WHY WE WORK IN HIGHER DIMENSIONS ?

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INTRODUCTION

In this article, with the help of tensor calculus and a little bit of general relativity, we shall try to find a genuine physical answer to the very basic question that *"why the higher dimensions are necessary to work for"*, while only three dimensions are visible.

In what follows, we shall briefly mention the results from tensor calculus and general theory of relativity that are necessary for answering our basic question (For a detailed discussions of tensors and general relativity, the reader is referred, respectively, to the references [1] and [2]).

In many areas of mathematical, physical and engineering sciences, it is often necessary to consider two types of quantities – namely ***scalars***, which have magnitude only. Mass, length, volume, density, work, electric charge, time, temperature, etc. are the examples of scalars; and ***vectors***, which have both magnitude and direction. Some of the examples of vectors are velocity, acceleration, force, momentum. Quite often the notion of vector is not sufficient to represent a physical quantity. For example

(i) the stress at a point depends upon two directions; one normal to the surface and the other that represents the force creating stress and thus stress can not be described by a vector quantity alone.

(ii) the measurement of charge density will depend upon the four velocity of the observer and thus can be represented by a vector, while the measurement of electric field strength in some direction will not only depends upon this direction but also on the four velocity of the observer and thus such measurement can not be described by a vector quantity alone.

These and similar other examples led to the generalization of a vector quantity to a quantity known as *tensor*.

The concept of ***tensors*** has its origin in the development of differential geometry by Gauss, Riemann and Christoffel. The systematic development of tensor calculus (also known as absolute calculus) as a branch of mathematics is due to Ricci and his pupil Levi-Civita who published the first memoir on this subject "Methods de calcul Differential Absolu et leurs Applications". The principal aim of tensor calculus is to investigate the relations which remain valid when we move from one coordinate system to another. The laws of physics (or nature) can not depend upon the frame of reference which the physicists choose for the description of such laws. Tensor analysis become popular when Albert Einstein (1879-1955) used it in his general theory of relativity. With the tools of Riemannian geometry, Einstein was able to formulate a theory that predicts the behavior of objects in the presence of gravity, electromagnetism and other forces. Now a days tensors have many applications in most of the branches of theoretical physics and engineering, such as classical mechanics, fluid mechanics, elasticity, plasticity and electromagnetism etc.

Transformation of coordinates

The fundamental notion involved in tensor analysis is that of a geometrical point which is defined by means of its coordinates. Thus in plane Euclidean geometry, a point is specified by its two Cartesian coordinates (X, Y) or its polar coordinates (r, θ). Consider two set of variables $(x^1, x^2, \dots, x^n)$ and $(x'^1, x'^2, \dots, x'^n)$ which determines the coordinates of a point in a n

dimensional space in two different frames of reference. These two set of variables are related to each other by means of equation

$$x'^i = f^i(x^1, x^2, \ldots, x^n), \; i = 1, 2, \ldots, n \tag{1}$$

where the functions f^i are single-valued, continuous differentiable functions of the coordinates. They are also independent. We can solve equation (1) for the coordinates x^j and we have

$$x^i = F^i(x'^1, x'^2, \ldots, x'^n), \quad i = 1, \; 2, \ldots, n \tag{2}$$

Equations (1) and (2) are said to define a *coordinate transformation.* If we take the differential of equation (1), then

$$dx'^i = \sum_{j=1}^{n} \frac{\partial x'^i}{\partial x^j} dx^j, \quad (i = 1, 2, \ldots \ldots n) \tag{3}$$

while the differential of equation (2) is

$$dx^i = \sum_{j=1}^{n} \frac{\partial x^i}{\partial x'^j} dx'^j, \quad (i = 1, 2, \ldots \ldots n) \tag{4}$$

Summation convention

If any index in a term is repeated, then a summation with respect to that index over the range $1, 2, \ldots, n$ is implied. This convention is known as *Einstein summation convention.* According to this convention, instead of the expression $\sum_{i=1}^{n} a_i x^i$, we simply write $a_i x^i$ and equations (3) and (4) with this convention may, respectively, be expressed as

$$dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j \text{ and } dx^i = \frac{\partial x^i}{\partial x'^j} dx'^j$$

Thus summation convention means the drop of summation symbol for the index appearing twice in the term. If a suffix (index) occurs twice in a term, once in the upper position and once in the lower position, then that suffix (index) is called a dummy suffix (index). Also

$$a_i^k x^i = a_1^k x^1 + a_2^k x^2 + \ldots + a_n^k x^n$$

and

$$a_j^k x^j = a_1^k x^1 + a_2^k x^2 + \ \ldots + \ a_n^k x^n$$

Thus $a_i^k x^i = a_j^k x^j$, which shows that a dummy suffix can be replaced by another dummy suffix not already appearing in the expression. A suffix (index) which is not repeated is called a real suffix (index). For example, k is a real suffix in $a_i^k x^i$. A real suffix can not be replaced by another real suffix as $a_i^k x^i \neq a_i^p x^i$.

Kronecker delta

The symbol δ_j^i defined by

$$\delta_j^i = \begin{cases} 1, & for\ i = j \\ 0, & for\ i \neq j \end{cases}$$

is called *Kronecker delta* and has the following properties:

(i) If $x^1, x^2, \ldots, x^n$ are independent varaiables, then

$$\frac{\partial x^i}{\partial x^j} = \delta_j^i$$

(ii) $\delta_k^i A^k = A^i$.

(iii) In n dimensions, $\delta_j^i = n$.

(iv) $\delta_j^i \delta_k^j = \delta_k^i$ and $\frac{\partial x^i}{\partial x'^j} . \frac{\partial x'^j}{\partial x^k} = \delta_k^i$.

Scalar, contravariant and covariant vectors

Scalars (invariant or tensor of rank zero): Quantities which do not change under coordinate transformation. Thus two functions $\phi(x)$ and $\phi'(x')$ are said to define a scalar if they are reducible to each other by a coordinate transformation, where $\phi(x)$ is the value of a scalar in one coordinate system and $\phi'(x')$ is its value in another coordinate system.

***Vectors*:** The set of n quantities A^i, which transform like the coordinate differentials

$$\mathrm{A}'^i = \frac{\partial x'^i}{\partial x^j} A^j, \quad i, j = 1, 2, \ldots, n$$

are called the components of a contravariant vector (contravariant tensor of rank one), while the set of n quantities A_i are called the components of a covariant vector (covariant tensor of rank one), if they which transform like

$$A'_i = \frac{\partial x^j}{\partial x'^i} A_j, \quad i, j = 1, 2, \ldots, n$$

Tensors of higher rank

Consider a set of n^2 quantities A^{kl} $(k,l = 1,2,\dots,n)$ in coordinate system x^i and let these quantities have the values A'^{ij} in another coordinate system x'^i. If these quantities obey the law

$$A'^{ij} = \frac{\partial x'^i}{\partial x^k}\frac{\partial x'^j}{\partial x^l}A^{kl}$$

then the quantities A^{kl} are said to be the components of a contravariant tensor of rank two. Similarly, the quantities A_{kl} are said to the components of a covariant tensor of rank two if

$$A'_{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j}A_{kl}$$

If the n^2 quantities A_l^k and $A_j'^i$ in the coordinate systems x^i and x'^i, respectively, are related to each other by the transformation law

$$A_j'^i = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}A_l^k$$

then the quantities A_l^k are said to be the components of a mixed tensor of rank two.

It may be noted that upper index of a tensor denotes the contravariant nature while the lower index indicates the covariant character of the tensor. The total number of upper and lower indices of a tensor is called the *rank* or *order* of the tensor. The rank of a tensor when raised as power to the number of dimensions gives the number of components of the tensor. Thus, a tensor of rank r in n dimensions has n^r components.

We can define the tensors of much higher rank as follows:

A set of quantities $A^{i_1 i_2 \dots\dots i_r}$ in a coordinate system x^i are said to be the components of a contravariant tensor of rank r if they satisfy the transformation law

$$A'^{j_1 j_2 \dots\dots j_r} = \frac{\partial x'^{j_1}}{\partial x^{i_1}}\frac{\partial x'^{j_2}}{\partial x^{i_2}}\dots\dots\dots\frac{\partial x'^{j_r}}{\partial x^{i_r}}\,A^{i_1 i_2 \dots\dots i_r}$$

where $A'^{j_1 j_2 \dots\dots j_r}$ are the quantities in x'^i coordinate system. Similarly, the quantities $A_{i_1 i_2 \dots\dots i_r}$ obeying the law

$$A'_{j_1 j_2 \dots\dots j_r} = \frac{\partial x^{i_1}}{\partial x'^{j_1}}\frac{\partial x^{i_2}}{\partial x'^{j_2}}\dots\dots\dots\frac{\partial x^{i_r}}{\partial x'^{j_r}}\,A_{i_1 i_2 \dots\dots i_r}$$

are said to form the components of a covariant tensor of rank r. While, the quantities $A^{i_1 i_2 \dots\dots i_r}_{j_1 j_2 \dots\dots j_s}$ satisfying the law

$$A'^{l_1 l_2 \dots\dots l_r}_{m_1 m_2 \dots\dots m_s} = \left(\frac{\partial x'^{l_1}}{\partial x^{i_1}}\frac{\partial x'^{l_2}}{\partial x^{i_2}}\dots\dots\dots\frac{\partial x'^{l_r}}{\partial x^{i_r}}\right)\left(\frac{\partial x^{j_1}}{\partial x'^{m_1}}\frac{\partial x^{j_2}}{\partial x'^{m_2}}\dots\dots\dots\frac{\partial x^{s}}{\partial x'^{m_s}}\right)A^{i_1 i_2 \dots\dots i_r}_{j_1 j_2 \dots\dots j_s}$$

are the components of a mixed tensor of rank $(r + s)$.

Algebra of tensors

In tensor algebra only those operations are allowed which when performed on tensors give rise to new tensors, Some of the algebraic operations on tensors are as follows:

Addition and subtraction. A linear combination of tensors of same type and same rank is a tensor of same type and same rank. Thus, if A_{ij} and B_{ij} are second rank covariant tensors and α and β are scalars, then $C_{ij} = \alpha A_{ij} \pm \beta B_{ij}$ is also a second rank covariant tensor.

Equality of tensors. Two tensors X and Y are said to be *equal* if their components are equal. That is $X^i_{jk} = Y^i_{jk}$ or $X^{ij}_k = Y^{ij}_k$ for all values of the indices.

Inner and outer products. Let A^i be a contravariant vector and B_i a covariant vector then the product $A^i B_i$ is a scalar. This scalar product is called the *inner product* (of a contravariant vector with a covariant vector). While on the other hand, the quantity $A^i B_j$ is called *outer product* of two vectors. The outer product is defined for any type of tensors, the total rank of the resulting tensor is the sum of the individual ranks of the tensors. For example, the product of A^{ij}_k and B^l_m produces a tensor of rank five. It may be noted that the product of a tensor by a scalar (multiplication of each component by the scalar) is again a tensor.

Contraction. The process of summing over a covariant and a contravariant index of a tensor to get another tensor such that the rank of this new tensor is lowered by two. For example, consider a mixed tensor A^i_{jkl} . Put $l = i$ we get the tensor A^i_{jki} whose rank is lowered by two as i appears as a dummy suffix. Moreover, contraction of a second rank mixed tensor A^i_j on setting $j = i$, leads to $A'^i_i = A^p_p$. This is an invariant, called the *trace* of A^i_j and has the same value in all coordinate systems.

The quotient law. In tensor analysis, we often come across quantities about which we are not certain whether they are tensor or not. The direct method requires to find the appropriate transformation law and in practice this is not an easy job. However, we do have a criterion which tells us about the tensorial nature of a set of quantities; it is known as *quotient law* and is stated as

"a set of quantities, whose inner product with an arbitrary covariant (or contravariant) tensor is a tensor, is itself a tensor".

Riemanian Space and Metric Tensor

Consider a Euclidean plane in which rectangular Cartesian coordinates exist. If (x, y) and $(x + dx, y + dy)$ are two neighbouring points in this plane, then by Pythagoras theorem the distance ds between two points is

$$ds^2 = dx^2 + dy^2.$$

This formula is called the *metric* of the Euclidean plane; when the polar coordinates (r, θ) are used, this metric takes the form

$$ds^2 = dr^2 + r^2 d\theta^2,$$

while for three dimensional case, we have

$$ds^2 = dx^2 + dy^2 + dz^2,$$

and

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \; d\phi^2.$$

Another way of defining the distance is

$$d(x, y) = |\, x - y \,| \text{ (the usual metric).}$$

Now we may move on to the general case of n-dimensional space. One way is to extend the dimensionality of this space from two/three to n, and a point in such a space will have coordinates $(x^1, x^2, \dots, x^n)$. For the other way, we assume that the distance between two neighboring points is given by

$$ds^2 = g_{ij} dx^i dx^j,$$

where $i, j = 1, 2, \dots, n$ and the summation convention is used. Here g_{ij}*'s* are the functions of the coordinates and may vary from point to point. This equation is called the *metric equation* and ds is the *interval* or *line-element.* The space which satisfies this equation is called the *Riemannian space*. Our three dimensional Euclidean space is a special case of Riemannian space. The function g_{ij}*'s* are n^2 in number, real and need not be positive but are

such that their determinant

$$g = \det g_{ij} = |g_{ij}| \neq 0.$$

Since dx^i and dx^j in $ds^2 = g_{ij}dx^i dx^j$ are contravariant vectors and ds^2 is an invariant for any arbitrary choice of dx^i and dx^j , from quotient law, it follows that g_{ij} is a covariant tensor of rank two. This tensor is known as *metric tensor* or *fundamental tensor*. In addition to the metric tensor there are two more fundamental tensors g^{ij} and δ_j^i which are defined as

$$g^{ij} = \frac{\text{cofactor of } g_{ij} \text{ in } \det g_{ij}}{\det g_{ij}},$$

This g^{ij} is called the *conjugate* or *reciprocal tensor* of g_{ij}. Also

$$g_{ij}g^{ik} = \delta_j^k.$$

It is a mixed tensor of rank two. The three tensors $g_{ij}, g^{ij}, \delta_j^i$ defined through above equations are called *fundamental tensors* and are of basic importance in general theory of relativity.

The metric tensor and its conjugate can be used for raising and lowering the indices of a tensor (vector). Thus, for a contravariant vector A^k, the corresponding covariant vector is $A_i = g_{ik}A^k$ and for covariant vector B_k the corresponding contravariant vector is $B^i = g^{ik}B_k$. From a second rank tensor A^{ij} , we have

$$A_k^i = g_{kj}A^{ij},\ A_{ik} = g_{ip}A_k^p = g_{ip}g_{kj}A^{pj} ,$$

and so on. Tensors obtained as a result of raising and lowering operations with the metric tensor are called *associated tensors*.

Christoffel symbols

From the metric tensor g_{ij} and its conjugate g^{ij}, we can construct two functions. These functions are not tensors but are used to define the differentiation of tensors. They are known as Christoffel symbols and are defined as

$$\Gamma_{kij} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right),$$

and

$$\Gamma_{jk}^i = \frac{1}{2}g^{im}\left(\frac{\partial g_{jm}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right) = g^{im}\Gamma_{mjk}.$$

The symbols $\Gamma_{kij}, \Gamma_{jk}^i$ are, respectively, called *Christoffel symbols of first* and *second kind.* Since g_{ij} is symmetric, so are the Christoffel symbols, i.e.,

$$\Gamma_{kij} = \Gamma_{kji}, \Gamma^i_{jk} = \Gamma^i_{kj},$$

Also

(i) $\Gamma_{ljk} + \Gamma_{jkl} = \frac{\partial g_{jl}}{\partial x^k},$

(ii) $\Gamma^j_{jk} = \frac{\partial}{\partial x^k} \log\sqrt{g} = \frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^k},$

(iii) $\frac{\partial g^{pq}}{\partial x^m} = -g^{pj}\Gamma^q_{jm} - g^{qj}\Gamma^p_{jm}.$

Equation of a Geodesic

What we mean by a straight line in Euclidean space? One meaning implied by the adjective "*straight*", is that its direction remain unchanged as we move along it. The other property associated with the straight line is that it represents the path of the shortest distance between any two given points. Here we shall find out what curves are implied by the later definition in a more general space - the Riemannian space (for the implication of the former definition, see [1]).

The path of a particle between two points P and Q is a geodesic and is determined by the condition that the interval between the two points P and Q given by $\int_P^Q ds$ be stationary. In other words, a geodesic is defined by the condition that

$$\delta \int_P^Q ds = 0.$$

where δ denotes the variation from the actual path (world line) between the two points P and Q on it, to any other path in the neighborhood of this line (actual path).

To obtain the equation of geodesic, consider the metric of the Riemannian space,

i.e., $ds^2 = g_{ij}dx^i dx^j$ and we have

$$\frac{d^2x^t}{ds^2} + \Gamma^t_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0.$$

This is the required condition for the given integral $\int_P^Q ds$ to be stationary. The "*straight line*" given by this equation is the *equation of geodesic*. For $t = 1, 2, 3, 4$ the equation of geodesic gives four equations which determine the geodesic.

Now since ds is an element of the world line, we can interpret these equations as the equations of motion of a particle which moves along a world line. When the components of the metric tensor are constant, then $\Gamma_{ij}^{t} = 0$ and equation of geodesic reduces to

$$\frac{d^2x^t}{ds^2} = 0,$$

which shows that the particle is moving with uniform speed along a straight line.

Covariant differentiation

If A^i is a vector, then what is the nature of its derivative $\frac{\partial A^i}{\partial x^j}$? Will it be a tensor or not? Consider a contravariant vector A^i which transforms as

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j.$$

Differentiate this equation partially with respect to x'^k, we get

$$\begin{aligned} \frac{\partial A'^i}{\partial x'^k} &= \frac{\partial}{\partial x'^k}\left(\frac{\partial x'^i}{\partial x^j} A^j\right), \\ &= \frac{\partial}{\partial x^p}\left(\frac{\partial x'^i}{\delta x^j} A^j\right)\frac{\partial x^p}{\partial x'^k}, \\ &= \frac{\partial x'^i}{\partial x^j}\frac{\partial x^p}{\partial x'^k}\frac{\partial A^i}{\partial x^p} + \frac{\partial^2 x'^i}{\partial x^p \partial x^j}\frac{\partial x^p}{\partial x'^k} A^j. \end{aligned}$$

If $\frac{\partial A'^i}{\partial x'^k}$ is a tensor then this equation should contain only the first term on the right hand side. But due to presence of the second term on the right hand side of this equation, the quantity $\frac{\partial A'^i}{\partial x'^k}$ does not behave like a tensor. That is, the outcome of the differentiation of a tensor is not a tensor; and in tensor analysis only those operations are allowed which when performed on a tensor lead to a tensor. Thus, the quantity $\frac{\partial A'^i}{\partial x'^k}$ will be a tensor only when the quantity $\frac{\partial^2 x'^i}{\partial x^p \partial x^j}$ vanish, i.e., if the coordinates x'^i are linear function of the coordinates x^j then this equation is the transformation law of a tensor.

The process of obtaining tensors through the process of partial differentiation is known as *covariant differentiation*. We have

covariant derivative of a contravariant vector:

$$A^{j}{}_{;p} = \frac{\partial A^{j}}{\partial x^{p}} + \Gamma^{j}_{qp}\, A^{q},$$

covariant derivative of a covariant vector:

$$A_{j;k} = \frac{\partial A_{j}}{\partial x^{k}} - \Gamma^{p}_{jk}\, A_{p},$$

covariant derivative of a covariant tensor of rank two:

$$A_{ij;p} = \frac{\partial A_{ij}}{\partial x^{p}} - \Gamma^{k}_{ip} A_{kj} - \Gamma^{k}_{pj} A_{ik},$$

covariant derivatives of contravariant and *mixed tensors of rank two* are defined through the following equations

$$A^{ij}{}_{;p} = \frac{\partial A^{ij}}{\partial x^{p}} + \Gamma^{i}_{pk} A^{kj} + \Gamma^{j}_{pk} A^{ik},$$

and

$$A^{i}_{j;p} = \frac{\partial A^{i}_{j}}{\partial x^{p}} + \Gamma^{i}_{pk} A^{i}_{k} - \Gamma^{k}_{pj} A^{i}_{k}.$$

Covariant derivative of tensors of higher rank :

The process of covariant differentiation can be applied to tensors of higher ranks and in general, for a mixed tensor of rank $(p + q)$, the covariant derivative is defined as

$$A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_q;k} = \frac{\partial A^{i_1 i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_q}}{\partial x^{k}} + \Gamma^{i_1}_{mk} A^{m i_2 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_q} + \Gamma^{i_2}_{mk} A^{i_1 m i_3 \ldots\ldots i_p}_{j_1 j_2 \ldots\ldots j_q} + \ldots + \Gamma^{i_p}_{mk} A^{i_1 i_2 \ldots\ldots i_{P-1} m}_{j_1 j_2 \ldots\ldots j_q}$$

$$-\Gamma^{m}_{j_1 k} A^{i_1 i_2 \ldots\ldots i_p}_{\mathrm{m j_2 \ldots\ldots j_q}} - \Gamma^{m}_{j_2 k} A^{i_1 i_2 \ldots\ldots i_p}_{\mathrm{j_1 m \ldots\ldots j_q}} - \ldots - \Gamma^{m}_{j_q k} A^{i_1 i_2 \ldots\ldots i_p}_{\mathrm{j_1 j_2 \ldots\ldots j_{q-1}\, m}}.$$

Remarks

(i) The covariant differentiation is denoted by a semi-colon $(;)$ while the partial differentiation is denoted by a comma $(,)$.

(ii) It may be noted that the covariant derivative is an operator which reduces to partial derivative in flat space (where g_{ij} are constant) with Cartesian coordinates but transforms as a tensor on an arbitrary manifold.

(iii) For a vector A^i, the covariant derivative $A^i{}_{;j}$, for each direction j, will be given by the partial derivative operator $\frac{\partial}{\partial x^j}$ plus a correction specified by Γ^i_{jk}.

(iv) From the defining equations of covariant derivatives, it may be noted that through the process of covariant differentiation, we get tensors of higher rank. Thus, the rank of a tensor can be raised by differentiating it covariantly, while the rank of the tensor is lowered by the process of contraction.

Rules for covariant differentiation

1. The covariant derivative of a linear combination of tensors, with constant coefficients, equals to the linear combination of these tensors after the covariant differentiation was performed. Thus, for example if A^i_j and B^i_j are two mixed tensors of rank two and a and b are scalars then

$$\left(aA^i_j \pm bB^i_j \right)_{;k} = aA^i_{j;k} \pm bB^i_{j;k}$$

2. The covariant derivatives of outer and inner products of tensors obey the same rules as that of the usual derivatives. For example

$$\left(A^i B_{jk}\right)_{;p} = A^i_{;p} B_{jk} + A^i B_{jk;m}$$

$$\left(A^i B_i\right)_{;k} = A^i_{;j} B_i + A^i B_{i;j}$$

$$\phi_{;i} = \frac{\partial \phi}{\partial x^i}.$$

3. The fundamental tensors are covariantly constant. That is the covariant derivative of fundamental tensors is zero.

Divergence of a vector field

Let A^i be a contravariant vector and $A^k_{;i}$ be its covariant derivative, then $A^i_{;i}$, a unique scalar (invariant), is called the *divergence of a vector field* A^i and is defined by

$$A^i_{;i} = \text{div}\, A^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(A^i \sqrt{g}\right).$$

Curl of a vector field

Using the definition of the covariant derivative of a covariant vector, we have

$$A_{i;j} - A_{j;i} = \left(\frac{\partial A_i}{\partial x^j} - \Gamma_{ij}^m A_m\right) - \left(\frac{\partial A_j}{\partial x^i} - \Gamma_{ji}^m A_m\right) = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$$

This difference is, of course, a tensor and does not involve Christoffel symbols. This tensor is skew symmetric and is known as the *curl* or *rotation* of the vector A_i. The curl operation is not applicable to contravariant vectors or tensors of higher rank. That is

$$A^i_{;j} - A^j_{;i} = \left(\frac{\partial A^i}{\partial x^j} + \Gamma_{pj}^i A_p\right) - \left(\frac{\partial A_j}{\partial x^i} + \Gamma_{pi}^j A_p\right) \neq \frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i}$$

The Riemann Curvature Tensor

For a covariant vector field A_i, using the definition of covariant differentiation, we have

$$A_{i;j;l} - A_{i;l;j} = R^k_{ijl} A_k \quad \text{(Ricci Identity)}$$

where

$$R^k_{ijl} = \frac{\partial}{\partial x^j}\Gamma^k_{il} - \frac{\partial}{\partial x^l}\Gamma^k_{ij} + \Gamma^m_{il}\Gamma^k_{mj} - \Gamma^m_{ij}\Gamma^k_{ml} \tag{5}$$

The rank four mixed tensor R^k_{ijl} is known as the *Riemann curvature tensor* or simply the *Riemann tensor*, and was first discovered by Riemann $(1826 - 1866)$ and then after many years by Christoffel $(1829 - 1900)$. This tensor plays a central role in the geometric structure of a Riemannian space. It may be noted, this tensor vanishes for Euclidean space. The Riemann curvature tensor is not only important in describing the geometry of the curved space, but also from this tensor we can construt other tensors which give a complete description of the gravitational field.

The covariant form of Riemann tensor is given by

$$R_{hijl} = \frac{1}{2}\left(\frac{\partial^2 g_{hl}}{\partial x^j \partial x^i} + \frac{\partial^2 g_{ij}}{\partial x^l \partial x^h} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^h} - \frac{\partial^2 g_{hj}}{\partial x^l \partial x^i}\right) + g_{km}(\Gamma^k_{ij}\Gamma^m_{hl} - \Gamma^k_{il}\Gamma^m_{hj})$$

The Riemann tensor satisfies the following properties:

(i) R^h_{ijl} and R_{hijl} is antisymmetric in the last two indices, i.e.,

$$R^h_{ijl} = -R^h_{ilj} \text{ and } R_{hijl} = -R_{hilj}.$$

(ii)R_{hijl} is antisymmetric in the first two indices h and i, i.e.,

$$R_{hijl} = -R_{ihjl}.$$

(iii) R_{hijl} is symmetric with respect to an interchange of the first pair of indices (hi) and second pair of indices (jl), without changing the order of the indices in each pair, i.e.,

$$R_{hijl} = R_{jlhi}.$$

(iv) $R^k_{ijl} + R^k_{jli} + R^k_{lij} = 0, \; R_{hijl} + R_{hjli} + R_{hlij} = 0.$

(v) The Riemann tensor also satisfies the identities

$$R_{hijk;l} + R_{hikl;j} + R_{hilj;k} = 0.$$

$$R^h_{ijk;l} + R^h_{ikl;j} + R^h_{ilj;k} = 0.$$

These equations are known as *Bianchi identities*.

The Riemann tensor can be used to obtain tensors of lower rank. Since Riemann tensor, defined by equation (5), is a mixed tensor of rank four, there are three possible contraction that can be performed on Riemann tensor; and we have

(i) Putting $k = i$ in equation (5), we get $R^i_{ijl} = 0.$

(ii) Putting $k = j$ in equation (5), we get

$$R^j_{ijl} = \frac{\partial \Gamma^j_{il}}{\partial x^j} - \frac{\partial \Gamma^j_{ij}}{\partial x^l} + \Gamma^m_{il}\,\Gamma^j_{mj} - \Gamma^m_{ij}\Gamma^j_{ml} = -R^j_{ilj} = -R_{il} \tag{6}$$

(iii) Putting $k = l$ in equation (5), we get

$$R^l_{ijl} = R_{ij} = \frac{\partial \Gamma^l_{il}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^l} + \Gamma^m_{il}\,\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{ml} \tag{7}$$

Equation (7), obtained by the contraction of first contravariant index with the last covariant index of the Riemann tensor, defines a second rank tensor known as *Ricci tensor*, while equation (6) is just the negative of Ricci tensor. Further, from Ricci tensor we can construct a scalar as

$$R = g^{ij}R_{ij}$$

This scalar R is called the *scalar curvature.*

Moreover, from the Bianchi identities, we have

$$\left(R^j_i - \frac{1}{2}g^j_i R\right)_{;j} = 0$$

If we take

$$G_i^j = R_i^j - \frac{1}{2} g_i^j R,$$

then

$$G_{i;j}^j = 0$$

The tensor G_i^j is called the *Einstein tensor*; since it is obtained by contracting the Bianchi identities, it is some times also known as the *contracted Bianchi identities*. This tensor G_i^j plays a fundamental role in the general theory of relativity.

General Theory of Relativity

General theory of relativity is the geometric theory of gravitation published by Albert Einstein. It generalizes special relativity and Newton's law of universal gravitation, providing a unified description of gravity as a geometric property of space and time, or space-time. In particular, the curvature of space-time is directly related to the four-momentum (mass-energy and linear momentum) of whatever matter and radiation are present. The relation is specified by the Einstein field equations-a system of partial differential equations. Some predictions of general relativity differ significantly from those of classical physics, especially concerning the passage of time, the geometry of space, the motion of bodies in free fall, and the propagation of light. Examples of such differences include gravitational time dilation, gravitational lensing, the gravitational red shift of light, and the gravitational time delay. The predictions of General relativity have been confirmed in all observations and experiments to date. Although general relativity is not the only relativistic theory of gravity, it is the simplest theory that is consistent with experimental data. However, unanswered questions remain, the most fundamental being how general relativity can be reconciled with the laws of quantum physics to produce a complete and self-consistent theory of quantum gravity.

Einstein's theory has important astrophysical implications. For example, it implies the existence of black holes-regions of space in which space and time are distorted in such a way that nothing, not even light, can escape. There is ample evidence that such stellar black holes as well as more massive varieties of black hole are responsible for the intense radiation emitted by certain types of astronomical objects such as active galactic nuclei or microquasars. The bending of light by gravity can lead to the phenomenon of gravitational lensing, where multiple images of the same distant astronomical object are visible in the sky. General relativity also predicts the existence of gravitational waves, which have since been measured indirectly; a direct measurement is the aim of projects such as LIGO and NASA/ESA Laser Interferometer Space Antenna. In addition, general relativity is the basis of current cosmological models of a consistently expanding universe.

General relativity has emerged as a highly successful model of gravitation and cosmology, which has so far passed every unambiguous observational and experimental test. Observational data that is taken as evidence for dark energy and dark matter could indicate the need for new physics. Even taken as it is, general relativity is rich with possibilities for further exploration. Mathematical relativists seek to understand the nature of singularities and the fundamental properties of Einstein's equations. The race for the first direct detection of gravitational waves continues a pace, in the hope of creating opportunities to test the theory's validity for much stronger gravitational fields than has been possible to date. More than ninety five years after its publication, general relativity remains a highly active area of research.

To establish a correspondance between the laws of mechanics and electrodynamics, Einstein formulated his special theory of relativity in 1905; and attempts to formulate the law of gravitation in relativistic form have led Einstein to put forward his general theory of relativity in 1915. The postulates and hypotheses of this theory denoted, respectively by P and H, are as follows:

(P) *Principle of Covariance.* The laws of physics remain invariant under any spactime coordinate transformation. Since tensors are the quantities which remain invariant under coordinate transformation, the laws of physics can be expressed in terms of tensorial equations.

(P) *Principle of Equivalence.* In the neighbourhood of a point it is always possible to choose a coordinate system such that the effects of gravity can be made negligible in the neighbourhood of that point. In other words, it is not possible to distinguish between the field (gravitational) produced by the attraction of masses and the field produced by accelerating the frame of references.

(H) To describe an event, a $4-$dimensional spacetime is needed-the metric of which is

$$ds^2 = g_{ij}dx^i dx^j, \qquad i,j = 1,2,3,4$$

(H) The path of a particle is a geodesic whose equation is

$$\frac{d^2x^t}{ds^2} + \Gamma_{ij}^t \frac{dx^i}{ds}\frac{dx^j}{ds} = 0.$$

The path of a light ray is a null geodesic, that is the one for which $ds = 0$.

(H) At large distances from the source, the line-element of general relativity reduces to the line-element of special relativity.

(H) Gravitation is a field phenomenon and the *field equations*, in the presence of matter, are

$$R_{ij} - \frac{1}{2} g_{ij} R = -kT_{ij} \tag{8}$$

where T_{ij} is the energy-momentum tensor. The left hand side of this equation describes the geometry of the spacetime, while the right hand side represents the physics of the spacetime.

Now multiplying equation (8) by g^{ij}, we get $R = kT$ and equation (8) thus reduces to

$$R_{ij} = -k(T_{ij} - \frac{1}{2} g_{ij} T)$$

so that when there is no matter $(T_{ij} = 0)$, this equation reduces to

$$R_{ij} = 0 \tag{9}$$

which are the *field equations for empty spacetime*. The word 'empty' here means that there is no matter present and also no physical fields except the gravitational field. The gravitational field does not disturb the emptiness of the space. While the other fields do. For the space between the planets in the solar system, the condition of emptiness holds in good approximation and equation (9) is applied in such case.

Why higher dimensions are necessary?

Through his general theory of relativity, Einstein redefined gravity. From the classical point of view, gravity is the attractive force between massive objects in three dimensional space. In general relativity, gravity manifests as curvature of four dimensional space-time. Conversely curved space and time generates effects that are equivalent to gravitational effects. J.A. Wheelar has described the results by saying *Matter tells spacetime how to bend and spacetime returns the complement by telling matter how to move.*

The general theory of relativity is thus a theory of gravitation in which gravitation emerges as the property of the space-time structure through the metric tensor g_{ij}. The metric tensor determines another object known as Riemann curvature tensor $[cf., equation\ (5)]$. At any given event this tensorial object provides all information about the gravitational field in the neighbourhood of the event. It may be interpreted as describing the curvature of the spacetime. The Riemann curvature tensor is the simplest non-trivial object one can build at a point; its vanishing is the criterion for the absence of genuine gravitational fields and its structure determines the relative motion of the neighboring test particles via the equation of geodesic deviation. These discussions clearly illustrates the importance of the Riemann curvature tensor in general relativity.

Moreover, from the Riemann tensor and the metric tensor, we can construct another 4-rank tensor C_{ijkl}, known as *Weyl conformal curvature tensor*. This tensor, (for $n > 2$), is defined by the equation

$$R_{ijkl} = C_{ijkl} + \frac{1}{n-2}\left(g_{il}R_{jk} + g_{jk}R_{il} - g_{jk}R_{jl} - g_{il}R_{ik}\right)$$
$$-\frac{R}{(n-1)(n-2)}\left(g_{il}g_{jk} - g_{ik}g_{jl}\right). \tag{10}$$

So that for $n = 4$, we have

$$R_{ijkl} = C_{ijkl} + \frac{1}{2}\left(g_{il}R_{jk} + g_{jk}R_{il} - g_{ik}R_{jl} - g_{jl}R_{ik}\right) - \frac{R}{6}\left(g_{il}g_{jk} - g_{ik}g_{jl}\right). \tag{11}$$

The Weyl tensor has the same symmetries as that of Riemann tensor except that

$$C^{m}_{imj} = 0 = g^{mn}C_{nimj,}$$

that is, the Weyl tensor is trace-less.

Also, for the empty spacetime $R_{ijkl} = C_{ijkl}$. Since R_{ijkl}characterize the gravitational field, therefore it is the Weyl tensor C_{ijkl} which describe the true graviational fields in a vacuum region.

Moreover, from the symmetries of Riemann tensor, the number of independent components of Riemann tensor in n-dimension is

$$\frac{1}{12}n^2(n^2 - 1).$$

Thus

(i) if $n = 1$, $R_{hijk} = 0$;

(ii) if $n = 2$, R_{hijk} has only one independent component which can be calculated from

$$R_{ijkl} = \frac{1}{2} R(g_{ik} g_{jl} - g_{il} g_{jk})$$

and is $R_{1212} = \frac{1}{2} gR;$

(iii) $if\ n = 3, R_{hijk}$ has six independent components. The Ricci tensor has also six independent components and thus R_{hijk} can be expressed in terms of R_{ij} as

$$R_{hijk} = g_{hj} R_{ik} + g_{ik} R_{hj} - g_{hk} R_{ij} - g_{ij} R_{hk} - \frac{1}{2}(g_{hj} g_{ik} - g_{hk} g_{ij})R$$

(iv) if $n = 4$, R_{hijk} has twenty independent components (ten of which are given by Ricci tensor and the remaining ten by the Weyl tensor C_{hijk}) and we have

$$R_{ijkl} = C_{ijkl} + \frac{1}{2}(g_{il} R_{jk} + g_{jk} R_{il} - g_{ik} R_{jl} - g_{jl} R_{ik}) - \frac{R}{6}(g_{il} g_{jk} - g_{ik} g_{il}).$$

From these discussions, we have

If $R_{ij} = 0$ (empty spacetime), then for $n = 1, 2, 3$; $R_{ijkl} = 0$ (no gravitational field) and for $n = 4, R_{ijkl} = C_{ijkl}$.

Thus according to general relativity, we can easily state that

"if we lived in a three dimensional Universe, gravity could not exist in a vacuum region".

So if there would be no gravity, the earth could not be going to move around the sun and therefore it had never been possible for me to write this article. Hence we have a genuine reason that why the higher dimensions are necessary to work for.

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