Quantum asymmetry between time and space

Article *in* Proceedings of The Royal Society A Mathematical Physical and Engineering Sciences · February 2015

DOI: 10.1098/rspa.2015.0670 · Source: arXiv

Joan Vaccaro
Griffith University, Nathan, Australia

Quantum asymmetry between time and space

Joan A. Vaccaro
Centre for Quantum Dynamics, Griffith University, Nathan 4111 Australia

In special relativity, time and space are necessarily interconvertible in order that the speed of light is an invariant. This places time and space on an equal footing which, perplexingly, does not carry over to other areas of physics. For example in quantum mechanics, time is treated classically whereas space is associated with a quantum description. Differences between time and space are also found in the violation of the discrete symmetries of charge conjugation, parity inversion and time reversal—the violations are inferred from the decay of particles over time irrespective of their position in space, and so they are associated with translations in time but not in space. Although these violations are clearly important, their wider implications are unknown. We show here that when the violations are included explicitly in a quantum formalism, remarkable differences arise between the character of quantum states in time and space. In particular, despite time and space having an equal footing at a fundamental level, we find that quantum states can be localised in space and yet have unbounded evolution in time. As such, the violations are shown to play a defining role in the asymmetry between time and space in quantum mechanics.

I. INTRODUCTION

There is nothing unphysical about matter being localised in a region of space; matter can simply exist at one location and not another. But for it to be localised in a finite period of time is altogether different. Indeed, as the matter would exist only for that period and no other, the situation would be a direct violation of mass conservation. In conventional quantum mechanics, this undesirable situation is avoided axiomatically by requiring matter to be represented by a quantum state vector whose norm is fixed over time. Time then becomes a classical parameter whereas the location of matter in space is treated by quantum variables and, as a consequence, the status of time and space are quite different from the very outset.

However, time and space could have an equivalent footing in quantum mechanics if their differences were to arise phenomenologically rather than being imposed axiomatically on the theory. Such a prospect is well worth pursuing because it would help us to understand the relationship between time and space. It would require finding an underlying mechanism that allows matter to be localised in space but not in time. As localisation entails a constraint on the corresponding translational degree of freedom, we need to look for the mechanism in terms of translations. The generators of translations in space and time are given by the momentum and Hamiltonian operators, respectively, and with them lies a difference that sets space and time apart in the quantum regime. In fact, the last fifty years [1–6] has shown that the Hamiltonian is not invariant to particular combinations of the discrete symmetry operations of charge conjugation (C), parity inversion (P) and time reversal (T), whereas the momentum operator is. The violations are accounted for in the Standard Model of particle physics by the Cabibbo-Kobayashi-Maskawa (CKM) matrix [7, 8] and the violation of CP invariance, in particular, is believed to have played a crucial role in baryogenesis in the early universe [9]. Here we explore the potential impact the violations may have for giving quantum states different representations in space and time.

Given the fundamental character of the issues involved, one should not be surprised to find that to make any progress we need to pay due attention to quite subtle mathematical details. For instance, while the concept of the limit of an infinite sequence has rigorous meaning in a mathematical context, there is no *a priori* reason to suppose that it automatically carries corresponding meaning in a theory that is designed to underpin experimental physics. After all, the accuracy of observations made in experimental physics are always restricted by finite resources. For example, consider a theory in which the limit point a of the convergent sequence a_1, a_2, a_3, ... (i.e. where $a_n \to a$ as $n \to \infty$) represents an experimental parameter, and let ϵ represent the experimental accuracy of measuring a for a given level of resources. The convergence of the sequence implies that there exists a natural number N that depends on ϵ for which $|a - a_n| < \epsilon$ for all $n > N$, and so it is not possible to physically distinguish (using the given resources) the limit point a from any of the terms a_n for $n > N$. Under such circumstances, the set $\{a_n : n > N\}$ would be a more complete representation of the physical situation than just the limit point a. Another mathematical subtlety concerns the violation of the C, P and T discrete symmetries. The C, P and T symmetry operations do not appear in conventional quantum mechanics in any fundamental way. If we wish to see how the associated violations can give rise to differences between space and time, then we need to take care not to inadvertently exclude the C, P and T symmetry operations from the quantum formalism, even if in some circumstances they appear to be redundant. Likewise, we need to take care not to overlook situations where the generators of translations in space and time might play a role, even if that role appears to be irrelevant in the conventional quantum formalism.

With this in mind, we begin in section II by considering the mathematical construction of quantum states

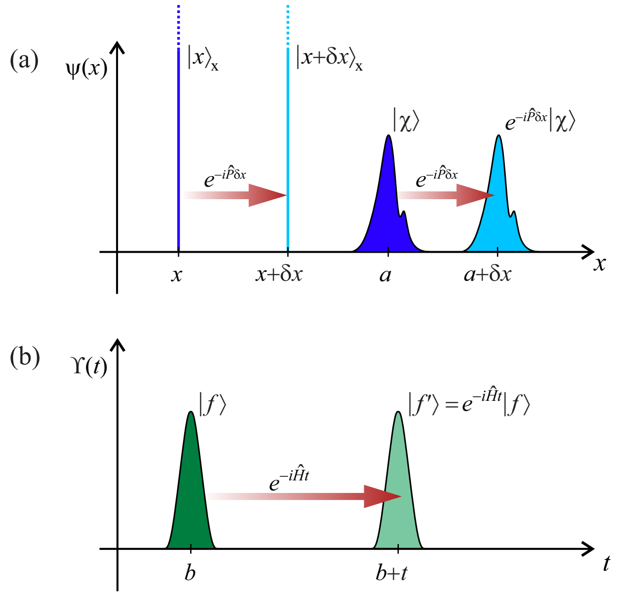

FIG. 1: Sketches illustrating the translation of wave functions along (a) the x axis and (b) the time axis. In (a) the wave functions represent the position eigenket $|x\rangle_{\rm x}$ and an arbitrary state $|\chi\rangle$ and the translation is by a distance δx. In (b) the wave function represents the state $|f\rangle$ and the translation is by an interval t.

that are distributed over space. We incorporate the parity inversion operation and translations in space into the construction and pay due consideration to fundamental limits of precision. This construction forms the basis of a new quantum framework that is used in the remainder of the paper. In section III, we replace parity inversion with time reversal, translations in space with those in time, and then apply the construction to quantum states that are distributed over time. We show that the presence of the violation of time reversal symmetry dramatically changes the quantum states from being localised in time to having unbounded time evolution. Following that, in section IV, we show how the conventional Schrödinger equation emerges as a result of coarse graining over time, and explore how the new formalism might be tested experimentally. We end with a discussion in section V.

II. MATHEMATICAL CONSTRUCTION OF QUANTUM STATES

We first need to find a mathematical construction of quantum states that is expressed explicitly in terms of the respective generator of translations and the C, P or T symmetry operations. We begin by considering a simple 1-dimensional model universe composed of a single "galaxy" and described by non-relativistic quantum mechanics. The galaxy is representative of any spatially localised physical system with mass and could in fact be a star, a planet or just a single particle; its details are not important for this study. Imagine that some precondition ensures that the centre of mass of the galaxy is described by a Gaussian wave function as follows:

$$|\psi\rangle \;\propto\; \int dx \exp(-\frac{x^2}{2\sigma_{\rm x}^2})|x\rangle_{\rm x} \tag{1}$$

where $|x\rangle_{\rm x}$ is an eigenstate of the x component of the centre of mass position with eigenvalue x and $\sigma_{\rm x}$ is a width parameter. Such a state gives the minimum of the product of the uncertainties in the centre of mass position and total momentum. It can be written explicitly in terms of spatial translations as

$$|\psi\rangle \propto \int dx \exp(-\frac{x^2}{2\sigma_{\rm x}^2}) \exp(-i\hat{P}x)|0\rangle_{\rm x} \tag{2}$$

where operator representing the total momentum of the galaxy, $\hat{P}$, generates spatial translations according to

$$\exp(-i\hat{P}\delta x)|x\rangle_{\rm x} = |x+\delta x\rangle_{\rm x}$$

as illustrated in Fig. 1(a). Here, and throughout this paper, we use units in which $\hbar = 1$. Inserting the resolution of the identity $\hat{\mathbf{1}} = \int dp|p\rangle_{\rm pp}\langle p|$ into Eq. (2) gives

$$|\psi\rangle \propto \iint dx\, dp \exp(-\frac{x^2}{2\sigma_{\rm x}^2}) \exp(-ipx)|p\rangle_{\rm pp}\langle p|0\rangle_{\rm x}\ ,$$

where $\{|p\rangle_{\rm p} : \hat{P}|p\rangle_{\rm p} = p|p\rangle_{\rm p}\}$ is the momentum basis. On carrying out the Fourier transform with respect to x, yields $|\psi\rangle \propto \exp(-\frac{1}{2}\hat{P}^2\sigma_{\rm x}^2)|0\rangle_{\rm x}$ and making use of the result

$$\exp(-A^2/2) = \lim_{N\to\infty} \cos^N(A/\sqrt{N}) \tag{3}$$

then leads to

$$|\psi\rangle \propto \lim_{N\to\infty} \frac{1}{2^N}\left[\exp(i\frac{\hat{P}\sigma_{\rm x}}{\sqrt{N}}) + \exp(-i\frac{\hat{P}\sigma_{\rm x}}{\sqrt{N}})\right]^N |0\rangle_{\rm x}\ . \tag{4}$$

Expanding the N-fold product in Eq. (4) gives a series of terms each of which comprise N translations (or "steps") of $\pm\sigma_{\rm x}/\sqrt{N}$ along the x axis. For example, a term of the form

$$\cdots \exp(-i\hat{P}a)\exp(-i\hat{P}a)\exp(i\hat{P}a)\exp(-i\hat{P}a)|0\rangle_{\rm x}\ ,$$

where $a = \sigma_{\rm x}/\sqrt{N}$, describes a path on the x axis from the origin 0 through the sequence of points a, 0, a, $2a$ and so on, as illustrated in Fig. 2(a). Equation Eq. (4) can be viewed, therefore, as a superposition of random paths away from the origin $|0\rangle_{\rm x}$ in the limit of infinitely small steps, and shares similarities with both quantum walks [10] and Feynman's sum over paths [11]. Note that here, however, the random path is traversed without reference to time, and so it should be considered to be traversed in a zero time interval. Each random path is, therefore, a generalisation of the virtual displacements used in classical mechanics [12]. For this reason each individual path

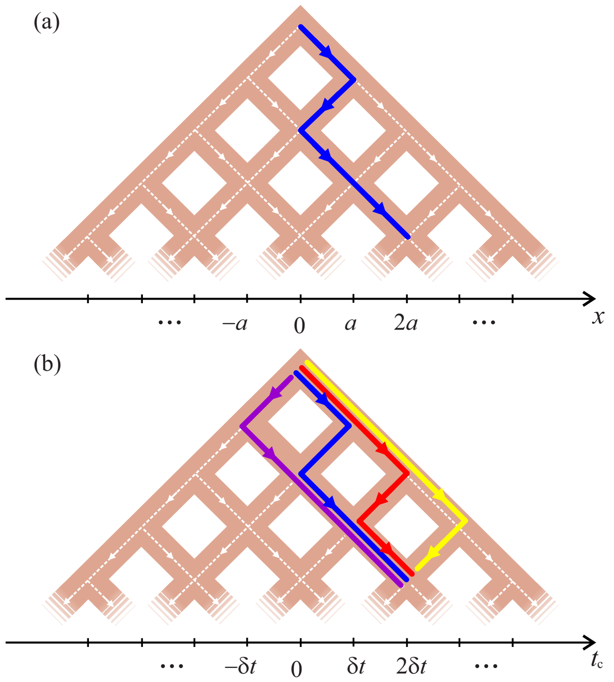

FIG. 2: Binary tree diagrams representing virtual paths in (a) space and (b) time. Each edge (white dashed line) in the tree represents a virtual displacement along the black horizontal axis. The thick blue edges in (a) represents a virtual path that passes through the sequence of points 0, a, 0, a, $2a$ on the x axis. In (b) four different virtual paths from 0 to $2\delta t$ on the t_c axis are represented in the tree by thick edges coloured yellow, red, blue and purple.

shall be called a *random virtual path* and the superposition of a set of random virtual paths like that in Eq. (4) shall be called a *quantum path*.

As $N \to \infty$ the step length $\sigma_x/\sqrt{N}$ in Eq. (4) will eventually breach the fundamental lower bound, say $\delta x_{\rm min}$, that is expected for physically distinguishable positions. For example, there are reasons [13] to believe that points in space are indistinguishable at the scale of the Planck length $\ell_P \approx 1.6 \times 10^{-35}$ m. Let $N^{(\rm space)}_{\rm min}$ be the value of N where the step length $\sigma_x/\sqrt{N}$ becomes equal to $\delta x_{\rm min}$, i.e. $N^{(\rm space)}_{\rm min} = \sigma_x^2/\delta x^2_{\rm min}$. This implies that the limit on the right side of Eq. (4) can be replaced by a term corresponding to any value of N larger than $N^{(\rm space)}_{\rm min}$ without any physically meaningful consequences. There are an infinite number of such terms, each of which has an *equal status* in representing the state of the universe. They form the set

$$\mathbf{\Psi} = \{|\psi\rangle_N : N \geq N^{(\rm space)}_{\rm min}\} \tag{5}$$

where

$$|\psi\rangle_N = \frac{1}{2^N}\left[\hat{\mathbf{P}}^{-1}\exp(-i\frac{\hat{P}\sigma_x}{\sqrt{N}})\hat{\mathbf{P}} + \exp(-i\frac{\hat{P}\sigma_x}{\sqrt{N}})\right]^N |0\rangle_x \ . \tag{6}$$

In Eq. (6) we have written the translations explicitly in terms of the parity inversion operator $\hat{\mathbf{P}}$. It has the property that

$$\exp(i\hat{P}x') = \hat{\mathbf{P}}^{-1}\exp(-i\hat{P}x')\hat{\mathbf{P}} \tag{7}$$

which expresses the fact that a translation along the x axis by $-x'$, corresponding to the left side of Eq. (7), can be produced by first performing a parity inversion, translating by x' and then reversing the parity inversion, as shown on the right side. Every element in the set $\mathbf{\Psi}$ can serve equally well as a representation of the state in Eq. (1) as far as the physically-distinguishable spatial limit allows; they all have equal status in this respect.

The mathematical construction represented by Eq. (5) and Eq. (6) is in the form of the explicit translations and discrete symmetry operations that we need for comparing the difference between quantum states in space and time. Although being equivalent to Eq. (1), we shall henceforth regard Eq. (5) and Eq. (6) as being a *more fundamental* description of the state of the galaxy due to this explicit form. Accordingly, any precondition for Eq. (1) now becomes a precondition for this construction. Note that the interpretation of Eq. (6) in terms of quantum paths does not hinge on the state $|0\rangle_x$ being the eigenstate of position with zero eigenvalue. In fact any state $|\chi\rangle$ with a variance in position very much smaller than $\sigma_x^2/2$ (and, correspondingly, a variance in total momentum very much larger than $1/2\sigma_x^2$) could be used in its place, in which case the steps in a path represent translations of $|\chi\rangle$ along the x axis, as illustrated in Fig. 1(a), rather than steps along the x axis itself.

III. APPLYING THE CONSTRUCTION TO QUANTUM STATES IN TIME

We now use our construction to explore the temporal analogy of Eq. (1) in which the galaxy is represented in time rather than space. We begin by recalling that the Hamiltonian $\hat{H}$ generates translations through time according to

$$\exp(-i\hat{H}t)|f\rangle = |f'\rangle$$

where $|f\rangle$ and $|f'\rangle$ represent states at times differing by t, as illustrated in Fig. 1(b). Next, we construct a set of states analogous to Eq. (5) but with each state representing a superposition of random virtual paths through *time* as

$$\mathbf{\Upsilon}_\lambda = \{|\Upsilon_\lambda\rangle_N : N \geq N^{(\rm time)}_{\rm min}, \lambda\} \tag{8}$$

where

$$|\Upsilon_\lambda\rangle_N \propto \frac{1}{2^N}\left[\hat{\mathbf{T}}^{-1}\exp(-i\frac{\hat{H}\sigma_t}{\sqrt{N}})\hat{\mathbf{T}} + \exp(-i\frac{\hat{H}\sigma_t}{\sqrt{N}})\right]^N |\phi\rangle \ . \tag{9}$$

Here λ distinguishes different physical situations that will be specified later, $N^{(\rm time)}_{\rm min} = \sigma_t^2/\delta t^2_{\rm min}$ is the value of

N for which the step size $\sigma_{\rm t}/\sqrt{N}$ reaches some fundamental resolution limit in time $\delta t_{\rm min}$ (e.g. taking the resolution limit as the Planck time would mean that $\delta t_{\rm min} = 5.4\times 10^{-44}$ s), and $\hat{\mathbf{T}}$ is Wigner's time reversal operator [14]. The state $|\phi\rangle$ plays the role of $|0\rangle_{\rm x}$ in Eq. (6) and is assumed to be sharply defined in time and, correspondingly, to have a broad distribution in energy [15]. More specifically, $|\phi\rangle$ must have a variance in energy that is very much larger than $1/2\sigma_{\rm t}^2$ in analogy with the requirement for any state $|\chi\rangle$ to be used in place of $|0\rangle_{\rm x}$. Other details of $|\phi\rangle$ are not crucial for our main results.

The violation of T invariance is expressed by $\hat{\mathbf{T}}^{-1}\hat{H}\hat{\mathbf{T}} \neq \hat{H}$ which implies that there are two versions of the Hamiltonian [16]. It is convenient to label the two versions as $\hat{H}_{\rm F} = \hat{H}$ and $\hat{H}_{\rm B} = \hat{\mathbf{T}}^{-1}\hat{H}\hat{\mathbf{T}}$, and set $\delta t = \sigma_{\rm t}/\sqrt{N}$ as the step in time. Using these definitions together with the fact [14] that $\hat{\mathbf{T}}^{-1} i\hat{\mathbf{T}} = -i$ then gives

$$|\Upsilon_\lambda\rangle_N \propto \frac{1}{2^N}\Big[\exp(i\hat{H}_{\rm B}\delta t) + \exp(-i\hat{H}_{\rm F}\delta t)\Big]^N |\phi\rangle \qquad (10)$$

which shows that $\hat{H}_{\rm F}$ and $\hat{H}_{\rm B}$ are responsible for translations in opposite directions of time.

This is an important point that warrants particular emphasis: in Eq. (10) a translation in time in the opposite direction to that given by $\exp(-i\hat{H}_{\rm F}t)$ is not produced by its *inverse* $\exp(i\hat{H}_{\rm F}t)$ but rather by its *time reverse*:

$$\exp(i\hat{H}_{\rm B}t) = \hat{\mathbf{T}}^{-1}\exp(-i\hat{H}_{\rm F}t)\hat{\mathbf{T}}\ .$$

Evidently we need to associate the operators $\exp(-i\hat{H}_{\rm F}t)$ and $\exp(i\hat{H}_{\rm B}t)$ with *physical evolution* in different directions of time according to Eq. (10). This leaves their respective inverses $\exp(i\hat{H}_{\rm F}t)$ and $\exp(-i\hat{H}_{\rm B}t)$ to be associated with the *mathematical operations of rewinding* that physical evolution. In short, physical time evolution is described by the former pair of operators, and not the latter.

In fact, this result follows from conventional quantum mechanics. For example, let $|f(t)\rangle$ represent the state of an arbitrary closed system at time t. Unitary evolution implies that $|f(t)\rangle = \exp(-i\hat{h}t)|f(0)\rangle$ where $|f(0)\rangle$ is the state at $t=0$ and $\hat{h}$ is the corresponding Hamiltonian. Recall that Wigner's time reversal operator $\hat{\mathbf{T}}$ reverses the direction of all momenta and spin [14]. Let the time-reversed states at times 0 and t be $|b(0)\rangle = \hat{\mathbf{T}}^{-1}|f(0)\rangle$ and $|b(-t)\rangle = \hat{\mathbf{T}}^{-1}|f(t)\rangle$, respectively. Using $\hat{\mathbf{T}}\hat{\mathbf{T}}^{-1} = \hat{1}$ and rearranging shows that $|b(-t)\rangle = \exp(i\hat{\mathbf{T}}^{-1}\hat{h}\hat{\mathbf{T}}t)\hat{\mathbf{T}}^{-1}|f(0)\rangle = \exp(i\hat{\mathbf{T}}^{-1}\hat{h}\hat{\mathbf{T}}t)|b(0)\rangle$ and so the time-reversed state $|b(-t)\rangle = \hat{\mathbf{T}}^{-1}|f(t)\rangle$ represents the evolution from the time-reversed state $|b(0)\rangle = \hat{\mathbf{T}}^{-1}|f(0)\rangle$ according to the Hamiltonian $\hat{\mathbf{T}}^{-1}\hat{h}\hat{\mathbf{T}}$ for the time $-t$. That is, evolving from the state $|f(0)\rangle$ for the time t with the Hamiltonian $\hat{h}$ is equivalent to evolving from the time-reversed state $|b(0)\rangle$ for the time $-t$ with the Hamiltonian $\hat{\mathbf{T}}^{-1}\hat{h}\hat{\mathbf{T}}$. In other words, $\hat{h}$ generates translations in one direction of time and $\hat{\mathbf{T}}^{-1}\hat{h}\hat{\mathbf{T}}$ generates translation is the opposite direction, which is consistent with Eq. (10).

If our model universe satisfied T invariance, $\hat{H}_{\rm F}$ and $\hat{H}_{\rm B}$ would be commuting operators and the terms in Eq. (10) would be able to be manipulated algebraically in exactly the same way as those in Eq. (6). Thus, for the temporal quantum path to be qualitatively distinct from the spatial one, the model universe must violate T invariance to the extent of giving a non zero commutator $[\hat{H}_{\rm F},\hat{H}_{\rm B}]$. We could model such a commutator using details of the T violation that has been observed in the decay of mesons [3–6] or that has been speculated for a Higgs field [17, 18]. However, the potential repercussions of T violation will be manifest most clearly for the simplest departure from time reversal invariance. Accordingly we shall imagine that our model universe contains an unspecified T-violating mechanism that is consistent with the commutator $i[\hat{H}_{\rm F},\hat{H}_{\rm B}] = \lambda$ for real valued λ. This is the origin of the parameter λ that appears in Eq. (8) and Eq. (9).

We have previously [16] shown that the operator on the right side of Eq. (10) can be expanded and reordered using the Zassenhaus formula [19] as follows

$$\begin{aligned}\Big[\exp(i\hat{H}_{\rm B}\delta t) &+ \exp(-i\hat{H}_{\rm F}\delta t)\Big]^N \qquad (11)\\ &= \sum_{n=0}^{N}\exp[i\hat{H}_{\rm B}(N-n)\delta t]\exp(-i\hat{H}_{\rm F}n\delta t)\\ &\times\sum_{v=0}^{m}\cdots\sum_{\ell=0}^{s}\sum_{k=0}^{\ell}\exp\Big[(v+\cdots+\ell+k)(\delta t^2[\hat{H}_F,\hat{H}_B]+\hat{Q})\Big]\end{aligned}$$

where $\hat{Q}$ is of order δt^3, in general. In the specific case here, $[\hat{H}_F,\hat{H}_B] = -i\lambda$, from which it can be shown that $\hat{Q}=0$. After these replacements have been made, the resulting expression can be further simplified using results in Ref. [16] to yield

$$|\Upsilon_\lambda\rangle_N \propto \sum_{n=0}^{N} I_{N-n,n}(\delta t^2\lambda)\exp[i\hat{H}_{\rm B}(N-n)\delta t]\exp[-i\hat{H}_{\rm F}n\delta t]|\phi\rangle \qquad (12)$$

where

$$I_{N-n,n}(z) = \exp[-in(N-n)z/2]\prod_{q=1}^{n}\frac{\sin[(N+1-q)z/2]}{\sin(qz/2)} \qquad (13)$$

is an *interference function* that takes account of the noncommutativity of $\hat{H}_{\rm F}$ and $\hat{H}_{\rm B}$.

To relate this to what an observer in the galaxy would see, imagine that the galaxy contains a clock that is constructed from T-invariant matter. We will refer to any time shown by the clock as "clock time" and use the symbol $t_{\rm c}$ to represent its value. Let the state $|\phi\rangle$ represents the clock showing the time $t_{\rm c}=0$. The state

$$\exp[i\hat{H}_{\rm B}(N-n)\delta t]\exp[-i\hat{H}_{\rm F}n\delta t]|\phi\rangle \qquad (14)$$

represents evolution by $\exp[-i\hat{H}_{\rm F} n\delta t]$ in one direction of time followed by $\exp[i\hat{H}_{\rm B}(N-n)\delta t]$ in the opposite direction which, by convention, first increases $t_{\rm c}$ by $n\delta t$ and then decreases it by $(N-n)\delta t$, respectively. The state in Eq. (14) would therefore represent the clock showing the net clock time of

$$t_{\rm c} = (2n - N)\delta t\ . \tag{15}$$

A. Time reversal invariance

It is useful to first consider the special case where the universe is invariant under time reversal. For this we set $\lambda = 0$, $\hat{H}_{\rm F} = \hat{H}_{\rm B} = \hat{H}$ in Eq. (12). The interference function for $\lambda = 0$ is the binomial coefficient $I_{N-n,n}(0) = \binom{N}{n}$ which is approximately proportional to the Gaussian function $\exp[-(N-2n)^2/2N]$ for large N. Substituting $\lambda = 0$ and $\hat{H}_{\rm F} = \hat{H}_{\rm B} = \hat{H}$ into Eq. (12) and using this result yields

$$|\Upsilon_0\rangle_N \mathrel{\substack{\propto\\\sim}} \sum_{n=0}^{N} \exp[-(N-2n)^2/2N] \exp[i(N-2n)\hat{H}\delta t]|\phi\rangle\ .$$

Re-expressing the summation in terms of the index $m = 2n - N$ and using the definition $\delta t = \sigma_{\rm t}/\sqrt{N}$ then yields

$$|\Upsilon_0\rangle_N \mathrel{\substack{\propto\\\sim}} \sum_{m\in S} \exp[-\frac{(m\delta t)^2}{2\sigma_{\rm t}^2}] \exp(-i\hat{H}m\delta t)|\phi\rangle \tag{16}$$

where $S = \{-N, -N+2, \ldots, N\}$. This state is a Gaussian weighted superposition of the time-translated states $\exp(-i\hat{H}m\delta t)|\phi\rangle$. It represents the galaxy existing in time only for a duration of the order of $\sigma_{\rm t}$ and is analogous to Eq. (2) which represents the centre of mass of the galaxy existing only in a spatial region with a size of the order of $\sigma_{\rm x}$. Our construction, therefore, allows for the same kind of quantum state in time as in space, in the absence of T violation. In other words, there is a symmetry between time and space for quantum states in this special case.

B. Violation of time reversal invariance

Next we examine the quite-different situation of T violation where $\lambda \neq 0$ and $\hat{H}_{\rm F} \neq \hat{H}_{\rm B}$. In that case the amplitudes for different virtual paths to the same point in time, as illustrated in Fig. 2(b), can interfere leading to undulations in $I_{N-n,n}(z)$ as a function of n. To find the values of n where the modulus of the interference function $I_{N-n,n}(z)$ is maximized it is sufficient to look for the position where $|I_{N-n,n}(z)|$ is unchanged for consecutive values of n, i.e. where $|I_{N-(n-1),n-1}(z)| = |I_{N-n,n}(z)|$. This condition reduces, on using Eq. (13) and performing some algebraic manipulation, to $|\sin[(N+1-n)z/2]| = |\sin(nz/2)|$. Note that Eqs. (12) and (13) imply $z = \delta t^2\lambda$ and recalling $\delta t = \sigma_{\rm t}/\sqrt{N}$ shows that z is inversely proportional to N; thus $z = \theta/N$ where

$$\theta = \sigma_{\rm t}^2\lambda$$

is the coefficient of proportionality (i.e. θ is independent of N). Hence we wish to know the values of n that satisfy $|\sin[(N+1-n)\theta/2N]| = |\sin(n\theta/2N)|$. Writing $x = \theta(N+1)/2N$ and $y = n\theta/2N$ transforms this equation into $|\sin(x-y)| = |\sin(y)|$ which has the solutions $y = (x-\pi)/2 + m\pi$ for integer m. Re-expressing the solutions in terms of n then gives

$$n = \frac{N+1}{2} + \frac{N(2m-1)\pi}{\theta}\ .$$

The modulus of the interference function reaches a maximum value at this value of n and one less (i.e for $n-1$). Taking the midpoint and choosing the particular values $m = 0, 1$ then gives the positions of two maxima (or "peaks") at $n = n_\pm$ where

$$n_\pm = N\left(\tfrac{1}{2} \pm \tfrac{\pi}{\theta}\right)\ . \tag{17}$$

Substituting $n_\pm$ for n in Eq. (15) gives the corresponding clock times as

$$\pm\, t_{\rm c}^{(\rm peak)} = (2n_\pm - N)\delta t = \pm\frac{2\pi\sigma_{\rm t}\sqrt{N}}{\theta} \tag{18}$$

where $t_{\rm c}^{(\rm peak)}$ is defined to be positive.

The modulus of the interference function Eq. (13) is shown in Appendix 1 to be approximately Gaussian about these maxima, which allows us to write $|\Upsilon_\lambda\rangle_N$ in Eq. (12) as a superposition of two states as follows:

$$|\Upsilon_\lambda\rangle_N \propto |\Upsilon_\lambda^{(+)}\rangle_N + |\Upsilon_\lambda^{(-)}\rangle_N \tag{19}$$

where

$$|\Upsilon_\lambda^{(\pm)}\rangle_N \propto \sum_{n=0}^{N} f_n^{(\pm)} g_n^{(\pm)} \exp[i\hat{H}_{\rm B}(N-n)\delta t] \exp[-i\hat{H}_{\rm F} n\delta t]|\phi\rangle \tag{20}$$

for $2\pi < \theta < 4\pi$. Here

$$f_n^{(\pm)} = \exp\{-i[n_+n_- - (n-n_\pm)^2]\theta/2N\}\ , \tag{21}$$
$$g_n^{(\pm)} = \exp[-(n-n_\pm)^2|\theta\tan(\theta/4)|/2N] \tag{22}$$

are a complex phase function and Gaussian weighting function, respectively. Figure 3 illustrates the accuracy of the approximation for different values of N. Keeping in mind the definition of the clock time Eq. (15) for the state Eq. (14), we find that $|\Upsilon_\lambda^{(\pm)}\rangle_N$ is a Gaussian-weighted superposition of states over a range of clock times with a mean of $t_{\rm c} = \pm t_{\rm c}^{(\rm peak)}$ and a variance of $(\Delta t_{\rm c})^2 \approx 2/|\lambda\tan(\theta/4)|$. In other words, the states $|\Upsilon_\lambda^{(+)}\rangle_N$ and $|\Upsilon_\lambda^{(-)}\rangle_N$ represent the universe localised in time for a duration of the order of $\Delta t_{\rm c}$ about the mean times $t_{\rm c} = t_{\rm c}^{(\rm peak)}$ and $t_{\rm c} = -t_{\rm c}^{(\rm peak)}$, respectively.

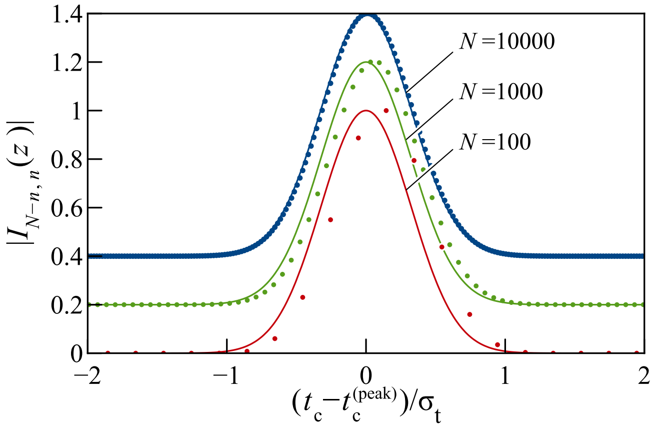

FIG. 3: $|I_{N-n,n}(z)|$ plotted as a function of the scaled clock time $(t_c - t_c^{(peak)})/\sigma_t$ where $t_c = (2n - N)\delta t$. The points $(|I_{N-n,n}(z)|, (t_c - t_c^{(peak)})/\sigma_t)$ are generated parametrically by varying n. The dots represent the exact values from Eq. (13) and the solid curves represent the approximation given by $|f_n^+ g_n^+|$ in Eq. (20). The numerical values used are $z = \theta/N$ where $\theta = 2.23\pi$ and $N = 100$ (red curve), $N = 1000$ (green) and $N = 10000$ (blue). For clarity, the functions have been scaled to give a maximum of unity, and the green ($N = 1000$) and blue ($N = 10000$) curves have been displaced vertically by 0.2 and 0.4, respectively.

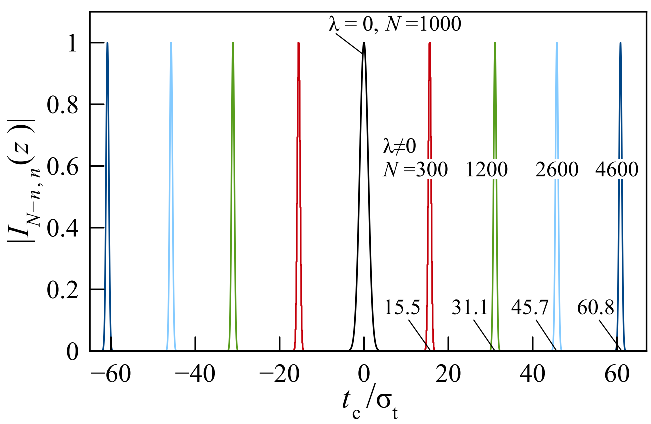

FIG. 4: $|I_{N-n,n}(z)|$ as a function of the scaled clock time t_c/σ_t where $t_c = (2n - N)\delta t$ for different values of λ and N. As in Fig. 3, the points $(|I_{N-n,n}(z)|, t_c/\sigma_t)$ are generated parametrically by varying n. For clarity, in each case straight lines connect consecutive discrete points of $|I_{N-n,n}(z)|$ to form a continuous curve. The black curve represents the T invariant case (i.e. $\lambda = 0$) and has been generated for $N = 1000$. It does not visibly change with increasing values of N. The remaining curves represent the T violation case (i.e. $\lambda \neq 0$) for $\theta = 2.23\pi$ and a range of N values as follows: red curve for $N = 300$ and $t_c^{(peak)} = 15.5\sigma_t$, green curve for $N = 1200$ and $t_c^{(peak)} = 31.1\sigma_t$, light blue curve for $N = 2600$ and $t_c^{(peak)} = 45.7\sigma_t$, and dark blue curve for $N = 4600$ and $t_c^{(peak)} = 60.8\sigma_t$. All curves have been scaled to give a maximum of unity.

The symmetry of the clock times associated with $|\Upsilon_\lambda^{(+)}\rangle_N$ and $|\Upsilon_\lambda^{(-)}\rangle_N$ about the time $t_c = 0$ reflects the symmetry of the construction Eq. (8) and Eq. (9) which has no bias toward one direction of time or the other. Moreover, if the state $|\phi\rangle$ is T invariant (i.e. if $\hat{\mathbf{T}}|\phi\rangle \propto |\phi\rangle$) and we shall assume that it is, then $\hat{\mathbf{T}}|\Upsilon_\lambda^{(+)}\rangle_N \propto |\Upsilon_\lambda^{(-)}\rangle_N$ and $\hat{\mathbf{T}}|\Upsilon_\lambda\rangle_N \propto |\Upsilon_\lambda\rangle_N$. This symmetry also arises in time symmetric cosmological and gravitational studies of the direction of time [20, 21]. As the time evolution in one component of the superposition in Eq. (19) is mirrored in the other, it suffices for us to consider just $|\Upsilon_\lambda^{(+)}\rangle_N$ and its corresponding value of $t_c^{(peak)} = 2\pi\sqrt{N}\sigma_t/\theta$. Accordingly, we will call this value of $t_c^{(peak)}$ the *representative clock time* and use it to label the whole state $|\Upsilon_\lambda\rangle_N$. The minimum representative clock time of a state in the set $\boldsymbol{\Upsilon}_\lambda$ is found, using Eq. (18) with $N = N_{\rm min}^{(\rm time)} = \sigma_t^2/\delta t_{\rm min}^2$ and $\theta = \sigma_t^2\lambda$, to be

$$t_{\rm c,min}^{(\rm peak)} = \frac{2\pi}{\lambda\delta t_{\rm min}} . \tag{23}$$

A discussion of the values of λ and $\delta t_{\rm min}$ in relation to $t_{\rm c,min}^{(\rm peak)}$ is given in Appendix 3.

Figure 4 illustrates the properties of $|I_{N-n,n}(z)|$ as a function of the clock time $t_c = (2n - N)\delta t$. The black curve corresponds to the time reversal invariance case where $\lambda = 0$ (and so $\theta = \sigma_t^2\lambda = 0$). All other curves correspond to the violation of time reversal invariance (i.e. $\lambda \neq 0$) and have been generated for $\theta = 2.23\pi$ which gives the minimum uncertainty in energy and time (see Appendix 2 for details). Clearly the inclusion of the violation of time reversal invariance *dramatically* changes the set $\boldsymbol{\Upsilon}_\lambda$ in (8) from one containing states that are all localised around the same time $t_c = 0$ to one containing states that are diverging in time.

For clarity, $|I_{N-n,n}(z)|$ is plotted in Fig. 4 only for a select few values of N for which the peaks in the corresponding curves are widely separated. To see how close they can be, consider the difference $\delta t_c^{(\rm peak)}$ in the representative clock times $t_c^{(\rm peak)}$ of states $|\Upsilon_\lambda\rangle_N$ with consecutive values of N, which is found from Eq. (18) to be

$$\delta t_c^{(\rm peak)} = \frac{2\pi\sigma_t\sqrt{N+1}}{\theta} - \frac{2\pi\sigma_t\sqrt{N}}{\theta} \approx \frac{\sigma_t\pi}{\theta\sqrt{N}}$$

for large N. Noting that $N \geq N_{\rm min}^{(\rm time)} = \sigma_t^2/\delta t_{\rm min}^2$ gives $\delta t_c^{(\rm peak)} \leq (\pi/\theta)\delta t_{\rm min}$ and as $2\pi < \theta < 4\pi$ we find

$$\delta t_c^{(\rm peak)} < \frac{1}{2}\delta t_{\rm min} .$$

Hence, for any given time $t > t_{\rm c,min}^{(\rm peak)}$, there is a state in the set $\boldsymbol{\Upsilon}_\lambda$ given by Eq. (8) whose representative clock time $t_c^{(\rm peak)}$ is equal to t to within the resolution limit $\delta t_{\rm min}$.

C. Impact for quantum states in time and space

These remarkable results manifest a fundamental difference between quantum states in time and space. Any one element of the set $\mathbf{\Psi}$ in Eq. (5) can represent the state of the universe in space and so, presumably, any one element of the set $\mathbf{\Upsilon}_\lambda$ in Eq. (8) can likewise represent the state of the universe in time. All the states in the set $\mathbf{\Upsilon}_{\lambda=0}$ associated with T invariance represent the galaxy existing only for a duration of order $\sigma_{\rm t}$ near $t_{\rm c}=0$. The fact that the states in the set $\mathbf{\Upsilon}_{\lambda=0}$ don't conserve mass is testament to mass conservation not being an explicit property of the construction defined by Eq. (8) and Eq. (9). But for a set $\mathbf{\Upsilon}_{\lambda'}$ associated with T violation with $\lambda'\neq 0$, for any given time $t\geq t_{\rm c,min}^{(\rm peak)}$ we have just seen that there is a state $|\Upsilon_{\lambda'}\rangle_N\in\mathbf{\Upsilon}_{\lambda'}$, whose representative clock time $t_{\rm c}^{(\rm peak)}$ is equal to t to within the resolution limit $\delta t_{\rm min}$. In other words, the set $\mathbf{\Upsilon}_{\lambda'}$ contains a state that represents the galaxy's existence at each corresponding moment in time. That being the case, it would not be unreasonable to regard the set as representing a *history* of the universe. It follows that the set $\mathbf{\Upsilon}_{\lambda'}$ represents the persistence of the mass of the galaxy over the same period of time, in so far as the Hamiltonians $\hat{H}_F$ and $\hat{H}_B$ conserve mass. This raises a subtle point regarding conservation laws; while they may be due to deep principles (such as Noether's theorem) they are not manifested in quantum mechanics unless the state persists over a period of time. The crucial point being that in conventional quantum mechanics, the persistence of the state is essentially *axiomatic* and ensured by adopting a compliant dynamical equation of motion whereas here it arises *phenomenologically* as a property of the set of states $\mathbf{\Upsilon}_{\lambda'}$. Finally, on comparing the two sets $\mathbf{\Upsilon}_{\lambda=0}$ and $\mathbf{\Upsilon}_{\lambda'\neq 0}$ one could even venture to say that T violation, in effect, *causes* the contents of the universe to be translated or, indeed, to *evolve*, over an unbounded period of time.

IV. EMERGENCE OF CONVENTIONAL QUANTUM MECHANICS

A. Coarse graining over time

The spread of the state $|\Upsilon_\lambda\rangle_N$ along the time axis, as illustrated by the plots of $|I_{N-n,n}(z)|$ in Fig. 4, represents a significant departure from conventional quantum mechanics for which states have no extension in time. Nevertheless, the conventional formalism can recovered in the following way. Imagine that observations of the galaxy are made with a resolution in time that is much larger than the width of the Gaussian weighting function $g_n^\pm$ in Eq. (20). Under such coarse graining, the summation in Eq. (20) can be replaced by the term corresponding to the maximum in $g_n^\pm$ and so, for example,

$$|\Upsilon_\lambda^{(+)}\rangle_N \propto\approx \exp[i\hat{H}_{\rm B}(N-n_+)\delta t]\exp(-i\hat{H}_{\rm F}n_+\delta t)|\phi\rangle\ .$$

We can re-express this state in terms of its representative clock time, $t_{\rm c}^{(\rm peak)}$, which we shall shorten to $t_{\rm c}$ for brevity, as

$$|\Upsilon_\lambda^{(+)}\rangle_N \propto\approx \exp(i\hat{H}_{\rm B}t_{\rm c}a_-)\exp(-i\hat{H}_{\rm F}t_{\rm c}a_+)|\phi\rangle \qquad (24)$$

where $a_\pm=n_\pm/(n_+-n_-)$ and we have used $t_{\rm c}=(2n_+-N)\delta t=(n_+-n_-)\delta t$ and $N-n_+=n_-$. At this level of coarse graining, the time step δt is effectively zero and $t_{\rm c}$ is effectively a continuous variable. Making use of the Baker-Campbell-Hausdorff formula [19] in Eq. (24) yields

$$|\Upsilon_\lambda^{(+)}\rangle_N \propto\approx \exp(\tfrac{1}{2}ia_+a_-t_{\rm c}^2\lambda)\exp[-i(\hat{H}_{\rm F}a_+-\hat{H}_{\rm B}a_-)t_{\rm c}]|\phi\rangle\ . \qquad (25)$$

The complex phase factor can be accommodated by transforming to a new state, $|\widetilde{\Upsilon}(t_{\rm c})\rangle$, as follows

$$\begin{aligned}|\widetilde{\Upsilon}(t_{\rm c})\rangle &= \exp(-\tfrac{1}{2}ia_+a_-t_{\rm c}^2\lambda)|\Upsilon_\lambda^{(+)}\rangle_N\\ &\propto \exp[-i(\hat{H}_{\rm F}a_+-\hat{H}_{\rm B}a_-)t_{\rm c}]|\phi\rangle\ . \qquad (26)\end{aligned}$$

On taking the derivative with respect to $t_{\rm c}$ we recover Schrödinger's equation,

$$\frac{d}{dt_{\rm c}}|\widetilde{\Upsilon}(t_{\rm c})\rangle \propto\approx -i(\hat{H}_{\rm F}a_+-\hat{H}_{\rm B}a_-)|\widetilde{\Upsilon}(t_{\rm c})\rangle\ . \qquad (27)$$

Here, the coarse-grained Hamiltonian $(\hat{H}_{\rm F}a_+-\hat{H}_{\rm B}a_-)$ is a linear combination of $\hat{H}_{\rm F}$ and $\hat{H}_{\rm B}$ owing to the fact that the quantum path involves contributions from both. It is useful at this point to divide the galaxy into two non-interacting subsystems, one whose Hamiltonian $\hat{H}^{(\rm i)}=\hat{\mathbf{T}}^{-1}\hat{H}^{(\rm i)}\hat{\mathbf{T}}$ is T-invariant and the remainder whose Hamiltonian $\hat{H}_{\rm F}^{(\rm v)}=\hat{\mathbf{T}}^{-1}\hat{H}_{\rm B}^{(\rm v)}\hat{\mathbf{T}}\neq\hat{H}_{\rm B}^{(\rm v)}$ is T-violating; in that case we can write

$$\hat{H}_{\rm F}=\hat{H}^{(\rm i)}\otimes\hat{\mathbf{1}}^{(\rm v)}+\hat{\mathbf{1}}^{(\rm i)}\otimes\hat{H}_{\rm F}^{(\rm v)}\ ,\ \hat{H}_{\rm B}=\hat{H}^{(\rm i)}\otimes\hat{\mathbf{1}}^{(\rm v)}+\hat{\mathbf{1}}^{(\rm i)}\otimes\hat{H}_{\rm B}^{(\rm v)} \qquad (28)$$

where the superscripts "i" and "v" label operators associated with the state space of the T-invariant and T-violating Hamiltonians, respectively, and $\hat{\mathbf{1}}^{(\cdot)}$ is an appropriate identity operator. Equation Eq. (27) can then be rewritten as

$$\frac{d}{dt_{\rm c}}|\widetilde{\Upsilon}(t_{\rm c})\rangle \propto\approx -i(\hat{H}^{(\rm i)}\otimes\hat{\mathbf{1}}^{(\rm v)}+\hat{\mathbf{1}}^{(\rm i)}\otimes\hat{H}_{\rm phen}^{(\rm v)})|\widetilde{\Upsilon}(t_{\rm c})\rangle \qquad (29)$$

where $\hat{H}_{\rm phen}^{(\rm v)}=\hat{H}_{\rm F}^{(\rm v)}a_+-\hat{H}_{\rm B}^{(\rm v)}a_-$ is the phenomenological Hamiltonian for the T-violating subsystem.

It is straightforward to show that the commutator of $\hat{H}_{\rm phen}^{(\rm v)}=\hat{H}_{\rm F}^{(\rm v)}a_+-\hat{H}_{\rm B}^{(\rm v)}a_-$ with its time reversed version is

$$[\hat{H}_{\rm phen}^{(\rm v)},\hat{\mathbf{T}}^{-1}\hat{H}_{\rm phen}^{(\rm v)}\hat{\mathbf{T}}]=-i\frac{\theta}{2\pi}\lambda$$

which is $\theta/2\pi$ times the commutator $[\hat{H}_{\rm F}^{(\rm v)},\hat{H}_{\rm B}^{(\rm v)}]$. Thus, in principle, the commutation relation could be used to distinguish the phenomenological Hamiltonians $\hat{H}_{\rm phen}^{(\rm v)}$ and $\hat{\mathbf{T}}^{-1}\hat{H}_{\rm phen}^{(\rm v)}\hat{\mathbf{T}}$ from the more elementary versions $\hat{H}_{\rm F}^{(\rm v)}$ and $\hat{H}_{\rm B}^{(\rm v)}$.

B. Conventional formalism and potential experimental test

These results are important because they not only show how the conventional formalism of quantum mechanics is recovered, but they also show how the construction introduced here may be verified experimentally. To see this consider the following three points. First, Eq. (29) shows that the T-invariant subsystem behaves in accord with the conventional Hamiltonian $\hat{H}^{(\mathrm{i})}$ with respect to clock time t_c. This means that conventional quantum mechanics is recovered for this subsystem. Second, Eq. (29) shows that, due to the coarse graining, the role of the clock time t_c has been reduced from being a physical variable that describes the location and uncertainty of the galaxy with respect to time as illustrated in Fig. 4, to being simply a parameter that labels a different state in the set $\boldsymbol{\Upsilon}_\lambda$ according to the time $t_\mathrm{c}^{(\mathrm{peak})} = t_\mathrm{c}$ of the maximum in g_n^+. Indeed, its demoted role is the very reason we are able to recover Schrödigner's equation. Third, any experiments involving T-violating matter that are performed by observers in the galaxy would give results that are consistent with Eq. (29) and so they would provide evidence of the phenomenological Hamiltonian $\hat{H}^{(\mathrm{v})}_{\mathrm{phen}}$ in the same way that experiments in our universe give evidence of the Hamiltonian associated with meson decay. Any demonstration that $\hat{H}^{(\mathrm{v})}_{\mathrm{phen}}$ differs from the more elemental Hamiltonians $\hat{H}^{(\mathrm{v})}_\mathrm{F}$ and $\hat{H}^{(\mathrm{v})}_\mathrm{B}$ represents a "smoking gun" that verifies the construction introduced here. Of course, this specific result can not be used in practice because it applies to the simple model of T violation chosen here for its clarity rather than accuracy, and also because present knowledge of T violating Hamiltonians is based on empirical results and so it is limited to the phenomenological version of the Hamiltonians. More realistic models of the universe and T violating mechanisms may provide experimentally testable predictions, such as novel deviations from exponential decay for T violating matter or local variations in clock time. But these are beyond the scope of the present work whose aim is to show, in the clearest way possible, how T violation may underlie differences between time and space.

V. DISCUSSION

We began by noting that conventional quantum mechanics assumes an asymmetry between space and time to the extent that space is associated with a quantum description whereas time is treated as a classical parameter. We set out to explore an alternate possibility by introducing a new quantum formalism that gives both space and time analogous quantum descriptions. In developing the formalism, we paid particular attention to subtle mathematical details that play no significant role in conventional quantum mechanics. These details involve explicitly taking into account the C, P and T symmetry operations, translations of states in space and time, and fundamental limits of precision. We incorporated them in a mathematical construction where quantum states are represented as a superposition of random paths in space or time. We found that with no C, P or T symmetry violations, quantum states had analogous representations in space and time. However with the violation of T symmetry, dramatic differences between the representation of quantum states in space and time arise through the quantum interference between different paths. The Schrödinger equation of conventional quantum mechanics, where time is reduced to a classical parameter, then emerges as a result of coarse graining over time. As such, T violation is seen in the new formalism as being responsible for the differences between space and time in conventional quantum mechanics.

The new formalism may also help resolve other perplexing issues associated with space and time. For example, the arrows of time indicate a preferred direction from "past" to "future" [22], but there is no analogous preferred direction of space. The new formalism appears to offer a basis for understanding why. Indeed the set of states in time, $\boldsymbol{\Upsilon}_\lambda$ for $\lambda \neq 0$ in Eq. (8), has a natural order over time in the following sense. First recall that our interpretation of Eq. (10) is that $\exp(-i\hat{H}_\mathrm{F}t)$ and $\exp(i\hat{H}_\mathrm{B}t)$ are associated with physical evolution in different directions of time, whereas the inverses $\exp(i\hat{H}_\mathrm{F}t)$ and $\exp(-i\hat{H}_\mathrm{B}t)$ are associated with the mathematical operations of rewinding that physical evolution. Within this context, the coarse-grained state $|\widetilde{\Upsilon}(t_\mathrm{c})\rangle$ in Eq. (26) is interpreted as resulting from evolution by $t_\mathrm{c}a_+$ in the positive direction of time and $t_\mathrm{c}a_-$ in the reverse direction, giving a net evolution of $t_\mathrm{c}(a_+ - a_-) = t_\mathrm{c}$ in time from the state $|\phi\rangle$. Correspondingly, the state $|\widetilde{\Upsilon}(t'_\mathrm{c})\rangle$ with $t'_\mathrm{c} > t_\mathrm{c}$ represents a *more-evolved state* than $|\widetilde{\Upsilon}(t_\mathrm{c})\rangle$. In fact writing

$$|\widetilde{\Upsilon}(t'_\mathrm{c})\rangle \propto \exp[-i(\hat{H}_\mathrm{F}a_+ - \hat{H}_\mathrm{B}a_-)\delta t]|\widetilde{\Upsilon}(t_\mathrm{c})\rangle \tag{30}$$

where $\delta t = t'_\mathrm{c} - t_\mathrm{c} > 0$ shows that $|\widetilde{\Upsilon}(t'_\mathrm{c})\rangle$ evolves from $|\widetilde{\Upsilon}(t_\mathrm{c})\rangle$. One might be tempted to argue that we could equally well regard $|\widetilde{\Upsilon}(t_\mathrm{c})\rangle$ as evolving from $|\widetilde{\Upsilon}(t'_\mathrm{c})\rangle$ because

$$|\widetilde{\Upsilon}(t_\mathrm{c})\rangle \propto \exp[i(\hat{H}_\mathrm{F}a_+ - \hat{H}_\mathrm{B}a_-)\delta t]|\widetilde{\Upsilon}(t'_\mathrm{c})\rangle\ , \tag{31}$$

but doing so would be inconsistent with our interpretation of Eq. (10). According to that interpretation, Eq. (31) represents the mathematical *rewinding* of the physical evolution represented by Eq. (30). Note that the state $|\widetilde{\Upsilon}(t_\mathrm{c})\rangle$ is a coarsed-grained version of the component $|\Upsilon^{(+)}_\lambda\rangle_N$ of $|\Upsilon_\lambda\rangle_N$ in Eq. (19); an analogous argument also applies to the coarse-grained version of the other component $|\Upsilon^{(-)}_\lambda\rangle_N$, and thus to the whole state $|\Upsilon_\lambda\rangle_N$. Hence, the set of states $\boldsymbol{\Upsilon}_\lambda$ for $\lambda \neq 0$ are ordered by the degree of time evolution from the state $|\phi\rangle$. This gives two preferred directions of time away from the origin of the time axis and so represents a *symmetric arrow* *of time*. Time-symmetric arrows have also been explored by Carroll, Barbour and co-workers [20, 21]. In stark contrast, there is no analogous ordering for the set, $\mathbf{\Psi}$ in Eq. (5), of states distributed over space. Indeed, all the states in $\mathbf{\Psi}$ are physically indistinguishable. Also the ordering of the set $\mathbf{\Upsilon}_\lambda$ vanishes at $\lambda = 0$ which corresponds to T symmetry. It appears, therefore, that T violation is also responsible giving time a direction (in the sense of orientating time away from the occurrence of $|\phi\rangle$).

In addition to these conceptual results, the new formalism was also found to have potential experimentally-testable consequences. Indeed, for a subsystem associated with T violation, the formalism predicts that the experimentally-determined Hamiltonian, $\hat{H}^{(\mathrm{v})}_{\mathrm{phen}}$ in Eq. (29), will be different to the Hamiltonians, $\hat{H}^{(\mathrm{v})}_{\mathrm{F}}$ or $\hat{H}^{(\mathrm{v})}_{\mathrm{B}}$ in Eq. (28), associated with conventional quantum mechanics. Further work is needed to develop feasible experiments for testing predicted departures from conventional theory like this. An experimental verification of the new formalism would have profound impact on our understanding of time.

In conclusion, the importance of Feynman's sums over paths for describing quantum phenomena is well beyond doubt [11]. A distinctive feature of the quantum paths in the new formalism is that they explicitly take into account the violation of the C, P and T symmetries. The new formalism has the advantage of giving *time and space an equal footing at a fundamental level* while allowing familiar differences, such as matter being localised in space but undergoing unbounded evolution in time, to arise *phenomenologically* due to the fact that violations of the C, P and T symmetries are a property of translations in time and not space. As such, the violation of the discrete symmetries are seen to play a defining role in the quantum nature of time and space.

Acknowledgment

The author thanks D.T. Pegg, H.M. Wiseman, M.J. Hall and T. Croucher for helpful discussions.

Appendix

1. Approximate shape of the maxima

An approximate form of the interference function near its maxima can be found by retaining terms of order $1/\sqrt{N}$ or larger as follows. Substituting $n = n_\pm + k$, where k is an integer, into Eq. (13) and using $N - n_\pm = n_\mp$ gives

$$I_{N-n,n}(z) = |I_{N-n_\pm,n_\pm}(z)| \exp[-i(n_\pm + k)(n_\mp - k)z/2] \prod_{r=1}^{k} \frac{\sin[(N+1-r-n_\pm)z/2]}{\sin[(r+n_\pm)z/2]} \ . \tag{32}$$

Next, substituting for $n_\pm$ and z using Eq. (17) and $z = \theta/N$, respectively, using trigonometric identities and performing some algebraic manipulations eventually shows that the iterated product in Eq. (32) can be written as

$$\prod_{r=1}^{k} \frac{\sin[(N+1-r-n_\pm)z/2]}{\sin[(r+n_\pm)z/2]} = (-1)^k \prod_{r=1}^{k} \frac{\cos(A)\cos(B) + \sin(A)\sin(B)}{\cos(A)\cos(C) - \sin(A)\sin(C)} \tag{33}$$

with $A = \theta/4$, $B = (r-1)\theta/2N$ and $C = r\theta/2N$. Note that k represents the number of steps in time, each of duration $\delta t = \sigma_{\mathrm{t}}/\sqrt{N}$, from the position of the maximum. To retain only features that are a finite distance in time from the maximum in the limit $N \to \infty$, we need to keep terms where k, and thus r, are of order $\sqrt{N}$. It follows that in Eq. (33) we can use the approximations $\cos(B) \approx 1$, $\cos(C) \approx 1$, $\sin(B) \approx B$ and $\sin(C) \approx C$ to first order in $1/\sqrt{N}$ and so

$$\prod_{r=1}^{k} \frac{\sin[(N+1-r-n_\pm)z/2]}{\sin[(r+n_\pm)z/2]} \approx (-1)^k \prod_{r=1}^{k} \frac{1 + B\tan(A)}{1 - C\tan(A)} \ .$$

As $B \approx C \approx r\theta/2N \ll 1$ to the same order of approximation, we can further approximate this as

$$\prod_{r=1}^{k} \frac{\sin[(N+1-r-n_\pm)z/2]}{\sin[(r+n_\pm)z/2]} \approx (-1)^k \prod_{r=1}^{k} \exp\left[\frac{r\theta}{N}\tan\left(\frac{\theta}{4}\right)\right] \ ,$$

and then on ignoring a term in the exponent of order $1/N$ we eventually find

$$\prod_{r=1}^{k} \frac{\sin[(N+1-r-n_\pm)z/2]}{\sin[(r+n_\pm)z/2]} \approx (-1)^k \exp\left[\frac{k^2\theta}{2N}\tan\left(\frac{\theta}{4}\right)\right] .$$

Substituting into Eq. (32) then gives

$$I_{N-n,n}(z) = |I_{N-n_\pm,n_\pm}(z)| \exp\left[-i(n_-n_+ - k^2)\frac{\theta}{2N}\right] \exp\left[\frac{k^2\theta}{2N}\tan\left(\frac{\theta}{4}\right)\right] .$$

The right-most factor is a Gaussian function of k provided $\theta\tan(\theta/4)$ is negative. To ensure that this is the case we set $2\pi < \theta < 4\pi$. It follows from $N - n_+ = n_-$ and the symmetry property $I_{N-m,m}(z) = I_{m,N-m}(z)$ that $I_{N-n_+,n_+}(z) = I_{N-n_-,n_-}(z)$. Thus, noting $k = n - n_\pm$, we find

$$I_{N-n,n}(z) \propto f_n^{(+)} g_n^{(+)} + f_n^{(-)} g_n^{(-)}$$

where $f_n^{(\pm)}$ and $g_n^{(\pm)}$ are defined by Eq. (21) and Eq. (22), respectively. Substituting this result into Eq. (12) then leads to Eq. (19).

2. Minimum uncertainty in energy and time

We have defined t_c as the time measured by clock devices that are constructed from T-invariant matter to avoid any difficulties that might arise in defining clocks that are constructed from T-violating matter. However, for the particular case here where $i[\hat{H}_\mathrm{F}, \hat{H}_\mathrm{B}] = \lambda$ there are no such difficulties and a clock constructed from both T-invariant and T-violating matter will consistently register the same clock time t_c irrespective of the path, and the value of t_c will be the same as for a clock that is entirely constructed from T-invariant matter. To see this, consider the two paths represented by $\hat{A}\hat{B}|\phi\rangle$ and $\hat{B}\hat{A}|\phi\rangle$ where $\hat{A} = \exp(-i\hat{H}_\mathrm{F} n\delta t)$ and $\hat{B} = \exp[i\hat{H}_\mathrm{B}(N-n)\delta t]$. It is straightforward to show using the Baker-Campbell-Hausdorff formula that [19]

$$\hat{A}\hat{B}|\phi\rangle = \exp[-i\lambda n(N-n)\delta t^2]\hat{B}\hat{A}|\phi\rangle \tag{34}$$

and so both paths result in the same state apart from a complex phase factor. If we regard the whole universe as being a device that registers clock time t_c and if $|\phi\rangle$ represents $t_\mathrm{c} = 0$ then Eq. (34) implies that both $\hat{A}\hat{B}|\phi\rangle$ and $\hat{B}\hat{A}|\phi\rangle$ represent it registering the clock time $t_\mathrm{c} = (2n - N)\delta t$. The clock time t_c is therefore representative of the whole universe in this case.

Although we do not have an operator corresponding to the clock time t_c, we can still estimate the uncertainty in t_c for the state $|\Upsilon_\lambda^{(\pm)}\rangle_N$ using the following heuristic argument. The sum over n in Eq. (20) means that, in addition to any intrinsic uncertainty in the time represented by clocks due to the state $|\phi\rangle$, there is an additional contribution due to the finite width of the Gaussian weighting function $g_n^{(\pm)}$. In fact, taking into account the relationship $t_\mathrm{c} = (2n-N)\delta t$, the variance in possible clock time values t_c will be at least $(2\delta t)^2$ times the variance in n due to $|g_n^{(\pm)}|^2$. Thus we can bound the uncertainty in clock time as $(\Delta t_\mathrm{c})^2 \gtrsim 4(\Delta n)^2\delta t^2$ where $(\Delta n)^2 = N/2|\theta\tan(\theta/4)|$, and so using $\delta t = \sigma_\mathrm{t}/\sqrt{N}$ and $\theta = \sigma_\mathrm{t}^2\lambda$ we find

$$(\Delta t_\mathrm{c})^2 \gtrsim \frac{2}{|\lambda\tan(\theta/4)|} \tag{35}$$

where $(\Delta t_\mathrm{c})^2 = \overline{t_\mathrm{c}^2} - \overline{t_\mathrm{c}}^2$ for averages calculated using $|g_n^{(\pm)}|^2$ as the probability distribution.

The variance in Eq. (35) depends on the value of θ. Rather than use any value in the allowed range $2\pi < \theta < 4\pi$, it would be useful to have one that has a particular physical meaning. One such value corresponds to minimal uncertainties in t_c, H_F and H_B. The first step in finding it is to use the Robertson-Schrödinger uncertainty relation [23] for the Hamiltonians $\hat{H}_\mathrm{F}$ and $\hat{H}_\mathrm{B}$:

$$(\Delta H_\mathrm{F})^2(\hat{H}_\mathrm{B})^2 \geq \tfrac{1}{4}|\langle\{\hat{H}_\mathrm{F}, \hat{H}_\mathrm{B}\}\rangle - 2\langle\hat{H}_\mathrm{F}\rangle\langle\hat{H}_\mathrm{B}\rangle|^2 + \tfrac{1}{4}|\langle[\hat{H}_\mathrm{F}, \hat{H}_\mathrm{B}]\rangle|^2$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ and $(\Delta A)^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$ is the variance in $\hat{A}$. As $[\hat{H}_\mathrm{F}, \hat{H}_\mathrm{B}] = -i\lambda$, the minimum of the right side occurs when the covariance is zero:

$$\langle\{\hat{H}_\mathrm{F}, \hat{H}_\mathrm{B}\}\rangle - 2\langle\hat{H}_\mathrm{F}\rangle\langle\hat{H}_\mathrm{B}\rangle = 0 . \tag{36}$$

Thus the minimum uncertainty is given by

$$(\Delta H_{\rm F})^2(\Delta H_{\rm B})^2 = \frac{|\lambda|^2}{4} \ . \tag{37}$$

With no bias towards one direction of time or the other, there is correspondingly no bias towards one version of the Hamiltonian or the other and so we take the minimum uncertainty condition for the energy as

$$(\Delta H_{\rm F})^2 = (\Delta H_{\rm B})^2 = \frac{|\lambda|}{2} \ . \tag{38}$$

Next we need to determine the relationship between $\Delta t_{\rm c}$, $\Delta H_{\rm F}$ and $\Delta H_{\rm B}$. Unfortunately, there has not been any previous study of the time-energy uncertainty relation for the case of T violation where two versions of the Hamiltonian operate, and it would be beyond the scope of this work to analyse it in detail here. Nevertheless, we can glean some insight into the problem as follows. Consider the operator defined by

$$\hat{H} = \tfrac{1}{2}(\hat{H}_{\rm F} + \hat{H}_{\rm B});$$

it is straightforward to show that $\hat{\mathbf{T}}^{-1}\hat{H}\hat{\mathbf{T}} = \hat{H}$ and so $\hat{H}$ is T invariant. We know from the discussion of Eq. (34) that the state $|\psi\rangle = \exp(-i\hat{H}_{\rm F}t_1)\exp[i\hat{H}_{\rm B}t_2]|\phi\rangle$ represents the time $t_{\rm c} = t_1 - t_2$. Rearranging using the Baker-Campbell-Hausdorff formula [19] shows that $|\psi\rangle \propto \exp[-i\hat{H}_{\rm F}(t_1+t_2)]\exp[i\hat{H}2t_2]|\phi\rangle$ and so it follows that the state $\exp[i\hat{H}2t_2]|\phi\rangle$ represents the time $t_{\rm c} = -2t_2$. Thus $\hat{H}$ is clearly a generator of translations in time. A similar argument shows that the operator $\frac{1}{2}(\hat{H}_{\rm F} - \hat{H}_{\rm B})$ does not generate translations in time. This implies that there is a meaningful uncertainty relation for the clock time $t_{\rm c}$ and $\hat{H}$. The uncertainty in $\hat{H}$ is related to that of $H_{\rm F}$ and $H_{\rm B}$ by

$$(\Delta H)^2 = \tfrac{1}{4}\left[(\Delta H_{\rm F})^2 + (\Delta H_{\rm B})^2 + \langle\{\hat{H}_{\rm F}, \hat{H}_{\rm B}\}\rangle - 2\langle\hat{H}_{\rm F}\rangle\langle\hat{H}_{\rm B}\rangle\right] \ ,$$

and if Eq. (38) holds, then so does Eq. (36) and we find

$$(\Delta H)^2 = \frac{|\lambda|}{4} \ . \tag{39}$$

It is easy to calculate the product of variances in $t_{\rm c}$ and $\hat{H}$ for a state like Eq. (16). In the limit $N \to \infty$ the sum over m in Eq. (16) becomes an integral over $t = m\delta t$ and so

$$\lim_{N\to\infty} |\Upsilon_0\rangle_N \propto \int dt \exp(-\frac{t^2}{2\sigma_{\rm t}^2})\exp(-i\hat{H}t)|\phi\rangle \tag{40}$$

which is the temporal analogy of Eq. (2). Replacing $\sigma_{\rm t}^2$ with $2(\Delta t_{\rm c})^2$ and performing the integral in Eq. (40) in the eigenbasis of $\hat{H}$ yields

$$\lim_{N\to\infty} |\Upsilon_0\rangle_N \propto \exp[-\hat{H}^2(\Delta t_{\rm c})^2]|\phi\rangle \ .$$

The state $|\phi\rangle$ is assumed to have a large variance in energy; in the limit that $|\phi\rangle$ is a uniform superposition of the eigenstates of $\hat{H}$, the probability distribution for $\hat{H}$ for the state on the right side becomes a truncated Gaussian [24] with a variance of $(\Delta H)^2 \approx (1 - 2/\pi)/4(\Delta t_{\rm c})^2$. Hence an approximate energy-time uncertainty relation for this particular class of states is

$$(\Delta H)^2(\Delta t_{\rm c})^2 \approx \frac{1}{4}(1 - \frac{2}{\pi}) \ .$$

We will presume that this result also applies to the states $|\Upsilon_\lambda^{(\pm)}\rangle_N$ in Eq. (20) without significant modification. In that case using Eq. (39) to replace $(\Delta H)^2$ gives

$$(\Delta t_{\rm c})^2 \approx \frac{1}{|\lambda|}(1 - \frac{2}{\pi}) \ . \tag{41}$$

This gives the least uncertainty in clock time for the case where the uncertainty in energy is minimized; to be clear, the uncertainty in $\Delta t_{\rm c}$ can be smaller than that given by Eq. (41) provided the uncertainty in energy is higher than the minimum represented by the equality in Eq. (37). Comparing Eq. (35) and Eq. (41) and keeping in mind that $2\pi < \theta < 4\pi$ shows that the minimum uncertainty in energy and time is given by $\tan(\theta/4) \approx -2/(1 - 2/\pi)$, i.e. for $\theta \approx 2.23\pi$.

Note that the uncertainty $\Delta t_{\rm c}$ is for each of the components $|\Upsilon_\lambda^{(\pm)}\rangle_N$ and not for the whole state $|\Upsilon_\lambda\rangle_N$ in Eq. (19). This uncertainty is appropriate from the point of view of an observer within the galaxy for whom the states $|\Upsilon_\lambda^{(\pm)}\rangle_N$ equally describe the state of the universe up to the symmetry given by $\hat{\mathbf{T}}|\Upsilon_\lambda^{(+)}\rangle_N \propto |\Upsilon_\lambda^{(-)}\rangle_N$.

3. Quantifying the T violation

The minimum representative clock time $t_{\rm c,min}^{\rm (peak)}$ for the set $\boldsymbol{\Upsilon}_\lambda$ defined in Eq. (23) and the uncertainty in the clock time $\Delta t_{\rm c}$ defined in Eq. (35) give important physical parameters. To estimate their values we need to quantify the minimum physically resolvable time given by $\delta t_{\rm min}$ and the degree of T violation represented by the value of λ. The Planck time, $t_{\rm P}=5.4\times 10^{-44}$ s, is widely used as the minimum resolvable time and so we will adopt it here and set $\delta t_{\rm min}=t_{\rm P}$.

Quantifying λ is a rather more difficult. One possibility is to assume that it has of the same order of magnitude as that of meson decay in our universe. The eigenvalue spectrum of the commutator $i[\hat{H}_{\rm F},\hat{H}_{\rm B}]$ for meson decay has been estimated to have a Gausian distribution with a mean of zero and a standard deviation of $\sqrt{f}\times 10^{57}$ s^{-2} where f is the fraction of the estimated 10^{80} particles in the visible universe that contribute to kaon-like T violation [16]. Accordingly we set $\lambda=\sqrt{f}\times 10^{57}$ s^{-2}. Using Eq. (23) with these values of λ and $\delta t_{\rm min}$ then gives the minimum representative clock time as

$$t_{\rm c,min}^{\rm (peak)}\approx f^{-1/2}\times 10^{-13}\ {\rm s}\ .$$

Thus Eq. (27) and Eq. (29) describe the coarse-grained time evolution of the model universe from this time onwards. The corresponding value of the uncertainty in the clock time $\Delta t_{\rm c}$ is, from Eq. (41),

$$\Delta t_{\rm c}\approx f^{-1/4}\times 10^{-29}\ {\rm s}\ .$$

Another way to quantify λ is to treat it as if its value is chosen by nature in order that the minimum representative clock time is equal to the minimum time resolution, i.e. to make $t_{\rm c,min}^{\rm (peak)}=\delta t_{\rm min}$. In that case we find, using Eq. (23), that

$$\lambda=\frac{2\pi}{\delta t_{\rm min}^2} \tag{42}$$

which becomes $\lambda\approx 10^{87}$ s^{-2} for $\delta t_{\rm min}=t_{\rm P}$. Then using Eq. (41) we find the corresponding uncertainty in the clock time is

$$\Delta t_{\rm c}\approx \tfrac{1}{4}\delta t_{\rm min}\ . \tag{43}$$

This represents the most extreme situation where Eq. (27) and Eq. (29) describe the coarse-grained time evolution for all times from $t_{\rm c}=0$ and the uncertainty in clock time is undetectable.

Finally, we should add that any non-zero value of λ will give rise to the qualitative behaviour described in the main text. However, according to Eq. (23), as the value of λ approaches zero, the minimum representative clock time $t_{\rm c,min}^{\rm (peak)}$ becomes correspondingly large and so the results are confined to ever larger times $t_{\rm c}^{\rm (peak)}$.

[1] Lee TD, Yang CN. 1956 Question of parity conservation in weak interactions. *Phys. Rev.* **104**, 254–258.

[2] Christenson JH, Cronin JW, Fitch VL, Turlay, R. 1964 Evidence for the 2π decay of the ${K_2}^0$ meson. *Phys. Rev. Lett.* **13**, 138–140.

[3] Angelopoulos A, *et al.* 1998 First direct observation of time-reversal non-invariance in the neutral-kaon system. *Phys. Lett. B* **444**, 43–51.

[4] Sehgal LM, van Leusen J, 1999 Violation of Time Reversal Invariance in the Decays $K_L\to\pi^+\pi^-\gamma$ and $K_L\to\pi^+\pi^-e^+e^-$. *Phys. Rev. Lett.* **83**, 4933–4936.

[5] Alvarez-Gaume L, Kounnas C, Lola S, Pavlopoulos P. 1999 Violation of time-reversal invariance and CPLEAR measurements. *Phys. Lett. A* **458**, 347–354.

[6] Lees JP, *et al.* 2012 Observation of time-reversal violation in the B^0 meson system. *Phys. Rev. Lett.* **109**, 211801.

[7] Cabibbo N, 1963 Unitary Symmetry and Leptonic Decays. *Phys. Rev. Lett.* **10**, 531–533.

[8] Kobayashi M, Maskawa T, 1973 CP-Violation in the Renormalizable Theory of Weak Interaction. *Prog. Theor. Phys.* **49**, 652–657.

[9] Sakharov AD. 1967 Violation of CP Symmetry, C Asymmetry and Baryon Asymmetry of the Universe. *JETP Lett.* **5**, 24–27.

[10] Venegas-Andraca SE. 2012 Quantum walks: a comprehensive review. *Quantum Information Processing* **11**, 1015–1106.

[11] Feynman RP. 1948 Space-time approach to non-relativistic quantum mechanics. *Rev. Mod. Phys.* **20**, 367–387.

[12] Goldstein H, Poole C, Safko J. 2002 *Classical mechanics*, 3rd edn. San Francisco, CA: Addison-Wesley.
[13] Amelino-Camelia G. 1999 Gravity-wave interferometers as quantum-gravity detectors. *Nature* **398**, 216–218.
[14] Wigner EP. 1959 *Group theory and its application to the quantum mechanics of atomic spectra*, New York, NY: Academic Press.
[15] Pegg DT. 1991 Time in a quantum mechanical world. *J.Phys. A* **24**, 3031–3040.
[16] Vaccaro JA. 2011 *Found. Phys.* **41**, 1569–1596.
[17] Lee TD. 1974 CP nonconservation and spontaneous symmetry breaking. *Phys. Rep.* **9**, 143–177.
[18] Ivanov IP, Lavoura L. 2013 Geometrical CP violation in the N-Higgs-doublet model. *Eur. Phys. J. C* **73**, 2416.
[19] Suzuki M. 1977 On the convergence of exponential operators—the Zassenhaus formula, BCH formula and systematic approximants. *Commun. Math. Phys.* **57**, 193–200.
[20] Carroll SM, Chen J. 2004 Spontaneous Inflation and the Origin of the Arrow of Time. arXiv:hep-th/0410270.
[21] Barbour J, Koslowski T, Mercati F. 2014 Identification of a Gravitational Arrow of Time. *Phys. Rev. Lett.* **113**, 181101.
[22] Price H. 1996 *Time's Arrow and Archimedes' Point*, New York, NY: Oxford Uni. Press.
[23] Robertson HP. 1934 An Indeterminacy relation for several observables and its classical interpretation. *Phys. Rev.* **46**, 794–801.
[24] Barr DR, Sherrilla ET. 1999 Mean and variance of truncated normal distributions. *Am. Stat.* **53**, 357–361.