)}(P^{(d+m-1)})| = \cdots = \\
&\quad = (d+m-1)(d+m-2)\cdots(d+1)d\,|K^{(d-1)} \\
&\qquad \times (\partial P^{(d)})| \\
&\qquad + 2\,|K^{(d+m-1)}(P^{(d+m-1)})| \\
&\qquad + 2(d+m-1)\,|K^{(d+m-2)}(P^{(d+m-2)})| + \cdots + \\
&\qquad + 2(d+m-1)(d+m-2)\ldots \\
&\qquad \times (d+1)\,|K^{(d)}(P^{(d)})|.
\end{aligned}$$

Observing that $|K^{(i)}(P^{(i)})| = i(i-1)\ldots(d+1)\,|K^{(d)}(P^{(d)})|$, $d+1 \leqslant i \leqslant d+m-1$ and collecting the resulting m equal terms, we finally obtain:

$$\begin{aligned}
&|K^{(d+m-1)}(\partial P^{(d+m)})| \\
&\quad = (d+m-1)(d+m-2)\ldots \\
&\qquad \times (d+1)d\,|K^{(d-1)}(\partial P^{(d)})| \\
&\qquad + 2m(d+m-1)\ldots(d+1)\,|K^{(d)}(P^{(d)})|, \qquad (29)
\end{aligned}$$

or, if $d \geqslant 1$:

$$|K^{(d+m-1)}(\partial P^{(d+m)})| = \frac{(d+m-1)!}{(d-1)!}\left\{|K^{(d-1)}(\partial P^{(d)})| + \frac{2m}{d}\,|K^{(d)}(P^{(d)})|\right\}. \qquad (30)$$

Notice that (29) and (30) are identities when $m = 0$.

Example 7 Take a point and apply a sequence of m translational extrusions. The resulting polyhedron is an m-dimensional hypercube. We have $d = 0$, $\partial P^{(0)} = \varnothing$ and $|K^{(0)}(P^{(0)})| = 1$, so that (29) becomes

$$|K^{(m-1)}(\partial P^{(m)})| = 2 \cdot m! \qquad (31)$$

(conversely $|K^{(m)}(P^{(m)})| = m!$ – see section 3). Formula (31) gives 48 for the 4D hypercube. In fact, it is well known that such an hypersolid is bounded by 8 cubes, and each of these is decomposed in 6 simplices in our triangulation.

If $\hat{P}^{(d+1)}$ linearly approximates with h steps the curve polyhedron $\tilde{P}^{(d+1)}$, obtained as rotational extrusion of $P^{(d)}$, the equation (28) is easily modified as follows:

$$|K^{(d)}(\partial\hat{P}^{(d+1)})| = h \cdot d \cdot |K^{(d-1)}(\partial P^{(d)})| + 2\,|K^{(d)}(P^{(d)})|. \qquad (32)$$

Example 8 For the 2-polyhedral star in figure 9 we have $d = 2$, $h = 30$, $|K^{(1)}(\partial P^{(2)})| = 8$, and $|K^{(2)}(P^{(2)})| = 6$.

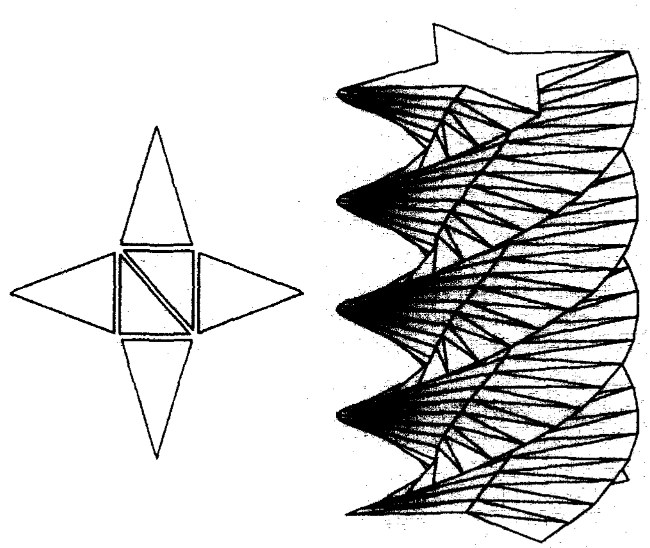

Figure 9. Star (simplices are scaled with respect to their centroid) and linear approximation $\hat{P}^{(3)}$ of $\tilde{P}^{(3)}$ with $\theta = 360°$ and h = 30

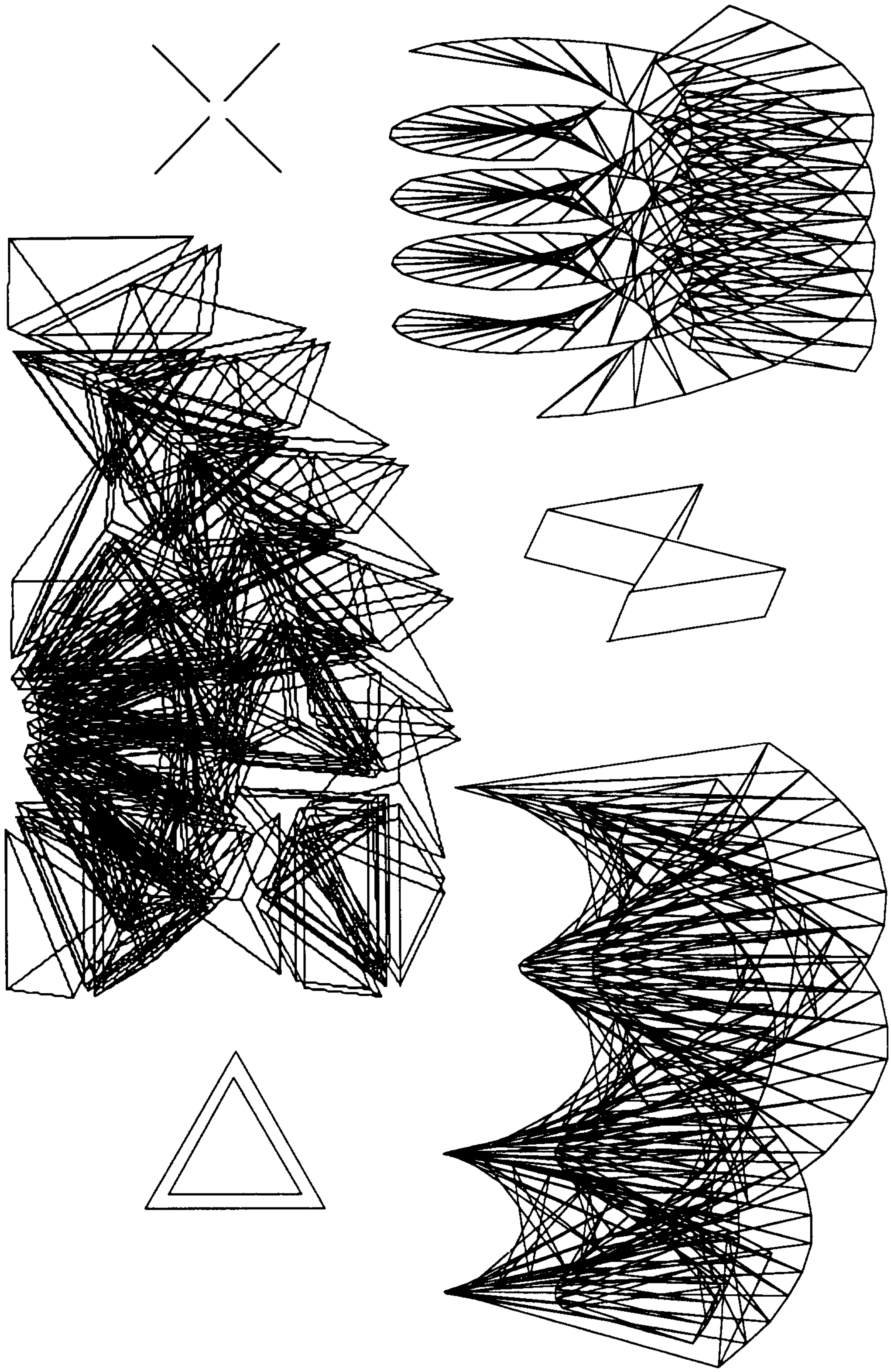

Figure 10. Polyhedral approximation of rotational extrusion of various order polyhedra. Intrinsic dimension of the generated solids is 2, 4, 3, respectively

Formula (32) gives $|K^{(2)}(\partial\hat{P}^{(3)})| = 30 \cdot 2 \cdot 8 + 2 \cdot 6 = 492$. In fact, $|K^{(0)}(P^{(2)})| = V = 8$, therefore $|K^{(0)}(\hat{P}^{(3)})| = (h + 1)V = 248$ (see section 4). But it is known that the number of triangles on the boundary of a simply connected 3-polyhedron $P^{(3)}$ is $2|K^{(0)}(P^{(3)})| - 4$. In our case $2|K^{(0)}(\hat{P}^{(3)})| - 4 = 492$, which is the result given by (32).

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