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Source: *The American Mathematical Monthly*, Vol. 90, No. 10 (Dec., 1983), pp. 697-701

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/2323537>

Accessed: 03-05-2016 01:56 UTC

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some d is a consequence of the following result, due essentially to Hausdorff [5, p. 280].

PROPOSITION 3. *If an ordinal $\sigma > 1$ is given the order topology, then the rank of the associated d is the smallest ordinal ρ such that $\omega^\rho \geq \sigma$.*

In order to describe $d^{\rho(d)}$ without reference to iteration (and also to describe a related concept), we need the following definitions. Let (S, d) be any topological space. Then d is *idempotent* if $d^2 = d$ and *hereditarily idempotent* if $(d_T)^2 = d_T$ for all $T \subseteq S$. For $A \in \mathcal{P}(S)$, let $c(A)$ denote the closure of A and let $p(A)$ denote the largest dense-in-itself subset of A (a subset B of S is dense-in-itself if $d(B) \supseteq B$). For (S, d) a T_D space, $d^{\rho(d)}$ is clearly the largest idempotent derived set function $\leq d$. Also, as was essentially known to Cantor and as is in any case easily verified, we have $d^{\rho(d)} = pc$ (where pc means “ c first, then p ”).

In a certain sense, $d^{\rho(d)}$ is not quite ultimate: although idempotent, it is not hereditarily idempotent in general (for instance, it is not in the case of the usual d on \mathbb{R}). All the same, there does exist a largest hereditarily idempotent derived set function $\leq d$, namely cp . In fact, cp is a hereditarily idempotent derived set function on any T_0 space S . One way to see this is as follows. First note that the scattered subsets of S form an ideal \mathfrak{S} ($A \subseteq S$ is scattered if and only if it contains no nonempty dense-in-itself set). Hence \mathfrak{S} may be minimally adjoined to the stock of closed subsets of S by means of the “localization” construction described by Vaidyanathaswamy [8, p. 171 et seq.]. Now, as Vaidyanathaswamy proves, the derived set function of the augmented topology is just cp [8, p. 183]). Moreover, since \mathfrak{S} is an adherence-ideal in the terminology of [8] (see pp. 177 and 183), it follows from results of [6] that cp is hereditarily idempotent and is furthermore the largest hereditarily idempotent derived set function $\leq d$.

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CROSS PRODUCTS OF VECTORS IN HIGHER DIMENSIONAL EUCLIDEAN SPACES

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Several times I have had students in undergraduate mathematics courses ask me the following question: Can one define a cross product of vectors in Euclidean n -space for $n > 3$ so that it will have properties similar to the usual cross product of vectors in 3-space? Of course the answer to this question will probably depend on which properties of the usual cross product one requires to hold in n -space; it is conceivable that there will be many different answers, depending on which properties are required to hold.

Fortunately the situation is not quite as chaotic as the foregoing sentences might suggest. If one requires only three basic properties of the cross product, properties which are explained in practically all undergraduate textbooks that discuss vector analysis, it turns out that a cross product of vectors exist only in 3-dimensional and 7-dimensional Euclidean space. To the best of the author's knowledge, the only textbook which contains a discussion of this fact is Hilton [8].

The purpose of this note is to give an explanation of this result which will be accessible to the average reader of this MONTHLY. We will actually give two theorems on this subject: the first uses purely algebraic techniques, and is based on a famous theorem proved by A. Hurwitz in 1898. The second theorem gives a stronger result; it depends on a deep theorem proved by J. F. Adams in 1958. At the end of the paper we discuss some other results in this area.

Our notation is standard: R^n denotes the real vector space consisting of n -tuples of real numbers,

$$x \cdot y = \sum x_i y_i$$

is the dot product of $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, and

$$|x| = (x \cdot x)^{1/2}$$

denotes the norm or length of the vector x .

THEOREM I. *Assume $n \geq 3$ and a cross product is defined which assigns to any two vectors $v, w \in R^n$ a vector $v \times w \in R^n$ such that the following three properties hold:*

- (a) $v \times w$ is a bilinear function of v and w .
- (b) The vector $v \times w$ is perpendicular to both v and w , i.e., $(v \times w) \cdot v = (v \times w) \cdot w = 0$.
- (c) $|v \times w|^2 = |v|^2 |w|^2 - (v \cdot w)^2$.

Then $n = 3$ or 7 .

REMARK. Note that condition (c) is the usual condition that the length of $v \times w$ shall be equal to the area of the parallelogram spanned by v and w .

Proof. The proof consists in showing that a cross product defined on R^n and having the three properties listed above implies the existence of a bilinear multiplication on R^{n+1} which has very special properties. We will consider R^{n+1} as an orthogonal direct sum:

$$R^{n+1} = R^1 \oplus R^n.$$

Thus an element of R^{n+1} consists of an ordered pair (a, v) , where a is a real number and $v \in R^n$. The required product is defined by the following formula:

$$(1) \quad (a, v)(b, w) = (ab - v \cdot w, aw + bv + v \times w).$$

This multiplication is obviously bilinear, and $(1, 0)$ is a 2-sided unit. An easy computation using properties (b) and (c) shows that the norm of the product of two elements of R^{n+1} is given by the following formula:

$$(2) \quad |(a, v)(b, w)|^2 = |(a, v)|^2 |(b, w)|^2$$

Now this is exactly the situation considered by A. Hurwitz [4] in 1898. Hurwitz proved that if we have a bilinear multiplication with a unit defined on R^q such that the norm of the product of two vectors is the product of the norms (condition (2) above), then q must be 1, 2, 4, or 8, and the multiplication is isomorphic to that of the real numbers, the complex numbers, the quaternions, or the octonions of Cayley and Graves. For a lucid exposition of a modern version of this theorem of Hurwitz, see Jacobson, [5, pp. 417–427].

Note that the uniqueness assertion of Hurwitz's theorem shows that conditions (a), (b), and (c) of Theorem I characterize the cross products on R^3 and R^7 uniquely up to isomorphism.

The interested reader is referred to pp. 408–409 of a paper by E. Calabi [2] for a list of additional properties of the cross product in R^7 and an actual multiplication table for this cross product in terms of an orthonormal basis of R^7 .

REMARK. The multiplication given by formula (1) has evidently been known for a long time. In 1942 B. Eckmann referred to it as "einer bekannten, elementaren Konstruktion" (see [3, p. 338]).

In our next theorem we show that we can significantly weaken conditions (a) and (c) of Theorem I without altering the conclusion.

THEOREM II. Assume $n \geq 3$ and that a cross product is defined which assigns to any two vectors $v, w \in R^n$ a vector $v \times w$ such that the following three properties hold:

- (a) $v \times w$ is a continuous function of the ordered pair (v, w) .
- (b) The vector $v \times w$ is perpendicular to both v and w , i. e., $(v \times w) \cdot v = (v \times w) \cdot w = 0$.
- (c) If v and w are linearly independent, then $v \times w \neq 0$.

Then $n = 3$ or 7 .

Proof. For any vectors $v, w \in R^n$, let

$$A(v, w) = [|v|^2 |w|^2 - (v \cdot w)^2]^{1/2}.$$

Then $A(v, w)$ is equal to the area of the parallelogram spanned by v and w ; it is obviously a continuous function of the ordered pair (v, w) . Using this area function, we define a function

$$f: R^n \times R^n \rightarrow R^n$$

by the formula

$$f(v, w) = \begin{cases} \frac{A(v, w)}{|v \times w|} (v \times w) & \text{if } v \times w \neq 0, \\ 0 & \text{if } v \times w = 0. \end{cases}$$

We assert that the function f thus defined is continuous. To prove this, it suffices to prove that if (v_k, w_k) is any infinite sequence of pairs of vectors such that

$$\lim_{k \rightarrow \infty} (v_k, w_k) = (v_0, w_0),$$

then

$$\lim_{k \rightarrow \infty} f(v_k, w_k) = f(v_0, w_0).$$

There are various cases to consider, depending on the two cases in the definition of $f(v, w)$, but the details are completely elementary.

Note that the function f thus defined satisfies the following two conditions:

- (3) $f(v, w) \cdot v = f(v, w) \cdot w = 0$,
- (4) $|f(v, w)|^2 = |A(v, w)|^2 = |v|^2 |w|^2 - (v \cdot w)^2$.

Exactly as before, we may consider R^{n+1} as the direct sum $R^1 \oplus R^n$, and define a function

$$\mu: R^{n+1} \times R^{n+1} \rightarrow R^{n+1}$$

by the formula

$$(5) \quad \mu[(a, v), (b, w)] = (ab - v \cdot w, av + bw + f(v, w))$$

(compare with (1)). Then μ is obviously continuous and $(1, 0)$ is a 2-sided unit in the sense that

$$\mu[(1, 0), (a, v)] = \mu[(a, v), (1, 0)] = (a, v).$$

Finally, we have the analog of formula (2):

$$(6) \quad |\mu(x, y)|^2 = |x|^2 |y|^2$$

for any $x, y \in R^{n+1}$. Now let S^n denote the unit n -dimensional sphere:

$$S^n = \{x \in R^{n+1} | |x| = 1\}.$$

In view of formula (6), we see that if x and y belong to S^n , then $\mu(x, y) \in S^n$ also. Thus μ defines a continuous multiplication with a 2-sided unit on the n -sphere S^n .

This raises the following question: For what values of n does the n -sphere admit a continuous multiplication with a 2-sided unit? This was a famous problem for many years in algebraic

topology. It was finally resolved by Frank Adams in 1958 (see [1]). The answer is that such a continuous multiplication exists on S^n only in the cases $n = 1, 3,$ and 7 . Examples of such multiplications arise from the multiplication of complex numbers, quaternions, and the Cayley–Graves octonions respectively, restricted to the unit sphere.

Thus we see that by referring to this theorem of Adams we can complete the proof of Theorem II.

We will conclude this note by considering other possible ways to generalize the definition of the cross product to higher dimensional Euclidean spaces. Let v and w be vectors in R^3 ; recall the formula for the components of $v \times w$ in terms of the components of v and w . According to this formula, if

$$v = (v_1, v_2, v_3)$$

$$w = (w_1, w_2, w_3)$$

then the k th component of $v \times w$ is the determinant of the 2×2 submatrix of the matrix

$$\begin{bmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

obtained by striking out the k th column. To be correct, we must multiply this determinant by $(-1)^k$.

Analogously, in R^n we can define a cross product $v_1 \times v_2 \times \cdots \times v_{n-1}$ of any ordered $(n-1)$ tuple of vectors by a similar process. Form a matrix whose successive rows are the vectors v_1, v_2, \dots, v_{n-1} . The k th component of $v_1 \times v_2 \times \cdots \times v_{n-1}$ is $(-1)^k$ times the determinant of the submatrix obtained by deleting the k th column. This generalized cross product enjoys many of the familiar properties of the cross product in 3-space: It is a multilinear, skew symmetric function. The norm of the product, $v_1 \times v_2 \times \cdots \times v_{n-1}$, is the $(n-1)$ -dimensional volume (or measure) of the parallelepiped spanned by the vectors v_1, \dots, v_{n-1} . The vector $v_1 \times v_2 \times \cdots \times v_{n-1}$ is perpendicular to v_k for $k = 1, 2, \dots, n-1$.

This raises the following question: given integers k and n such that $2 < k < n-1$, can we define a cross product of any k -tuple of vectors v_1, v_2, \dots, v_k in R^n having similar properties? To make the question precise, let us demand the following properties, similar to those in Theorem II:

- (a) $v_1 \times v_2 \times \cdots \times v_k$ is a continuous function of the ordered k -tuple (v_1, \dots, v_k) .
- (b) $(v_1 \times v_2 \times \cdots \times v_k) \cdot v_i = 0$ for $i = 1, 2, \dots, k$.
- (c) If the vectors v_1, v_2, \dots, v_k are linearly independent, then $v_1 \times v_2 \times \cdots \times v_k \neq 0$.

The answer, strangely enough, is that such a cross product does not exist, with a single exception: $n = 8$ and $k = 3$. For the proof of this, the reader is referred to a paper by George Whitehead [6]; for explicit formulas for a cross product in this case, see Zvengrowski, [7].

Another property of the cross product of vectors in 3-space is the following: For any rotation r (i.e., orthogonal transformation of determinant $+1$) of 3-space and vectors v and w ,

$$(7) \quad r(v \times w) = (rv) \times (rw).$$

(Note that this equation is *not* true if r is an orthogonal transformation of determinant -1). One can now prove the following:

PROPOSITION. *Assume that $n > 2$ and a cross product product is defined in R^n which is bilinear and satisfies equation (7) for any rotation of R^n ; then $n = 3$.*

The proof depends on a knowledge of the real representations of the special orthogonal group $SO(n)$; we do not have space in this note to go into details.

This preservation of the cross product by rotations, expressed by equation (7), is less well known than the usual properties which are treated in our first theorem. It is normally only treated in advanced texts in theoretical physics or geometry.

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A "COUNTEREXAMPLE" FOR THE SCHWARZ-CHRISTOFFEL TRANSFORM

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The Schwarz-Christoffel Transform is a formula for a one-to-one analytic function that maps the upper half of the complex plane onto the inside of a polygon. As such the Schwarz-Christoffel Transform can be used to translate problems set in polygonal domains to more manageable problems set in the upper half plane. Such applications of the Schwarz-Christoffel Transform are common in problems involving two-dimensional flows, diffusion, potentials, etc. (see [1], [3], [5]).

A statement of the Schwarz-Christoffel Transform can be found in most text books on elementary complex variables. One such statement is [4, p. 178]: *The functions $w = F(z)$ that map the upper half plane conformally onto polygons with interior angles $\pi\alpha_k$ ($k = 1, 2, \dots, n$) are of the form*

$$(1) \quad F(z) = A \int_0^z \prod_{k=1}^n (\zeta - x_k)^{-\beta_k} d\zeta + B$$

where $x_1 < x_2 < \dots < x_n$ are points on the real axis, $\beta_k = 1 - \alpha_k$ ($k = 1, 2, \dots, n$) and A, B are complex constants. Since the sum of the exterior angles of a polygon is 2π , we have:

$$(2) \quad \beta_1 + \beta_2 + \dots + \beta_n = 2 \quad \text{with} \quad -1 \leq \beta_k \leq 1.$$

(The polygons with one or more $\beta_k = \pm 1$ are those with some vertices at ∞ and/or some interior angles of 2π).

It is well known that conditions (1) and (2) are necessary for $F(z)$ to map the upper half plane conformally onto a polygon. However it is rarely stressed that these conditions are not sufficient for such a mapping and there appear to be few (if any) examples to this effect. In this paper we produce an example of a function $F(z)$ that satisfies (1) and (2) but is not one-to-one in the upper half plane. We start with a third condition (involving the choice of the x_k 's) that is necessary for $F(z)$ to be univalent.

The following theorem gives some guidelines for the choice of the real numbers x_k mentioned in the statement of the Schwarz-Christoffel Transform.

THEOREM. *Let $\{\beta_k\}_{k=1}^n$ be given with $-1 \leq \beta_k \leq 1$ and let $x_1 < x_2 < \dots < x_n$ be real. If the function $F(z)$ defined by (1) is univalent in the upper half plane, then*

$$(3) \quad \left| \sum_{k=1}^n \frac{\beta_k}{z_0 - x_k} \right| \leq \frac{3}{\text{Im } z_0}$$

whenever $\text{Im } z_0 > 0$.

The theorem is an immediate consequence of the following lemma which gives a necessary