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Einstein and the Kaluza–Klein particle

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Abstract

In his search for a unified field theory that could undercut quantum mechanics, Einstein considered five-dimensional classical Kaluza–Klein theory. He studied this theory most intensively during the years 1938–1943. One of his primary objectives was finding a non-singular particle solution. In the full theory this search got frustrated, and in the x^5-independent theory Einstein, together with Pauli, argued it would be impossible to find these structures.

Keywords: Einstein; Unified field theory; Kaluza–Klein theory; Quantization; Solitons

1. Introduction

Albert Einstein's later work was mostly involved with the unification of electromagnetism and gravity in a classical field theory. However, after finding in 1915 the final version of the general theory of relativity, he did not immediately start to address the issue that would characterize much of his later work. At first, it was still an open question to him whether relativity and electrodynamics together would cast light on the problem of the structure of matter (Einstein, 1916a). In a 1916 paper on gravitational waves, he actually anticipated a different development: since the electron in its atomic orbit would radiate gravitationally, something that cannot "occur in reality", he expected quantum theory would have to change not only the "Maxwellian electrodynamics, but also the new theory of gravitation" (Einstein, 1916b; our translation). Einstein's position, however, gradually changed. From about 1919 onwards, he took a strong interest in the

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unification program.[1] In later years, after about 1926, he hoped that he would find a particular classical unified field theory that could undercut quantum theory. Such a theory would have to contain the material objects—both the sources and the fields—and their dynamics. He would even expect the distinction between these concepts to fade: "a complete field theory knows only fields and not the concepts of particle and motion" (Einstein & Rosen, 1935). We will study how he wanted to realize these principles in classical Kaluza–Klein theory, and try to see what his objectives and results were.

In 1920, when giving his inaugural lecture in Leyden, Einstein for the first time publicly commented positively on the unification program:

> It would be a great step forward to unify in a single picture the gravitational and electro-magnetic fields. Then there would be a worthy completion of the epoch of theoretical physics begun by Faraday and Maxwell… (Einstein, 1920b, as cited in Vizgin, 1994)

Wolfgang Pauli had at first been a strong proponent of classical unification, in particular of Weyl's conformal theory (see Vizgin, 1994). However, during 1920 his opinion shifted to a more critical position, and in a public discussion during the 86th "Naturvorscherversammlung" in Nauheim, he addressed Einstein:

> I would like to ask Einstein if he agrees that one can only expect a solution to the problem of matter by a modification of our ideas about space (and perhaps also time) and electric fields in the sense of atomism… or should one hold on to the rudiments of the continuum theories? (Pauli, 1920; our translation)

Einstein would then not decide either way and replied with a subtle epistemological remark, concerning the situation when a scientific concept or theory is in opposition with nature.[2] He would, however, keep working on continuum theories, letting future success decide which approach would be justified (see Einstein, 1921). Pauli formulated his criticism in a very clear way in his 1921 encyclopedia article on relativity, which also summed up the most important characteristics of the unified field theory program.

> It is the aim of all continuum theories to derive the atomic nature of electricity from the property that the differential equations expressing the physical laws have only a discrete number of solutions which are everywhere regular, static, and spherically symmetric… [however] the existence of atomicity should in theory be interpreted in a simple and elementary manner and should not… appear as a trick in analysis. (Pauli, 1958)

[1] For an account of the early years of the unified field theory program, see Vizgin (1994).

[2] He made a distinction between a procedure where a concept is dropped from a physical theory, and a procedure where the system of arrangement of concepts to events is replaced by a more complicated one that still refers to the same concepts (Einstein, 1920a).

One sees here the essential role particle solutions played in the unified field theory program. To resolve the problem of matter, Pauli felt one needed to abandon the approach characteristic of classical field theories: "New elements foreign to the continuum must be added to the basic structure of the theory before one can arrive at a satisfactory solution of the problem of matter" (Pauli, 1958). Pauli exhibits a clear sense for future developments concerning the quantized nature of matter. Yet Einstein would not give up on classical field theory: "Before we seriously start considering such far-fetched possibilities we have to find out whether it really follows from the recent efforts and facts that it is impossible to succeed with partial differential equations" (Einstein, 1923; our translation).

2. Kaluza's theory

In 1921 Einstein presented to the Prussian Academy a paper by Theodor Kaluza, entitled "Zum Unitätsproblem der Physik", in which the gravitational and electro-magnetic field are geometrically unified in five dimensions (Kaluza, 1921). Einstein had known of Kaluza's idea already in 1919, and had commented positively on it in a letter to Kaluza (for correspondence of Einstein to Kaluza, see De Sabbata & Schmutzer, 1983).

Kaluza started out by taking as action the five-dimensional Ricci scalar, and he imposed a "cylinder condition": the components of the metric g_{IJ} should not depend on the space-like fifth dimension. He then interpreted the 5×5 metric as follows:[3]

$$g_{IJ}^{(5)} = \begin{pmatrix} g_{\mu\nu}^{(4)} & \alpha A_\nu \\ \alpha A_\mu & 2V \end{pmatrix} \quad \text{with } \partial_5 g_{IJ} = 0, \tag{1}$$

where $g_{\mu\nu}^{(4)}$ represents the metric of the four-dimensional spacetime, A_μ is the gauge field from electrodynamics, and α is a coupling constant, related to the Newton constant κ via $\alpha = \sqrt{2\kappa}$. There is a new field, $V(x)$, often called the dilaton. Kaluza evaluated the Ricci tensor for linearized fields:

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu}, \quad R_{5\nu} = -\alpha\partial^\mu F_{\mu\nu}, \quad R_{55} = -\Box\, V, \tag{2}$$

and continued by studying the specific example of a charged particle in five-dimensional space. This taught him that the momentum in the fifth direction needs to be interpreted as the electric charge: $j^\nu = \kappa T^{\nu 5}$. The charge is a conserved quantity, by virtue of the translation symmetry on x^5. From the five-dimensional line element, Kaluza recovered for the equation of motion of a charged particle:

$$m\left(\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\beta\gamma}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\right) = e\, F^\mu_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} - \frac{1}{2\kappa}\frac{e^2}{m}\partial^\mu V, \tag{3}$$

[3] Capital Latin indices run over 0,1,2,3,5, Greek indices run over the usual dimensions 0,1,2,3. Later we will use *lower*-case Latin indices running over the space-like directions 1,2,3.

which gives the Lorentz force law under the assumption of small specific charge e/m. However, for the electron this assumption breaks down and, contrary to experience, the interaction with the scalar field becomes the leading term in the equation of motion—as was pointed out to Kaluza by Einstein (Kaluza, 1921). To Kaluza, this was an important shortcoming of the theory.

The theory Kaluza formulated was, up to the point where he introduced the particle, a vacuum theory. It did not contain an additional term in the action besides the five-dimensional Ricci scalar: the field equations are given by setting the Ricci tensor (2) equal to zero. But once the particle is introduced, and consequently its energy-momentum tensor, one has left the vacuum theory. So, when Kaluza in his example writes $R_{MN} = \kappa T_{MN}$, one wonders where the source term should come from, if it is not to be introduced as an additional matter-term in the Lagrangian. In fact, Einstein asked the same question in a paper with Jakob Grommer:

> ... Kaluza introduces besides the quantities g_{IJ} another tensor representing the material current. But it is clear that the introduction of such a tensor is only intended to give a provisional, sheer phenomenological description of matter, as we presently have in mind as ultimate goal a pure field theory, in which the field variables produce the field of 'empty space' as well as the electrical elementary particles that constitute 'matter'. (Einstein & Grommer, 1923; our translation)

We will see that Einstein would want the sources to come from the geometry. Einstein and Grommer found that Kaluza's vacuum field equations (i.e., $R^{(5)}_{IJ} = 0$) cannot produce non-singular rotation symmetric particle solutions. It seems plausible that because of this argument Einstein would, for about five years, no longer work on Kaluza's theory. A strong indication is the following passage from a letter to Weyl, written in 1922:

> Kaluza seems to me to have come closest to reality, even though he too, fails to provide the singularity free electron. To admit to singularities does not seem to me the right way. I think that in order to make progress we must once more find a general principle conforming more truly to nature.[4]

3. Klein's compactification

In 1926 Oskar Klein returned to Kaluza's theory and wrote two classic papers (Klein, 1926a, b). His more elaborate paper in the *Zeitschrift für Physik* is divided into a review of Kaluza's classical theory and a study of the connection between the

[4] A. Einstein, Letter to H. Weyl, 6 June 1922, AE 24-71. We have studied the correspondence of Einstein in the Einstein Duplicate Archive at the Firestone library of Princeton University. The number given pertains to the file number of the relevant document in that archive. As will be the case whenever we quote Einstein's correspondence, the translation is our own.

quantum-mechanical wave-equation and five-dimensional (null-)geodesics. In the first part, Klein puts forward a more fruitful interpretation of the five-dimensional metric

$$g_{IJ}^{(5)} = \begin{pmatrix} g_{\mu\nu}^{(4)} + VA_\mu A_\nu & VA_\nu \\ VA_\mu & V \end{pmatrix}. \tag{4}$$

He continues by assuming the dilaton V is a constant, as this does not violate four-dimensional general covariance or the electro-magnetic gauge symmetry. He then varies the action and recovers the full field equations of general relativity, with the energy-momentum tensor of the electro-magnetic fields, and the source-free Maxwell equations:[5]

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \quad \nabla_\mu F^{\mu\nu} = 0. \tag{5}$$

So Kaluza's vacuum theory with the cylinder condition contains the full field equations.

Klein would continue by taking as a given fact the quantized nature of the electric charge. This would lead him to the conclusion that the fifth direction needs to be compact. In his German article, Klein explicitly studies a wave field, which he writes as

$$U = \exp(\mathrm{i}\hbar^{-1}\Phi(x^I)) \tag{6}$$

and for which he, in effect, postulates the massless wave equation in five dimensions

$$\frac{1}{\sqrt{-g}}\partial_I(\sqrt{-g}g^{IJ}\partial_J)U = 0. \tag{7}$$

With the identification

$$p_I = \partial_I \Phi, \tag{8}$$

Klein deduces that, in leading order ($\sim \hbar^{-2}$), the wave equation can be interpreted as a null geodesic in five dimensions: $p_I p^I = 0$. This null geodesic is then reinterpreted as the geodesic for a charged particle in a four-dimensional space, for which one has to put $p_5 = q/c\sqrt{2\kappa}$ with q the electric charge. Klein takes the value of the fundamental charge e for q, arguably equivalent to writing the wave field U as an eigenstate with eigenvalue $p_5 = e/c\sqrt{2\kappa}$:

$$U = \exp(\mathrm{i}e(\hbar c\sqrt{2\kappa})^{-1}x^5)\psi(x^\mu). \tag{9}$$

After differentiating with respect to x^5, the wave-equation takes the familiar Klein–Gordon form: $D_\mu D^\mu \psi - m^2\psi = 0$ with $m = e/\sqrt{2\kappa c^2}$ and the gauge field, A_μ, incorporated into the covariant derivative in the usual way. One can reinterpret the particle's momentum in the fifth direction as a rest mass in four dimensions, since it moves along a five-dimensional null geodesic and thus does not have a rest mass in five dimensions. The mass is of the order of the Planck mass. Klein realizes that the discreteness of the charge spectrum, via the de Broglie relation, leads to a discrete

[5] The inconsistency in putting the dilaton to a constant *after* varying, as noted for instance by Thiry (1948), thus does not surface. The field equation for the dilaton would then force $F^2 = 0$. For more on the role of the dilaton, see O'Raifeartaigh and Straumann (2000).

wavelength in the fifth direction:

$$p_5 = \frac{\hbar}{\lambda_5} = \frac{ne}{c\sqrt{2\kappa}}. \tag{10}$$

This then shows that the fifth direction needs to be compact, and periodic with period $2\pi\lambda_5$. Putting in the values of the parameters reveals that the scale of the fifth direction is about the Planck size

$$\lambda_5 = \frac{hc\sqrt{2\kappa}}{e} = 0.8 \times 10^{-30} \text{ cm}. \tag{11}$$

> The small value of this length together with the periodicity in the fifth dimension may perhaps be taken as a support of the theory of Kaluza in the sense that they may explain the non-appearance of the fifth dimension in ordinary experiments as the result of averaging over the fifth dimension. (Klein, 1926b)

With (9), one sees that translations $x^5 \to x^5 + \Lambda(x)$ correspond to gauge transformations of the wave-function. The gauge parameters take values on the circle, so the gauge group of electrodynamics is now $U(1)$. The topology of space has changed as the fifth direction is no longer represented by an infinite straight line, that one can scale arbitrarily, but must be thought of as a circle, with a fixed scale.

Einstein, shortly after Klein, published a paper that essentially reproduced Klein's results in the classical theory (which he acknowledges in an appendix), but does not mention at all the possible compactification or the quantization of charge (Einstein, 1927). We see here implicitly that, in unified field theory, Einstein did not start out by taking as an hypothesis or axiom a fact from nature with which he would consequently deduce or construct—a method he had been successful with in the past. He did not want to start out by using typical quantum relations, but rather, he wanted the classical theory to produce the quantum results.

In a five-dimensional theory of relativity, in which one takes the fifth direction to be compact, one can expand the metric's components in a Fourier series

$$g_{IK} = \sum_n g_{IK}^{(n)}(x^\mu)\mathrm{e}^{inx^5/\lambda_5}. \tag{12}$$

This decomposition one generally finds in more recent literature (see, for instance, Duff, Nilsson, & Pope, 1986). In this form, one can identify the charged sources with the higher-order Fourier components of the metric. Klein certainly did not write the metric in the above form. However, his introduction of the wave field is to some extent equivalent: one can put $T_{IK}^{(n)} = (\partial_I U^{(n)})^* \partial_K U^{(n)}$ for waves representing charge *ne*, and identify these terms with the higher Fourier component contributions to the five-dimensional Ricci tensor derived from the metric in (12). Pushing this analogy a bit further, one can show that the discreteness of charge is reproduced by a priori quantizing the field theory, i.e., turning continuous functions into counting operators.

As an example of this, consider a two-dimensional space upon which a massless Klein–Gordon field ϕ lives, which we take to represent the (diagonal) components of the metric. The terms depending on x^5 are of higher order

$$\phi(x^0, x^5) = \mathrm{e}^{-\mathrm{i}mx^0} \sum_n A^{(n)} \mathrm{e}^{\mathrm{i}nx^5/\lambda_5}, \quad g^{AB} \sim \begin{pmatrix} -(A^{(0)})^{-1} & 0 \\ 0 & (A^{(0)})^{-1} \end{pmatrix}. \tag{13}$$

Charge is the component of momentum in the x^5 direction:

$$P^5 = \int \mathrm{d}x^5\, T^{05} = \int \mathrm{d}x^5\, g^{A0} g^{B5} (\partial_A \phi)^* \partial_B \phi \tag{14}$$

and, using (13) gives

$$P^5 = m(A^{(0)})^{-2} \sum_n n(A^{(n)})^* A^{(n)}. \tag{15}$$

The frequency m is interpreted as mass when going down one dimension (e.g., from five to four). But in the higher-dimensional space, the wave propagates on a null geodesic, and thus $c = m\lambda_5$. Using this relation, one can rewrite the momentum in the fifth direction:

$$P^5 = c/\lambda_5 \times (A^{(0)})^{-2} \sum_n n(A^{(n)})^* A^{(n)} = const./\lambda_5. \tag{16}$$

The coefficients $A^{(n)}$ are just complex numbers that can take a continuous range of values, so classically the compactification does not give a discrete spectrum for the charge and there is no minimum value for the constant. After quantization, the discrete charge spectrum does imply a compact fifth direction and vice versa. In other words, imposing the de Broglie relation with integer charges is consistent with turning the constants $(A^{(n)})^*$ into creation operators.

The tower of charged spin-2 particles one retrieves from the 5×5 metric (12) upon quantization have very high masses $m_n = ne/c^2\sqrt{2\kappa}$ (their gravitational interaction would be of the same order of magnitude as the electro-magnetic interaction). As effective theory for low-energy physics, one can retain just the $n = 0$ mode in the Fourier expansion. This gives for the action:

$$R^{(0)(4)} + \frac{1}{4} V^{(0)} F^{(0)}_{\mu\nu} F^{(0)\mu\nu} - \frac{2}{V^{(0)1/2}} \Box\, V^{(0)1/2}, \tag{17}$$

which is just the action of the theory with the cylinder condition in place (referred to as *Kaluza's theory*; the theory with higher Fourier components—in other words, with periodic x^5 dependence— will be called *Kaluza–Klein theory*). Note that the Kaluza theory has forgotten about the original periodicity in x^5 (see Duff, Nilsson, & Pope, 1986).

4. Einstein on classical Kaluza–Klein

We now make a leap in time and turn to the study Einstein made of classical Kaluza–Klein theory in the years 1938–1943.

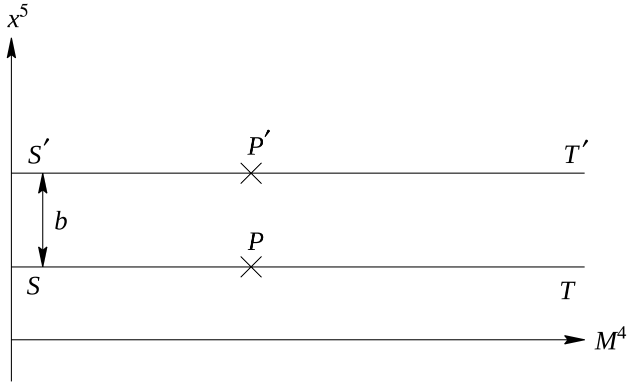

Fig. 1. Einstein and Bergmann gave the above two-dimensional image of the compactified space they studied: "We imagine the strip curved into a tube so that ST coincides with $S'T'$." (Einstein & Bergmann, 1938).

4.1. Einstein, Bergmann, and Bargmann

The first paper of this period starts with a re-evaluation of Kaluza's theory, where the dilaton has been set to a constant (thus $L = \sqrt{-g}(R + cF^2)$). Einstein, together with Peter Bergmann, makes the following remarks:

> Many fruitless efforts to find a field representation of matter free from singularities based on this theory have convinced us, however, that such a solution does not exist. Footnote: We tried to find a rigorous solution of the gravitational equations, free from singularities, by taking into account the electromagnetic field. We thought that a solution of a rotation symmetric character could, perhaps, represent an elementary particle. Our investigation was based on the theory of 'bridges'… We convinced ourselves, however, that no solution of this character exists. (Einstein & Bergmann, 1938)

This makes it very plausible that, because Einstein had not been able to find a non-singular object, he sought to expand the contents of the theory he studied.[6]

Einstein and Bergmann now proposed a theory in which the cylinder condition is relaxed: (periodic) dependence on x^5 would again be allowed. They do note, however, that spacetime appears to us as four-dimensional, and they explain this as follows (with a two-dimensional analogy, see also Fig. 1):

> If the width of the strip, that is the circumference of the cylinder (denoted by b), is small, and if a continuous and slowly varying function $\phi(x^0, x^1)$ is given, that is if

[6] Possibly, Einstein and Bergmann did not succeed in formulating the modification of the field equations as proposed by Einstein and Rosen (i.e., $g^2(R_{\mu\nu} + T_{\mu\nu}) = 0$) in the context of Kaluza's theory. This modified form of the field equations was essential in the interpretation of the Einstein–Rosen bridge as a non-singular object (Einstein & Rosen, 1935). See also Earman and Eisenstaedt (1999), in particular for discussion of Einstein's handling of the Schwarzschild singularity.

> $b\partial\phi/\partial x^a$ is small compared with ϕ, then the values of ϕ belonging to the points on the segment PP' [i.e., the line connecting the periodically identified points P and P'] differ from each other very slightly and ϕ can be regarded, approximately, as a function of x^1 only (Einstein & Bergmann, 1938).

Not only do they take the scale of the extra compact direction to be small, they also make the tacit assumption that the $n \neq 0$ Fourier coefficients ($A^{(n)}$) are small relative to the $n = 0$ coefficient.[7] However, they cannot, in their classical theory, explicitly calculate the value of the scale parameter $b = \lambda_5$ from the electron charge. They do not a priori quantize and, therefore, cannot use the value of $\hbar$, like Klein did when he used the de Broglie relation.

By relaxing the cylinder condition, it is again possible to have classical fields in the theory that carry a momentum in the fifth direction. These then represent charged matter sources, which were absent in Kaluza's theory with constant dilaton. Einstein and Bergmann find the following field equations (with $g_{55} = const.$):

$$\begin{aligned}
&\alpha_1(R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R) + \alpha_2(2F_{\mu\alpha}F^{\alpha}_{\nu} - \tfrac{1}{2}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}) \\
&\quad + \alpha_3(-2g_{\mu\nu,55} + 2g^{\alpha\beta}g_{\mu\alpha,5}g_{\nu\beta,5} - g^{\alpha\beta}g_{\alpha\beta,5}g_{\mu\nu,5} - \tfrac{1}{2}g^{\alpha\beta}_{,5}g_{\alpha\beta,5}g_{\mu\nu}) \\
&\quad + \alpha_4 g_{\mu\nu}(\tfrac{1}{2}(g^{\alpha\beta}g_{\alpha\beta,5})^2 + 2g^{\alpha\beta}g_{\alpha\beta,55} + 2g^{\alpha\beta}_{,5}g_{\alpha\beta,5}) = 0,
\end{aligned} \tag{18}$$

$$\int \mathrm{d}x^5\, \sqrt{-g}\{\alpha_1(g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta,5} - g^{\alpha\mu}\Gamma^{\beta}_{\alpha\beta,5}) - 4\alpha_2 F^{\mu\alpha}_{;\alpha}\} = 0. \tag{19}$$

One sees clearly in the last expression (19), i.e., Maxwell's equations, that the x^5-dependent metric fields play the role of charged matter sources. The α_i are coupling constants—that in this paper are still arbitrary, because in this version of the theory five-dimensional general covariance is not imposed. In a second paper on this theory, they do impose general covariance which fixes the constants. Einstein and Bergmann, who are by then joined by Valentin Bargmann, are at first uncomfortable about the arbitrariness in the values of these constants.[8] When this arbitrariness is lifted by general covariance, one faces a new problem because the electro-magnetic and gravitational interaction are then of the same order of magnitude (Einstein, Bargmann, & Bergmann, 1941).

In any event, we must now ask: Why did Einstein set out to study this generalization of Kaluza's five-dimensional theory? What made him study Kaluza–Klein as a classical field theory? Was it simply to formulate a unified theory, given the evident appeal that unifying the forces has? And now that Einstein had formulated field equations, which contain geometrically construed source terms, what was his next goal?

In our search for an answer to these questions, we have looked at the correspondence Einstein had with Bergmann, Bargmann, and also Wolfgang Pauli. Firstly, one learns that Einstein was quite happy with the results obtained so far: "It is a pity that our paper starts with Kaluza… But this is just a minor detail; I have a

[7] Einstein and Bergmann did not explicitly write the Fourier expansion of the metric.
[8] See A. Einstein, Letter to V. Bargmann, 9 July 1939, AE 6-207.

lot of confidence in the theory as such".[9] Bergmann then writes to Einstein: "Regardless of the difficulties that have surfaced I too very much confide in the theory because of its unity and the existence of "wave type" fields".[10] It is natural to wonder how Einstein would hope to undercut quantum theory with classical Kaluza–Klein theory. One expects this to be the primary objective for Einstein to study this theory. However, in none of the papers, or in his correspondence, is there any explicit mention of how this should come about. At certain points in the discussion with Bergmann, there is nevertheless mention of a "de Broglie frequency". But it seems that the few relations that are introduced which involve this frequency are merely used as a consistency check; for example, to see if the equations allow solutions with negative masses. We will, however, return to this point later to see in what ways Einstein may have wanted to derive typical quantum results. The wave field Bergmann refers to above probably corresponds to a field that has a wave propagating in the fifth direction, giving an electrically charged source term in the classical field equations.[11]

In September 1938, Pauli wrote Einstein:

> Bargmann[12] was recently here and reported on your work on the closed 5-dimensional continuum. But that is an old idea of O. Klein, including the circumstance that the ψ-function gets a factor of e^{ix_5} and he has always emphasized that changing the origin of x_5 corresponds to so-called gauge group transformations. Besides our old, more fundamental differences of opinion, your Ansatz appears to me too special, because it is hardly necessary for the ψ-function to be a symmetric tensor.[13]

Pauli continues by saying that, in his opinion, Bargmann is more talented in mathematics than in physics. Then: "In this context I would like to end with a little malicious remark: considering your current involvement with theoretical physics, this should hardly make a bad impression on you". Einstein replied:

> Your remark is well founded, and could easily be answered likewise.[14] But the new work is only superficially similar to Klein's. It is simply a logical improvement of Kaluza's idea, that deserves to be taken serious and examined accurately.[15]

[9] A. Einstein, Letter to P. Bergmann, 5 August 1938, AE 6-271.

[10] P. Bergmann, Letter to A. Einstein, 15 August 1938, AE 6-272.

[11] "We had put… $\sim\gamma_{\mu\nu} = \psi_{\mu\nu}\, e^{2i\pi x^5/\lambda} + \bar{\psi}_{\mu\nu}\, e^{-2i\pi x^5/\lambda} =$ real quantities, bar means complex conjugate… for our stationary problem… $\psi_{\mu\nu} = \chi_{\mu\nu} e^{i\omega t}$." (P. Bergmann, Letter to A. Einstein, 4 August 1938, AE 6-270.)

[12] Valentin Bargmann had joined Bergmann and Einstein in their work on Kaluza–Klein theory around this time.

[13] W. Pauli, Letter to A. Einstein, September 6, 1938, AE 19-174 (see also Pauli, 1985).

[14] Possibly Einstein is referring here to the work Pauli had done on the five-dimensional theory in 1933 (Pauli, 1933a, b).

[15] A. Einstein, Letter to W. Pauli, 19 September 1938, AE 19-175 (see also Pauli, 1985).

4.2. *"Ich hoffe dass dies nun glatt gehen wird, ohne dass uns die ewige Göttin erneut die Zunge herausstrecken wird"*[16]

From his correspondence, one learns how Einstein wanted to test the generalization of Kaluza's theory: he wanted to find a non-singular charged particle. Just after writing the paper with Bergmann, published in the *Annals of Mathematics* in July of 1938, Einstein started his attempts to realize this non-singular object. The discussion lasted for at least a year. One equation under consideration was the following:

> … in our case the lowest power… is
>
> $$+\frac{2\pi}{\lambda}\omega[\alpha(3\alpha+2\beta)] = \alpha_2\left(\phi'' + \frac{2}{\rho}\phi'\right) \tag{20}$$
>
> i.e., the sign of the charge (term on the right-hand side) and of ω/λ also depends on the sign of the square bracket which is different for both particles.[17]

Here, λ is the scale of the fifth direction, ω the frequency of a wave traveling in the fifth direction, α_2 is a coupling constant, and ρ is a radial coordinate ($\rho^2 = r^2 + a^2$ with a a constant and r the usual radial coordinate). ϕ is the electrostatic potential which is differentiated with respect to ρ. This is the 0-component of the Maxwell equations (19), i.e., the Poisson equation, written in the leading terms at large distances. The solution of the full equation would correspond to an object represented by a wave propagating in the fifth direction and curving the four-dimensional spacetime. The constants α and β stem from the four-dimensional curved metric. One then has a body sitting somewhere in space, with a momentum in the fifth direction that yields the electric charge.

What type of exact solution they hope to retrieve can be discerned from the following note from Einstein to Bergmann:

> The treatment of our centrally symmetric problem is still mysterious to me. We in any case need a factor that is essentially singular at infinity… Maybe we will have to introduce ρ again. But I still do not have a comprehensive view of the situation, particularly as ϕ has to go as ε/r. One would need to know a lot about differential equations to find this solution. Perhaps Bargmann will find a way. It is certainly not easy. Anyhow, one first needs to find a representation at large r and then strive for a convergent representation at the origin.[18]

One sees that Einstein wants a particular solution that drops off as ε/r at large distances; as one is far away from the object, one should see an electric monopole charge sitting somewhere in space. Indeed, it should be emphasized that Einstein wants a non-singular, convergent series for the potential at the location of the

[16] A. Einstein, Letter to P. Bergmann, 4 July 1938, AE 6-253.
[17] P. Bergmann, Letter to A. Einstein, 4 August 1938, AE 6-270.
[18] A. Einstein, Letter to P. Bergmann, 5 August 1938, AE 6-271.

source. (At some point, they expected that substituting ρ instead of r would give them a convergent series.)

In the case of the charged wave solution, one could allow the relation $E = h\nu$, and with the de Broglie relation, the five-dimensional photon carries a charge given by $p_5 = h/\lambda$. It is possible this consideration would be related to the following remark by Einstein:

> We have demanded that the frequency ν is equal to a de Broglie frequency. So far, this had only been supported by physical plausibility, not founded on the mathematical problem. However, it has now become clear to me this choice will have to be made.[19]

The quantized nature of the electric charge can be in accordance with the compactification of the fifth direction. But then one has the exact line of reasoning Klein followed. However, that is a priori only conclusive once one makes the transition from classical fields to quantum fields, i.e., from numbers to operators. Without quantizing, the P_5 of the wave-field would have a continuous range of values: $const./\lambda = m\sum_n n(A^{(n)})^* A^{(n)}(A^{(0)})^{-2}$ (cf. Eq. (16)). So, without quantization, the compactification loses its explanatory power in relation to the discreteness of the charge spectrum. As Einstein nowhere in his correspondence, nor in his articles, assumes a discrete charge spectrum, it becomes less probable that one should interpret the above remark as giving in to Klein's reasoning.

It is more likely that he hoped that the generalized Poisson equation (20), with the boundary conditions he imposed and combined with the generalized Einstein equations (18), would produce a discrete spectrum of charged solutions; i.e., that the charges P_5 take on a discrete range of values because the field equations and boundary conditions would restrict the integral to a discrete set of values. Note that because of the compactification, there would then be a fundamental constant in the theory with the dimension of angular momentum: $P_5\lambda = h$. One could set this to the value of Planck's constant and arrive at a value for λ. Then there would not be some ad hoc quantization implemented. Instead, the discretely charged solutions of the classical field equations, together with the compactification radius, would yield Planck's constant. So there would not be an arbitrary quantization invoked, but the geometric scheme—compactification—and the regularity conditions would now produce the existence of a minimal quantum of action h. Yet, we emphasize that there is no explicit indication in either his correspondence or his articles that the quantization of charge is an actual result he was looking for, and that the above argument would have been his method for rederiving h.

For at least a year, Einstein, Bergmann, and Bargmann tried to find a non-singular charged object in the classical Kaluza–Klein theory (with constant dilaton). Finally,

[19] A. Einstein, Letter to P. Bergmann, 21 June 1938, AE 6-249.

one can read in their joint paper[20] that this search had become frustrated:

> It seems impossible to describe particles by non-singular solutions of the field equations. As no arbitrary constants occur in the equations, the theory would lead to electro-magnetic and gravitational fields of the same order of magnitude. Therefore, one would be unable to explain the empirical fact that the electrostatic force between two particles is so much stronger than the gravitational force. This means that a consistent theory of matter could not be based on these equations. (Einstein et al., 1941)

5. Solitons in Kaluza's theory

We have seen the emphasis Einstein put on non-singular particle descriptions (see also Earman & Eisenstaedt, 1999), in particular in classical Kaluza–Klein theory. In 1941 he would also publish a short paper on the non-existence of non-singular massive objects in four-dimensional relativity theory (Einstein, 1941). The argument would later be extended to Kaluza's theory in a joint publication of Einstein and Pauli (1943). They argued that solitons cannot exist in this theory, a result that Einstein may have been disappointed with. Nevertheless, in the 1980s solitons were described in the Kaluza theory. We will briefly discuss these, and compare them with the Einstein–Pauli theorem.

5.1. Non-singular particles and Noether currents in relativity

We briefly discuss Einstein's 1941 argument for the non-existence of non-singular particle solutions of the vacuum field equations of four-dimensional relativity. This theory contains the conserved current $\aleph^{\mu}$:

$$\aleph^{\mu}_{,\mu} = (\sqrt{-g}[g^{\alpha\mu}\delta\Gamma^{\beta}_{\alpha\beta} - g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta}])_{,\mu} = 0. \tag{21}$$

Einstein derives the conservation law (21) by considering the variation

$$\delta R_{\mu\nu} = -(\delta\Gamma^{\alpha}_{\mu\nu})_{;\alpha} + (\delta\Gamma^{\alpha}_{\mu\alpha})_{;\nu} \tag{22}$$

which he rewrites as

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \aleph^{\alpha}_{,\alpha}, \tag{23}$$

with $g_{\mu\nu} + \delta g_{\mu\nu}$ the variation of a line-element of an arbitrary, everywhere regular field. He then assumes that both the varied and original fields satisfy the vacuum field equations throughout the spacetime ($R_{\mu\nu} = 0$), from which it follows that the variation (23) should vanish. Finally, the integral over spacetime of (21) should vanish for non-singular spaces.

[20] This came out in 1941 but was probably written sooner, because the slightly different approach laid out in this work—viz., basing the theory on five-dimensional general covariance—is already mentioned in a letter from Einstein in 1939 (Einstein to Bargmann, 9 July 1939, AE 6-207).

Einstein's argument is meant to hold under the following quite general conditions:

Assumption A.

> Let us consider now a solution without singularities of the gravitational equations plunged in an Euclidean (or Minkowski) space, which we assume either to be independent of x^4 or periodic or quasi periodic with respect to x^4. Such a solution would be a theoretical representation of a body, or, respectively a system of bodies, which in the average is at rest with respect to the coordinate system. At great distances from the origin of coordinates, the field of such a system may always be replaced by that of a resting point mass… (Einstein, 1941)

Asymptotically, one has $g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}$, for which the linearized field equations give

$$\gamma_{ik} = -\frac{2m}{r}\delta_{ik}, \quad \gamma_{44} = -\frac{2m}{r}. \tag{24}$$

The leading term in the variations will be

$$\aleph^i = -2(\delta m)\eta^{ik}\left(\frac{1}{r}\right)_{,k}, \quad \aleph^4 = 0. \tag{25}$$

One finds with Einstein's assumption for the time-dependence and the assumption of regularity (T is time)

$$\int \mathrm{d}^4x\, \aleph^\mu_{,\mu} = \int \mathrm{d}t \int \mathrm{d}^2\Omega_i\, \aleph^i = 8\pi T\delta m = 0. \tag{26}$$

"Result: two infinitely close solutions without singularities necessarily have the same mass" (Einstein, 1941). On the other hand, if (24) solves the equations of motion, then, on the basis of general covariance, there should also be a solution for r replaced by $(1+c)^{-1}r$, i.e., a non-singular solution with mass $(1+c)m$, where one can take c infinitely close to 0. So there should be non-singular solutions with infinitely close masses. Apparently, we have here a contradiction in terms: "This contradiction can only arise from the inexactitude of the hypothesis that there exists a solution free of singularities belonging to a total mass different from zero" (Einstein, 1941).[21]

[21] André Lichnerowicz generalized Einstein's result in 1946 (Lichnerowicz, 1946, 1955). Einstein's result asserts that there can be no masses (identified as the constant that multiplies the $1/r$ term in the metric) in non-singular spaces, while Lichnerowicz showed that higher-order multipole-terms cannot be present either. He could show that, relying on Einstein's argument, a non-singular space with *Minkowskian* signature necessarily has a trivial time-component, and, consequently, the curvature should be realized in the three space-like dimensions. But in three dimensions, if $R_{ij} = 0$ then $R^k_{ijl} = 0$, and the three-manifold is flat. Actually, Lichnerowicz had not seen Einstein's publication. At the time it was not available in France, as he would write Pauli later (Lichnerowicz to W. Pauli, 6 October 1945, see Pauli, 1985). Nevertheless, he was able to deduce its argument from its five-dimensional analogue in the paper Einstein and Pauli published in 1943. Lichnerowicz had formulated a similar singularity theorem in his 1939 thesis and publication (Lichnerowicz, 1938) and had sent this to Einstein the same year. It had not arrived however (see Tipler et al., 1980). Finally, when in 1945 Lichnerowicz corresponded with Pauli, he informed both Einstein and Pauli of his generalization.

5.2. Einstein and Pauli on non-singular solutions in Kaluza's theory

Einstein's argument would be repeated, by himself and Pauli, in an article on Kaluza's theory. The theory in which they reformulate the argument is the vacuum Kaluza–Klein theory where the higher modes have been dropped—a theory equivalent to a five-dimensional theory of relativity with a Killing vector $\partial/\partial x^5$ and in which a priori the extra dimension need not be compact. They start off as follows:

> …whether in a five-dimensional metric continuum (of signature 1) the equations $R_{IK} = 0$ admit of non-singular stationary solutions with a field g_{IK} asymptotically given by
>
> $$\begin{matrix} B & 0 & 0 & 0 & D \\ 0 & A & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & A & 0 \\ D & 0 & 0 & 0 & C \end{matrix} \tag{27}$$
>
> where at least one of the quantities A, B, C, D has the form $\pm 1 + const./r$ with a non-vanishing constant. This is the asymptotic form of a field representing a particle whose electric and ponderable masses do not both vanish.[22] (Einstein & Pauli, 1943)

Einstein and Pauli do not demand spherical symmetry. Note that for the first time since Klein, the corner component g_{55} is again retained. The possible non-constancy of this component, and its interpretation as a 'variable' gravitational constant, would be emphasized by Jordan (and would have been published by him in 1945, but then the *Physikalische Zeitschrift* had ceased publication). We learn from an article by Bergmann that Pauli at that time knew about this work, and he further informs us that he himself and Einstein had considered the possibility as well, "several years earlier" (Bergmann, 1948). Thus, it is perhaps not unexpected that this component would again be considered a variable by Einstein and Pauli in 1943.[23]

Einstein and Pauli re-establish relation (23), but now in five dimensions:

$$\aleph^S_{,S} = 0 \tag{28}$$

with $\aleph^S$ as in (21) for the metric of the five-dimensional space. One imposes time-independence and the cylinder condition:

$$\frac{\partial g_{IK}}{\partial x^\varepsilon} = 0, \quad \varepsilon = 0, 5. \tag{29}$$

[22] Note that introducing a Coulomb field in g_{05} would lead to a term proportional to $1/r^2$ in A and B as in the Reissner–Nordström metric. However, this term can be disregarded at large distances, as it is of higher order.

[23] Also Thiry in 1948 would re-introduce the g_{55} component as a variable (Thiry, 1948). He mentions that Lichnerowicz's theorems should apply to the resulting theory; see also (Lichnerowicz, 1955). For Jordan's work, see Jordan (1947).

With Gauss' theorem and (29), one finds for non-singular spaces

$$\oint \aleph^i n_i \, \mathrm{d}^2\Omega = 0. \tag{30}$$

For singular spaces, one has

$$\oint_{F_1} \aleph^i n_i \, \mathrm{d}^2\Omega = \oint_{F_2} \aleph^i n_i \, \mathrm{d}^2\Omega, \tag{31}$$

where F_1 is the inner surface surrounding the singularity and F_2 is an outer boundary. The variations are performed via coordinate transformations, ($x^I \to x^I + \xi^I$), that leave (29) invariant. There are two types of these variations. The first is independent of the coordinates x^ε with $\varepsilon = 0, 5$. These, and their derivatives, can be set to zero on the inner surface F_1, and on the basis of (30) one would not be able to decide whether or not the space is singular. The other type is given by

$$\xi^i = 0, \quad \xi^\varepsilon = c^\varepsilon_\rho \xi^\rho, \quad i = 1, 2, 3, \quad \varepsilon, \rho = 0, 5 \tag{32}$$

with constant coefficients c^ε_ρ. With these transformations, (30) would single out non-singular solutions. By calculating the variations $\delta\Gamma^I_{AB}$, the condition for non-singular spaces reduces to

$$c^\varepsilon_\rho \oint \sqrt{|g^5|} g^{\rho A} \Gamma^i_{\varepsilon A} n_i \, \mathrm{d}^2\Omega = 0. \tag{33}$$

One can drop the c^ε_ρ and the implied summation because the constants can be chosen arbitrarily anyway. Again, solving the linearized field equations for the asymptotic fields (27) yields

$$g_{IK} = \begin{pmatrix} -1 + \frac{m_{00}}{r} & 0 & 0 & 0 & \frac{m_{05}}{r} \\ 0 & 1 + \frac{m}{r} & 0 & 0 & 0 \\ 0 & 0 & 1 + \frac{m}{r} & 0 & 0 \\ 0 & 0 & 0 & 1 + \frac{m}{r} & 0 \\ \frac{m_{05}}{r} & 0 & 0 & 0 & 1 + \frac{m_{55}}{r} \end{pmatrix} \quad \text{as } r \to \infty \tag{34}$$

with

$$m + m_{55} = m_{00}. \tag{35}$$

Putting this into condition (30) leads to

$$\oint \sqrt{|g|} g^{\rho A} \Gamma^i_{\varepsilon A} n_i \, \mathrm{d}f = \pm 2\pi m_{\rho\varepsilon}. \tag{36}$$

For non-singular spaces, one finds $m_{\rho\varepsilon} = 0$, and Einstein and Pauli conclude that the only non-singular solution of the vacuum field equations is five-dimensional Minkowski space.

5.3. Two examples of solitons in Kaluza's theory

Despite Einstein and Pauli's result, solitons have been described in the vacuum Kaluza theory in the 1980s (Gross & Perry, 1983; Sorkin, 1983). Generally, these are instantons from Euclidean gravity, incorporated into Kaluza's theory in the metric's components in the space-like directions, and for the remainder one takes $g_{0A} = -\delta_{0A}$. We describe two of these objects in more detail here and discuss why the Einstein–Pauli proof does not apply to them.

We will disregard the x^5 dependent higher Fourier modes of the Kaluza–Klein theory, so we are working in the $n = 0$ Kaluza theory as did Einstein and Pauli. If one has a compact dimension, this is a principal bundle of $U(1)$ over the four-dimensional spacetime. In the $n = 0$ theory, as argued before, there may be no a priori reason for the theory to have a compact extra dimension. We will first study the magnetic monopole in this theory, that by virtue of a regularity condition forces the fifth dimension to be compact. This is an example of a non-trivial bundle: basically, it is the Dirac monopole (Dirac, 1931) in five dimensions. In fact, it is a generalization of the Euclidean Taub-NUT space (see Misner, 1963; Hawking, 1977) to five dimensions, and thus the Kaluza–Klein theory shows the equivalence of these two objects. For both objects, the topological S^3-structure and its consequential description with the Hopf-map had already been uncovered (for the Dirac monopole, this was done by Wu and Yang (1975), and in case of the Taub-NUT space by Misner (1963); for an overview, see Bais (1983)).

A natural restriction to put on the $n = 0$ theory when looking for solitons representing particle solutions is to require that these solutions be static. (Einstein and Pauli also imposed this restriction in their theorem on particle solutions, and the same restriction was imposed in the 1980s when Kaluza–Klein theory was again intensively studied.) This reduces the field equations to the equations of Euclidean gravity on surfaces of constant time, and the fifth dimension now plays the same role as the Euclidean time in the four-dimensional theory (Gross & Perry, 1983). With the latter theory, a number of gravitational instantons had been described by Hawking (1977) and Gibbons and Hawking (1979). Consequently, these were incorporated as solitons in the five-dimensional Kaluza theory by adding, in a topologically trivial way, the time-coordinate to the instanton metric. This was carried out for the monopole by Gross and Perry (1983) and Sorkin (1983), and for a number of other solitonic objects also by Gross and Perry.

5.3.1. Gravitational nut and Kaluza–Klein monopole

The gravitational instanton is defined by Stephen Hawking as follows (analogous to its definition in Yang-Mills theory): "a gravitational instanton… [is]… a solution of the classical field equations which is non-singular on some section of complexified spacetime and in which the curvature tensor dies away at large distances" (Hawking, 1977). Our first example is the Euclidean Taub-NUT gravitational instanton, a solution of the vacuum Euclidean Einstein equations. Its metric is

$$ds^2 = V(d\tau + 4m(1 - \cos\theta)\,d\phi)^2 + \frac{1}{V}(dr^2 + r^2(d\theta^2 + \sin\theta^2\,d\phi^2)), \tag{37}$$

$$\frac{1}{V} = 1 + \frac{4m}{r}. \tag{38}$$

Reminiscent of the Dirac-string singularity, this metric is singular when $\theta = \pi$;

$$g_{\tau\phi} = VA_\phi, \quad A_\phi = 4m\left(\frac{1 - \cos\theta}{r\sin\theta}\right). \tag{39}$$

One can remove this singularity by introducing two coordinate patches that are related to one another via the coordinate transformation (Misner, 1963)

$$\tau_N = \tau_S - 8m\phi. \tag{40}$$

$A_\phi^N = g_{\phi\tau_N}/V$ transforms in the following way:

$$A_\phi^S = A_\phi^N - \frac{1}{r\sin\theta}\frac{\partial(8m\phi)}{\partial\phi}, \quad \text{so } A_\phi^S = 4m\left(\frac{-1 - \cos\theta}{r\sin\theta}\right), \tag{41}$$

which is singular at $\theta = 0$. The singularities in the A_ϕ's are then obviously coordinate singularities, and one can restrict oneself to the (τ_N, A_ϕ^N)-description on the northern hemisphere (which excludes $\theta = \pi$) and to the (τ_S, A_ϕ^S)-description on the southern hemisphere (which excludes $\theta = 0$). With the two patches combined, one has a complete non-singular description of the (Euclidean) Taub-NUT spacetime. Where the two patches overlap, they are related by (40). Because of this condition, τ must now be considered a periodic coordinate with period $16\pi m$, for ϕ is periodic with 2π. In the Kaluza theory, this same regularity condition forces the fifth dimension to be compact.

The Taub-NUT space has topology $S^3 \times R$, where R represents the non-compact radial dimension. Its curvature goes to zero at large distances (Misner, 1963), but as its topology at infinity is not the trivial bundle (i.e., not $S^1 \times S^2$), we cannot refer to the object as asymptotically Minkowskian. To show that the Taub-NUT instanton is regular at $r = 0$, one can rewrite the metric with $r = (1/16m)\rho^2$ as follows:

$$\begin{aligned} ds^2 = &\frac{\rho^2}{\rho^2 + 64m^2}(d\tau + 4m(1 - \cos\theta)\,d\phi)^2 \\ &+ \frac{\rho^2 + 64m^2}{64m^2}\left(d\rho^2 + \frac{\rho^2}{4}(d\theta^2 + \sin\theta^2\,d\phi^2)\right). \end{aligned} \tag{42}$$

In this form, it is clear there is no singularity as $r \to 0$: the degeneracy at $r = 0$ is analogous to the degeneracy in polar coordinates. It is superfluous if one periodically identifies τ with period $16\pi m$. The fixed point of $\partial/\partial\tau$ defines what is called by Gibbons and Hawking (1979) a "nut". It is clear the object satisfies Hawking's definition of an instanton. One can include the object as a soliton in Kaluza's theory, in the manner argued before, by letting x^5 play the role of the Euclidean time τ and introducing a trivial time dependence. This then yields

$$\begin{aligned} ds^2 = & -dt^2 + V(dx^5 + 4m(1-\cos\theta)\,d\phi)^2 \\ & + \frac{1}{V}(dr^2 + r^2\,(d\theta^2 + \sin\theta^2\,d\phi^2)), \end{aligned} \tag{43}$$

$$\frac{1}{V} = 1 + \frac{4m}{r}. \tag{44}$$

The result is a magnetic monopole: A_ϕ has regained its interpretation as a component of the electro-magnetic vector-potential as in (4), and the magnetic field is given by

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{4m\vec{r}}{r^3}. \tag{45}$$

One hereby sees that the magnetic monopole charge m is quantized ($m = (1/8)kR$, with integer valued k), as the transition function in the fiber takes the value $e^{i(8m/R)\phi}$, which has to be single-valued on the equator.[24] Most importantly for our story, x^5 transforms as

$$x^5 \to x^5 + 8m\phi. \tag{46}$$

So the condition of regularity on the monopole spacetime forces the fifth dimension to be compact, with period $16\pi m$: it constrains the fiber of the $n = 0$ x^5-independent Kaluza theory to have the structure of $U(1)$, whereby the object reveals itself as a non-trivial bundle.

5.3.2. Gravitational bolt as soliton

The topology (i.e., Hopf-fibering) of Taub-NUT spacetime does not have a trivial asymptotically Minkowskian structure. This is at least one point where the monopole escapes Einstein's theorems. In both his papers on particle solutions in Kaluza's theory (i.e., the paper with Pauli (Einstein & Pauli, 1943), as well as the earlier one with Grommer (Einstein & Grommer, 1923)), Einstein focussed on objects that are not just Ricci flat at infinity, but asymptotically have the structure of Minkowski space. In what follows, we shall give one more example of a $n = 0$ Kaluza soliton, one that *does* have the right asymptotic behavior. In fact, it fulfills all but one of the demands that Einstein and Pauli put on particle solutions.

Again, this Kaluza soliton can be thought of as a trivial extension of a particular gravitational instanton. Now we are dealing with the ''bolt''-type instanton, which is just the Schwarzschild solution continued to the complex plane (Hawking, 1977, 1978; Gibbons and Hawking, 1979):

$$ds^2 = (1 - 4m/r)\,d\tau^2 + (1 - 4m/r)^{-1}\,dr^2 + r^2\,d\Omega^2. \tag{47}$$

---

[24] We have absorbed Kaluza's $\alpha = \sqrt{2\kappa}$ into the definition of $\vec{A}$; redefining it yields a reparametrisation: $B \to \alpha B$. Allowing the de Broglie relation Klein uses, together with (11), would give the Dirac quantization condition $m = k(hc)/e$, and thus explain the quantization of electric charge. But classically (i.e., without the de Broglie relation) the magnetic monopole implies nothing about the quantization of electric charge (e.g., one could have continuous electric fields in the background of the monopole spacetime).

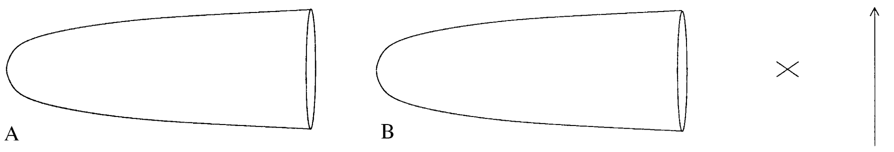

Fig. 2. The bolt solution (A) The Euclidean black hole; the periodic direction depicts the Euclidean time. (The θ and ϕ direction have been suppressed.); (B) The Kaluza bolt soliton; the periodic direction is now the direction of compactification x^5. (The Kaluza bolt can be interpreted as a Euclidean black hole multiplied by a trivial time coordinate.)

This is a positive definite metric for $r > 4m$. The intrinsic singularity from Schwarzschild space, located at $r = 0$, is excised from the Euclidean space. There is an apparent singularity at $r = 4m$, but this is like the apparent singularity at the origin of polar coordinates, as becomes clear upon introducing a new radial coordinate $x = 8m(1 - 4mr^{-1})^{1/2}$. The metric then becomes

$$\mathrm{d}s^2 = \left(\frac{x}{8m}\right)^2 \mathrm{d}\tau^2 + \left(\frac{r^2}{16m^2}\right)^2 \mathrm{d}x^2 + r^2\,\mathrm{d}\Omega^2. \tag{48}$$

If τ is regarded as an angular coordinate with period $16\pi m$, the above metric will be regular at $x = 0$, $r = 4m$. The corresponding regular coordinate system of this space would be the Kruskal system.[25] Again, we can incorporate this gravitational instanton as a soliton in the $n = 0$ Kaluza–Klein theory, in the same manner as before, just by (trivially) adding the time component (Gross & Perry, 1983; Chodos & Detweiler, 1982)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (1 - 4m/r)\,(\mathrm{d}x^5)^2 + (1 - 4m/r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2. \tag{49}$$

Just as for the monopole, the regularity condition (at $r = 4m$) requires the space to have a compact fifth dimension with x^5 having a period of $16\pi m$. At $r = 4m$ the spacetime is pinched off as the radius of the fifth dimension shrinks to zero. The set of fixed points of $\partial/\partial x^5$ at $r = 4m$ defines what is called a 'bolt'. As $r \to \infty$, the radius of the fifth direction remains of finite size (see Fig. 2).

There is one feature of Kaluza solitons that should be noted: all the $n = 0$ Kaluza–Klein solitons written down by Gross and Perry exhibit peculiar gravitational properties. For instance, the bolt has an inertial mass given by

$$P^0 = \int \mathrm{d}^3x\,\mathrm{d}x^5\sqrt{-g^{(5)}}\,R^{00} = \frac{8\pi}{\kappa}m. \tag{50}$$

However, both examples of solitons we have considered are 'flat' in the time direction, so in these spaces there are time-like geodesics that correspond to a particle sitting at rest relative to the soliton ($\mathrm{d}x^A/\mathrm{d}\tau = k\delta^{A0}$). The Newtonian force that a test particle experiences (i.e., $F \propto \frac{1}{2}\nabla g_{00}$) vanishes for these spaces. Hence the

[25] Writing the metric in the Kruskal form immediately shows the periodicity in τ. In addition, it again explains why the $r = 0$ singularity is not included in the space and, of course, in this system it is obvious that the metric is not singular at $r = 4m$. For this argument, see Gibbons and Hawking (1977).

solitons have a zero gravitational mass, $M_{\mathrm{grav}} = 0$. An observer at a fixed point in space cannot distinguish whether or not a soliton is present. This is argued not to be a violation of the (five-dimensional) principle of equivalence, but of the (four-dimensional) Birkhoff theorem—in the sense that not every rotation symmetric 5×5 metric has, at large distances, the structure of the Schwarzschild solution in its $g^{(4)}_{\mu\nu}$ part (Gross & Perry, 1983). The absence of gravitational mass actually makes it possible for these spaces to be non-singular: the Hawking–Penrose singularity theorems can be extended to five dimensions (see, e.g., Beem et al., 1996), but they do not apply because no trapped surface in the interior of some horizon is formed.

5.4. Scale invariance

Now why would the Einstein–Pauli argument not seem to apply to these soliton spaces? Equivalently, one may wonder how the gravitational instantons compare with Einstein's 1941 four-dimensional argument. Note that Einstein's Assumption A allows a Euclidean signature and a periodic time dependence. But Einstein tacitly assumed the period and the mass parameter to be independent. However, in the case of the gravitational instantons we have considered, the period *is* related to the mass parameter. Therefore, Einstein's contradiction is escaped; for replacing m by $(1 + c)m$ corresponds to performing a dilation on the Euclidean time τ. In the case of the Euclidean black hole, the dilation produces a conical singularity with an angular deficit of $16\pi cm$ at the horizon. So the dilation is not a symmetry of these solutions. In other words: the Euclidean black hole solution has action $S = 16\pi m^2$ (see Gibbons & Hawking, 1979) and, consequently, under the dilation $m \to (1 + c)m$, the action is not stationary.

For basically the same reason, the Einstein–Pauli argument falls short for the five-dimensional solitons described here. Notably, the Kaluza bolt solution (47) has the right asymptotics to fit their argument (34), viz.

$$\mathrm{d}s^2 \sim -\mathrm{d}t^2 + \left(1 - \frac{4m}{r}\right)(\mathrm{d}x^5)^2 + \left(1 + \frac{4m}{r}\right)(\mathrm{d}x^i)^2 \quad \text{as } r \to \infty, \tag{51}$$

which complies with (34) and (35) taking $m_{00} = m_{05} = 0$ and $m_{55} = -m$. The value of m_{55} had been shown to be necessarily zero by Einstein and Pauli on basis of the equation

$$c^5_5 \oint \sqrt{|g^5|}\, g^{5A} \Gamma^i_{5A} n_i \mathrm{d}^2\Omega = 0 \tag{52}$$

following from the assumed symmetry transformation

$$x^5 \to x^5 + \xi^5 = x^5 + c^5_5 x^5 \tag{53}$$

i.e., a dilation in the fifth dimension. If the bolt is to be considered non-singular, one must have a compact fifth direction: x^5 is taken periodic with period $16\pi m$. As a consequence, one has lost scale invariance and dilations in x^5 are no longer symmetry transformations because the invariant radius R changes under these transformations:

$$R = \int \mathrm{d}x^5 \sqrt{g_{55}} = 2\pi m \sqrt{g_{55}(r)} \to (1 + c)R. \tag{54}$$

A particular solution can have a period m or a period $(1 + c)m$ at infinity, but if the periods differ, they label distinct solutions that are not equivalent by virtue of global scale symmetry. In the bolt solution, dilations on x^5 would introduce conical singularities. These transformations therefore cannot be considered a symmetry of this soliton: if one were to allow the scale invariance and the associated conservation law Einstein and Pauli use, the fifth direction could not be compact and the bolt metric could only be considered singular. The Kaluza bolt has no curvature in the time direction, so its action is proportional to the instanton action: the action of the five-dimensional object is thus again not stationary in dilations in the compact direction.

The same statement can be made in case of the nut (i.e., the Kaluza–Klein monopole—note that it does not have the asymptotics (27)), with an action, in the four space-like dimensions, given by $16\pi m^2$ (see Gibbons & Hawking, 1979). Here also, when giving up the compactification of the fifth direction to allow dilation invariance, one again has to face the Dirac string singularity. The compactification is required in order for both spaces to be regular, but this violates the dilation invariance on which the Einstein–Pauli argument is based.[26]

One could look at the Einstein–Pauli theorem as, in fact, a positive, rather than a negative result, as considered by Einstein. From this perspective, it shows that there are no non-singular particle solutions if one retains the vacuum Kaluza theory and that, in order to find these objects, one needs a more general theory—a theory like classical Kaluza–Klein as described by Einstein and Bergmann. Nevertheless, we expect Einstein would have been discouraged by the apparent impossibility of solitons in Kaluza's theory. He never worked on a five-dimensional theory again, and it is quite clear that, also in the full Kaluza–Klein theory, Einstein's search for a non-singular particle had become frustrated.

6. Conclusion

We have seen that central to Einstein's work on either Kaluza theory—or the full classical Kaluza–Klein theory—was his desire to describe an electro-magnetically interacting object that would be a non-singular solution of the field equations. In his papers he expressed the desire to find such an object, just as he mentions this search was repeatedly frustrated. When in Kaluza's theory (with constant dilaton) he does not find a non-singular bridge type solution, he starts working on the expanded

[26] Time dilation is still a symmetry in five dimensions because the fifth dimension is not compact. It is the conservation law associated with this symmetry that made Einstein and Pauli conclude that $m_{00} = 0$. This can explain why there are no solitons in Kaluza's theory (for which one requires general covariance in the four spacetime dimensions) that have a gravitational mass (which is not terribly surprising, since this mass would introduce a horizon).

Kaluza–Klein theory (with constant dilaton) (Einstein & Bergmann, 1938). Here again we learned, in particular by studying the correspondence with his assistants, that he foremost wanted to find a non-singular electrically charged particle. He did not succeed (Einstein et al., 1941), which is probably one of the primary reasons why he abandoned working on this theory. In 1943 he argued, together with Pauli, that in Kaluza's theory (now with variable dilaton) it would be impossible in principle to find a non-singular particle. Einstein never worked in five dimensions again.

Nevertheless, it is striking that solitons *do* exist in this theory. Of course, the description of electro-magnetically interacting solitons requires a quite modern point of view (i.e., for the magnetic monopole, Kaluza's theory has to be viewed as a non-trivial principal bundle). Taking that position, one can argue that Einstein, in his discussions with Bergmann and in his paper with Pauli, imposed overly strict boundary conditions. One can only guess whether these objects are the exact sort of particles Einstein was looking for, however much in close contact the Einstein–Pauli argument stands to these modern day solitons. Indeed they are regular, massive non-singular particle objects that solve the classical field equations, and they show Einstein's intuitions were not far off. With these solitons, one can realize classical particles in a non-singular way—and there is no quantum state in the description of the particle involved.

Einstein's search for such solitonic objects cannot be separated from his negative attitude towards quantization and his possible plans to re-derive the quantum relations. Indeed, the most important lesson one learns about Einstein, when working on Kaluza–Klein, is that he would never even mention the result Klein had arrived at. Klein could explain the discrete charge spectrum by virtue of the de Broglie relation, combined with the compactification of the fifth direction. Einstein would never use the de Broglie relation that is so central to Klein's reasoning, not even to fix the scale of the fifth direction. Instead, he might have wanted to reason the other way around: after having found the desired solitonic objects, that by virtue of the boundary conditions would give a discrete charge spectrum, it would be possible to show that there had to be a minimum of action h. In any case, by his continual ignoring of Klein's argument, one sees that, when unifying the existence of discretised matter with the ideal of continuous fields, Einstein would want the quantum relations to follow, not to guide.

Einstein's attitude regarding quantization did not change after he had left all the varieties of the Kaluza–Klein theory. His assistant Bergmann, however, would change his position concerning quantization. He would later work on quantum gravity, on a theory of gravity in which even the gravitational interaction would be quantized in advance. In their later correspondence, Einstein exhibits again his discomfort with the ad hoc quantization procedures. In 1949 (just when Einstein was recovering from a laparotomy) Bergmann, now in Syracuse, wrote Einstein and asked if they could have a discussion sometime:

> As anyone can only be a crank about his own ideas, and as you are someone who combines steadfastness with the ability to acknowledge his hypothesis could go wrong (usually one can only find just one of these qualities, mostly the latter) I

> would appreciate very much talking to you and hearing your observations; whether we appreciate the same or not, what we want is sufficiently related that we could easily come to an understanding.[27]

Einstein replies:

> You are looking for an independent and new way to solve the fundamental problems. With this endeavor no one can help you, least of all someone who has somewhat fixed ideas. For instance, you know that on the basis of certain considerations I am convinced that the probability concept should not be primarily included in the description of reality, whereas you seem to believe that one should first formulate a field theory and subsequently 'quantize' it. This is in keeping with the view of most contemporaries. Your effort to abstractly carry through a field theory without having at your disposal the formal nature of the field quantities in advance, does not seem favorable to me, for it is formally too poor and vague.[28]

In 1954, Bergmann asks for funding at the National Science Foundation. Einstein is asked to evaluate the request, entitled "quantum theory of gravitation":

> The application of Dr. Bergmann concerns a problem of central significance for modern physics. All physicists are convinced of the high truth value of the probabilistic quantum theory and of the general relativity theory. These two theories, however, are based on independent conceptual foundations, and their combination to a unified logical system has so far resisted all attempts in this direction… If the decision were mine I should grant the funds asked for by Dr. Bergmann, in view of the central importance of the problem and the qualifications of the candidate. Even though in my opinion the probability of attaining the great goal seems rather small at this point, the financial risk incurred is on the other hand so modest that I should have no qualms to grant the application.[29]

Finally, in 1955 Einstein writes:

> Is it conceivable that a field theory permits one to understand the atomistic and quantum structure of reality? Almost everybody will answer this question with 'no.' But I believe that at the present time nobody knows anything reliable about it. This is so because we cannot judge in what manner and how strongly the exclusion of singularities reduces the manifold of solutions. We do not possess any method at all to derive systematically solutions that are free of singularities… we cannot at present compare the content of a nonlinear field theory with

[27] P. Bergmann, Letter to A. Einstein, 24 January 1949, AE 6-282.

[28] A. Einstein, Letter to P. Bergmann, 26 January 1949, AE 6-283. Einstein continues with: "Jedenfalls kann ich Sie also für die absehbare Zukunft nicht zu einem Besuch einladen und versichere Ihnen nochmals, dass einem bei der Eierlegerei niemand helfen kann, sondern das Geschöpf in Gottesnamen allein dasteht".

[29] A. Einstein, Letter to National Science Foundation, 18 April 1954, AE 6-313. (English as in original.)

experience… I see in this method [i.e., quantization] only an attempt to describe relationships of an essential nonlinear character by linear methods. (Einstein, 1955).

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