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# Germ of a synthesis: space–time is spinorial, extra dimensions are time-like

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A pressing issue for modern physics is the possibility of extra dimensions of space–time. Here, a novel approach to this question is put forward, with three facets:

First, an integral transform is introduced into Einstein's general relativity that is non-local and spinorial. For Minkowskian space–time, the transform intertwines three spaces of six dimensions, which *a priori* are on an equal footing, linked by the octavic triality of Cartan. Two of these spaces are interpreted as null twistor spaces; the third may be regarded as giving space–time two extra time-like dimensions, for which the ordinary space–time is an axis of symmetry.

Second, it is suggested that the extra dimensions perdure for a general space–time: the overall structure is controlled by a generalized Fefferman tensor. Accordingly, it is posited that the additional time-like dimensions arise naturally and constitute an aspect of space–time reality that ultimately will be amenable to experimental investigation. Conceivably, devices such as the Large Hadron Collider will uncover this reality.

Third, it is argued that the structure hints at a synthesis of ideas deriving from general relativity, string theory, condensed matter physics, category theory and non-commutative geometry.

**Keywords: integral transforms; triality; twistor theory**

## 1. Introduction

Since antiquity, from Pythagoras of Samos and Euclid of Alexandria to Galileo Galilei to Isaac Newton to Immanuel Kant to Hermann Minkowski to Albert Einstein and David Hilbert, the question of the nature of space and time has occupied scientists and philosophers (Euclid 300BC; Galilei 1638; Newton 1687; Kant 1783; Malchus 1886; Minkowksi 1908; Einstein 1916; Hilbert 1916). At least since Pythagoras, the squared interval between spatial points was expressed as the sum of squares of the increments in three perpendicular directions. Following Einstein, Minkowski added a new concept, that of event, a new dimension, that of time, and expressed the squared interval between events in terms of a sum of the three squared displacements of Pythagoras, combined with a new term representing the square of the time displacement, which enters with an overall negative sign and which utilizes the absolute character of the speed of light/

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gravity to convert dimensions appropriately. In particular, non-zero displacements with squared interval zero are allowed and determine the future and past light cones and the causal properties of space–time, which is now hyperbolic rather than Euclidean or Riemannian.

One possible path to a unified theory of the physical interactions of nature going beyond Einstein's theory of gravity involves the use of additional dimensions (Nordström 1914; Weyl 1918; von Kaluza 1921; Klein 1926; Witten 1981*a*,*b*; Moreschi & Sparling 1986; Green *et al.* 1987, 1988; Antoniadis 1990; Antoniadis *et al.* 1998; Arkani-Hamed *et al.* 1998; Dienes *et al.* 1998; Randall & Sundrum 1999*a*,*b*; Allanach *et al.* 2002; Benslama 2005). Typically, the extra dimensions are taken to be space-like (more squares with positive signs) rather than time-like (more squares with negative signs), so that the higher-dimensional theory remains hyperbolic rather than ultra-hyperbolic. But there need be no contradiction if time-like extra dimensions are used: for example, in the work of Randall & Sundrum (1999*b*), consistency can be achieved by replacing their parameters $r_c^2$ and $\Lambda$ by $-r_c^2$ and $-\Lambda$, respectively. Despite their prevalence in the physics literature, there are apparently no decisive *a priori* arguments for choosing only hyperbolic higher-dimensional theories: ultimately this is an issue that has to be settled by experiment.

Here, a new spinorial theory of physics is developed, built on Einstein's general relativity and using the unifying triality concept of Cartan: the triality links space–time with two twistor spaces (Brauer & Weyl 1935; Penrose 1967, 1975, 1976, 2005; Penrose & MacCallum 1970; Newman 1976; Penrose & Rindler 1984, 1986; Baez 2002). Each of these twistor spaces is six dimensional with an ultra-hyperbolic geometry of signature (3, 3). Unification entails that space–time also have six dimensions of signature (3, 3). Thus, it must acquire two extra time-like dimensions, each of which must be time-like. Fortunately, the experimental device known as the Large Hadron Collider, which is just now coming online, may be sensitive to higher dimensions and, if so, may be able to detect their signature and thereby put this prediction to the ultimate test (Allanach *et al.* 2002; Benslama 2005).

In attempting to construct a theory, one sometimes has very little to go on. One takes strands of thought from different disciplines and has to try to weave them into a coherent whole, a concinnity. One also has to be prepared to make major conceptual adjustments on the fly, as one brings in newer seminal ideas. One should be attuned to the efforts of others and try to incorporate the essence of their best ideas, even if, in the end, one goes in a slightly different direction. One should wield William of Ockhamus's Razor (Ockhamus 1495), but with parsimony.

Consider the simple act of taking a pencil and throwing it, spinning, into the air, so that it rotates completely around three times, before catching it again. Common sense suggests that, *ceteris paribus*, the pencil is the same at the end of this experiment as at the beginning. However, one knows that the pencil is composed mostly of a variety of fermions and that under an odd number of complete rotations, amazingly, the wave function of each of these fermions *changes sign*. Only if the experiment is rerun, can one be sure to restore the pencil to its pristine state.

Mathematically, this behaviour depends on the fact that the rotation group in three dimensions is not simply connected, but has a simply connected double

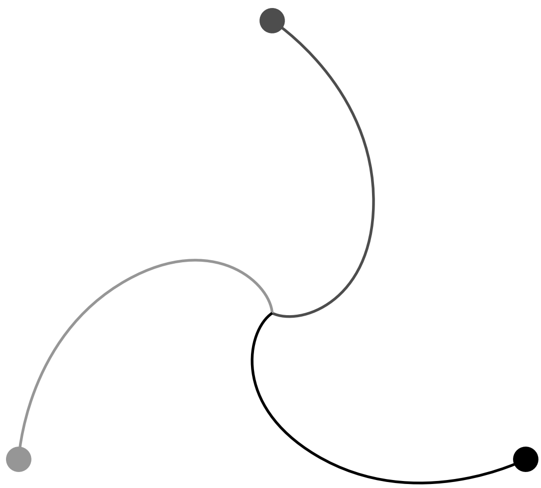

Figure 1. Cartan's triality symbol links two twistor spaces and space–time.

cover, the group $\mathbb{SU}(2, \mathbb{C})$ of $2\times2$ unitary matrices, with complex entries, of unit determinant, whose topology is that of the real three-sphere, $\mathbb{S}^3$. The lift of a single complete rotation to the group $\mathbb{SU}(2, \mathbb{C})$ is a curve connecting its identity element to its negative. A second complete rotation is then required to return to the identity. Quantities that transform with respect to the group $\mathbb{SU}(2, \mathbb{C})$ are called spinors; in particular, the most known elementary particles are spinorial; these somehow transcend space and time (Cartan 1925; Dirac 1928*a*,*b*).

Conventionally, spinors arise locally in space–time as elements of a fibre bundle. It has been a long-term goal to construct a theory which uses spinors in an essentially non-local way. This was partially achieved by the twistor theory of Sir Roger Penrose, Roy Kerr, Ivor Robinson, Ted Newman, Sir Michael Atiyah and a small band of others, including the present author (Brauer & Weyl 1935; Penrose 1967, 1975, 1976, 2005; Penrose & MacCallum 1970; Newman 1976; Penrose & Rindler 1984, 1986; Baez 2002). However, the twistor theory never fully encompassed general relativity. Also its underlying philosophy was to replace space–time by twistor space, which was to be regarded as more fundamental. In the present work, the philosophy is modified: the guiding ontology is the one present in many philosophies from the earliest times: it is the trinity, three entities, which combine harmoniously, forming the concinnity. The three entities are initially conceptualized as space–time, twistor space and dual twistor space (figure 1).

In the earlier manuscript 'A primordial theory', written by the present author and Philip Tillman, relevant *geometrical* and *algebraic* ideas were developed, the principal objects being the exceptional algebra of twenty-seven dimensions of Pascual Jordan, associated to the split octaves, and the associated fifty-six dimensional phase space of Freudenthal (Jordan *et al.* 1934; Freudenthal 1985; Sparling & Tillman 2004). Surprisingly, it emerges that there is a vast *analytical* component that until now seems to have been overlooked, although, for the case of flat space, Hughston (1990) had a similar approach.

## 2. The $\Xi$-transform

The analytic structure in question is a *transform*, which will be named the $\Xi$-transform. It is perhaps most easily expressed using the two-component complex spinor formalism for relativity. The constituents of the transform are as follows.

- The space–time $\mathbb{M}$, which is a smooth real manifold of dimension four.
- The phase space of $\mathbb{M}$ is the cotangent bundle $\mathbb{T}^*\mathbb{M}$ of $\mathbb{M}$, consisting of all pairs $(x, p)$ with $x$ in $\mathbb{M}$ and $p$ a co-vector at $x$. Denote by $\alpha = \theta \cdot p$, the contact one-form on $\mathbb{T}^*\mathbb{M}$; here $\theta$ is the vector-valued canonical one-form of $\mathbb{M}$, pulled back to $\mathbb{T}^*\mathbb{M}$. Also $p$ is the tautological co-vector-valued function on $\mathbb{T}^*\mathbb{M}$, whose value at $(x, p)$ is $p$ and the dot denotes the dual pairing of a vector with a co-vector.
- A Lorentzian metric $g$ for $\mathbb{M}$, of signature $(1, 3)$, such that $\mathbb{M}$ is space and time orientable and has a chosen spin structure.
- The co-spin bundle $\mathbb{S}^*$ is the set of all pairs $(x, \pi)$, where $\pi$ is a primed co-spinor at the point $x$ in $\mathbb{M}$; so $\mathbb{S}^*$ is a complex vector bundle of two complex dimensions over $\mathbb{M}$ (so as a real vector bundle $\mathbb{S}^*$ has four-dimensional fibres). Recall that $\mathbb{S}^*$ is equipped with a complex symplectic form $\epsilon$, a global section of the exterior product of $\mathbb{S}^*$, with itself, such that $g = \epsilon \otimes_{\mathbb{C}} \bar{\epsilon}$.
- Assume given the spin connection (a covariant exterior derivative), denoted as $d$, which is real, torsion free and annihilates $\epsilon$ and $g$.
- Each co-spinor $\pi$ gives rise to a real co-vector $p_\pi = \pi \otimes_{\mathbb{C}} \bar{\pi}$, which is zero if $\pi$ is zero and is otherwise null and future-pointing. Thus, $\mathbb{S}^*$ behaves like a 'square root' of the bundle of future-pointing null cotangent vectors.

Note that $p_{t\pi} = |t|^2 p_\pi$, for any spinor $\pi$ and any complex number $t$, so that the co-vector $p_\pi$ is insensitive to the overall phase of the spinor $\pi$. Very few constructions directly depend on this phase, the Fefferman tensor $\mathsf{F}$, to be described below, being the most crucial (Fefferman 1974; Witten 1981*a*,*b*; Sparling 2001).

Henceforth delete the zero section from $\mathbb{S}^*$, so that all spinors are taken to be non-zero.

- The bundle of co-spin frames, $\mathbb{B}$, consists of all ordered triples $(x, \pi_+, \pi_-)$, where $x$ is in $\mathbb{M}$ and the $\pi_\pm$ are primed co-spinors at $x$, which are normalized against each other by the equation $\pi_+ \otimes \pi_- - \pi_- \otimes \pi_+ = \epsilon$. Note that $\mathbb{B}$ is a principal $\mathbb{SL}(2, \mathbb{C})$-bundle over $\mathbb{M}$.
- Note also that if the non-zero co-spinor $\pi_-$ is given at $x \in \mathbb{M}$, the space of all normalized pairs $(\pi_+, \pi_-)$ at $x \in \mathbb{M}$ is a one-dimensional complex affine space, and so has two real dimensions. If $(\pi_0, \pi_-)$ is normalized, for some co-spinor $\pi_0$, then the general normalized pair $(\pi_+, \pi_-)$ can be written as $(\pi_0 + \lambda\pi_-, \pi_-)$, with $\lambda$ an arbitrary complex number.
- There are natural maps $\Pi_\pm : \mathbb{B} \to \mathbb{S}^*$, which map $(x, \pi_+, \pi_-) \in \mathbb{B}$ to $(x, \pi_\pm) \in \mathbb{S}^*$.
- The (future-pointing) null geodesic spray on the null cotangent bundle lifts naturally to the space $\mathbb{S}^*$ to give a vector field $\boldsymbol{N}$. The trajectories of $\boldsymbol{N}$ represent an affinely parametrized future-pointing null geodesic on $\mathbb{M}$,

together with a co-spinor $\pi$, parallelly propagated along the geodesic, such that $g^{-1}(p_\pi)$ is the (normalized) tangent vector to the geodesic.

— A quantity on $\mathbb{S}^*$ that is invariant along the vector field $\boldsymbol{N}$ is called a twistor quantity. In particular, a function $f(x, \pi)$ killed by $\boldsymbol{N}$ is called a twistor function. Such a function is said to be real homogeneous of (integral) degree $k$ in $\pi$, if $f(x, t\pi)=t^k f(x, \pi)$ for any non-zero *real* number $t$. If $f$ is complex-valued, one says that $f(x, \pi)$ is complex homogeneous of degree $k$ in $\pi$, if $f(x, t\pi)=t^k f(x, \pi)$ for any non-zero *complex* number $t$. Then the real homogeneous twistor functions depend on six real variables, whereas the complex homogeneous twistor functions depend on five real variables.

— The space $\mathsf{B}$ naturally carries *two* horizontal vector fields, $\boldsymbol{N}_\pm$, corresponding to the (horizontal) lifts of the vector field $\boldsymbol{N}$ along the maps $\Pi_\pm$. The trajectories of these vector fields represent an affinely parametrized null geodesic in space–time, together with a normalized spin frame $(\pi_+, \pi_-)$, parallelly propagated along the null geodesic, such that for $\boldsymbol{N}_\pm$, the vector $g^{-1}(p_{\pi_\pm})$ is a normalized tangent vector to the null geodesic.

Now one can proceed to the $\Xi$-transform.

— Let $\beta$ be a given differential three-form on the space $\mathbb{S}^*$.

— Let $\gamma$ be a given horizontal future-pointing null geodesic curve (i.e. an integral curve of the vector field $\boldsymbol{N}$) in $\mathbb{S}^*$.

— Pull $\beta$ back to the space $\mathbb{B}$ along the natural projection $\Pi_+$ to give the three-form $\beta_+ = \Pi_+^*(\beta)$ on the space $\mathbb{B}$.

— Let $\gamma_- = \Pi_-^{-1}(\gamma)$ denote the inverse image under the map $\Pi_-$ of the horizontal curve $\gamma$. One thinks of the space $\gamma_-$ as a 'fattened' version of the null geodesic $\gamma$, in that a two-real-dimensional affine space is located at each point of the null geodesic, rather than just the spinor $\pi_-$ itself. In particular, $\gamma_-$ has real dimension three.

— Integrate the three-form $\beta_+$ over the space $\gamma_-$ to give a number, denoted $\Xi(\beta)(\gamma)$. As the curve $\gamma$ varies, this yields, by definition, the $\Xi$-transform $\Xi(\beta)$ of the three-form $\beta$ as a function on the space $\mathbb{S}^*$, which is invariant along the null geodesic spray $\boldsymbol{N}$, so $\Xi(\beta)$ is a twistor function.

This completes the general description of the $\Xi$-transform.

The special case that is relevant for the remainder of this work is the case that $\beta = \mathrm{i}f(x,\pi)\epsilon^{-1}(\pi, d\pi)\bar{\epsilon}^{-1}(\bar{\pi}, d\bar{\pi})\theta p_\pi$, where $f(x, \pi)$ is a real-valued twistor function, a real homogeneous of degree minus four in the variable $\pi$. For this case, one writes the transform as $f \rightarrow \Xi(f)$. Then one can show that $\Xi(f)$ is itself a real twistor function, which is real homogeneous of degree minus two and that the transform is conformally invariant. Note that $\beta$ can be written also as the multiplication of the function $f$ by the pull back to $\mathbb{S}^*$ of the three-form $1/2\omega(p, g(\theta), dp, dp)$, where $\omega$ is the contravariant alternating orientation tensor associated to the metric $g$ and $dp$ is the tautological co-vector valued one form on the cotangent bundle of $\mathbb{M}$ that incorporates the Levi-Civita connection of $\mathbb{M}$.

Summarizing this key particular case of the $\Xi$-transform gives a conformally invariant operator taking twistor functions of degree minus four to twistor functions of degree minus two. In particular, both the input and output functions are functions with six real degrees of freedom.

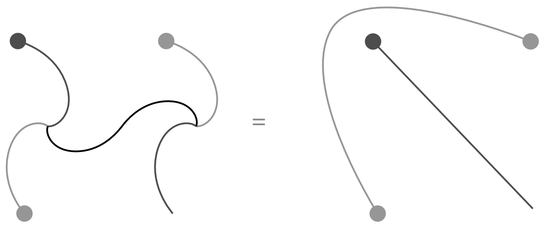

Figure 2. The basic identity obeyed by the $\mathbb{O}(4, 4)$-triality symbol.

Consider now the specialization to the conformally flat case. To understand this more fully, the transform that has just been constructed will be called, temporarily, $\Xi_1$ and two other transforms, denoted $\Xi_2$ and $\Xi_3$, at first sight unrelated to $\Xi_1$, will be introduced.

— The transform $\Xi_2$ is given by the following integral formula:

$$\Xi_2(f)(g, h) = \int_{p \in \mathbb{G}} f(p, g^{-1}ph)\omega_p.$$

— Here $g$, $h$ and $p$ belong to a compact Lie group, $\mathbb{G}$, $\omega_p$ is Haar measure for $\mathbb{G}$ and $f$ is a smooth function on $\mathbb{G}\times\mathbb{G}$. The integral is taken over all $p\in\mathbb{G}$. For the present purposes, one restricts to the case that $\mathbb{G}=\mathbb{SU}(2, \mathbb{C})$. Then $\mathbb{G}$ is topologically a real three-sphere $\mathbb{S}^3$, so $\Xi_2$ maps functions of six real variables, specifically functions on the product space $\mathbb{S}^3\times\mathbb{S}^3$, to themselves.

The transform $\Xi_3$ explicitly uses $\mathbb{O}(4, 4)$-triality, which will first be outlined briefly (Cartan 1925; Baez 2002; Sparling & Tillman 2004). It uses three real eight-dimensional vector spaces $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, say, each equipped with an $\mathbb{O}(4, 4)$ dot product, together with a certain real trilinear form mapping $\mathbb{A}\times\mathbb{B}\times\mathbb{C}$ to the reals, denoted by $(xyz)$, for $(x, y, z)\in\mathbb{A}\times\mathbb{B}\times\mathbb{C}$.

Dualizing this trilinear form gives rise to three real bilinear maps $\mathbb{A}\times\mathbb{B}\to\mathbb{C}$, $\mathbb{B}\times\mathbb{C}\to\mathbb{A}$ and $\mathbb{C}\times\mathbb{A}\to\mathbb{B}$, denoted by parentheses, such that, for example, $((xy)x)=x\cdot xy$, and $(xy)\cdot z=(zx)\cdot y=(yz)\cdot x=(xyz)$, for any $x$, $y$ and $z$ in $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, where the dot product is the appropriate $\mathbb{O}(4, 4)$ inner product (figure 2). The whole theory is then symmetrical under permutations of the three vector spaces.

One says that $x$ in $\mathbb{A}$ and $y$ in $\mathbb{B}$ are *incident* if both are non-zero and yet $(xy)=0$ (note that this concept does not arise in the context of $\mathbb{O}(8)$ triality). Then both $x$ and $y$ are null; also given a null $y\neq0$ in $\mathbb{B}$, the space of all $x$ in $\mathbb{A}$, such that $(xy)=0$ is a real, totally null, self-dual, four-dimensional vector space.

— The transform $\Xi_3$ now proceeds as follows. Let $f(x)$ be a smooth real-valued function homogeneous of degree minus four, defined for all non-zero null $x\in\mathbb{A}$. Then $f(x)x\wedge dx\wedge dx\wedge dx$ is a closed three-form on the null cone of $\mathbb{A}$ taking values in $\Omega^4(\mathbb{A})$, the fourth exterior product of $\mathbb{A}$ with itself. Then define a function $\xi_3(f)$, on the null cone of $\mathbb{B}$, taking values in $\Omega^4(\mathbb{A})$, by the formula, valid for any null vector $y\neq0$ in $\mathbb{B}$

$$\xi_3(f)(y) = \int_{x \text{ incident with } y} f(x)x\wedge dx\wedge dx\wedge dx.$$

Here, the integral is taken over the natural homology three-sphere in the complement of the origin of the space of all $x$ such that $(xy)=0$.

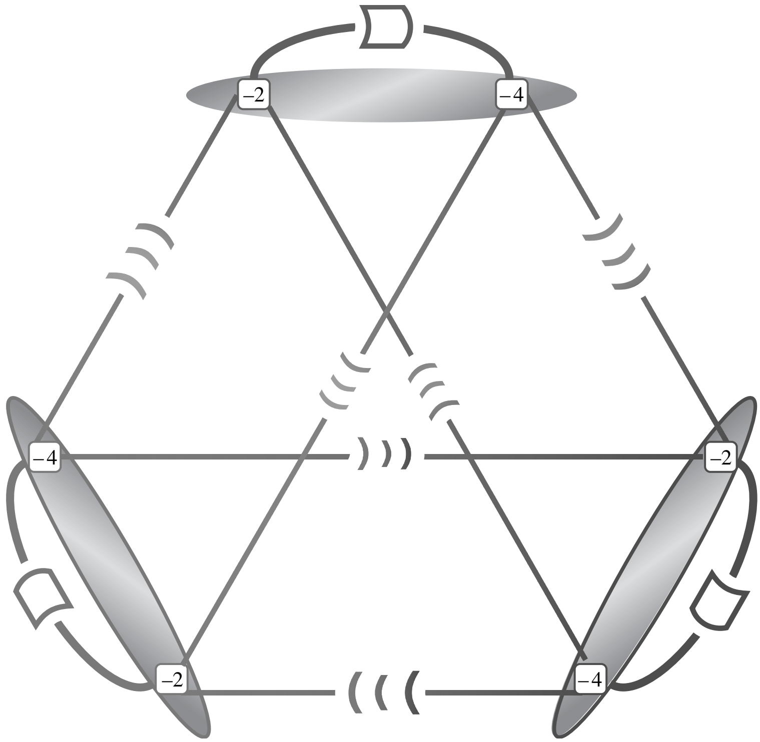


Figure 3. The six transforms.

For this transform, there is a beautiful additional subtlety. First, one shows that the output takes values in the self-dual part of $\Omega^4(\mathbb{A})$. Next, one observes that $\Omega^4(\mathbb{A})$ has real dimension 70, so the self-dual part has real dimension 35. But this is exactly the real dimension of symmetric trace-free tensors of valence two in $\mathbb{B}$ and one shows that there is a natural isomorphism between the two spaces. For $y \in \mathbb{B}$, which is null, the tensor $y \otimes y$ is symmetric and trace-free. Let $\epsilon(y \otimes y) \in \Omega^4(\mathbb{A})$ denote the (self-dual) image of $y \otimes y$ under this isomorphism. Then one shows that the output $\xi_3(f)(y)$ naturally *factorizes*

$$\xi_3(f)(y) = \Xi_3(f)(y)\epsilon(y \otimes y).$$

Since $\xi_3(f)(y)$ is, from its definition, homogeneous of degree zero in $y$, it follows that $\Xi_3(f)$ is a real-valued function on the space of all null non-zero vectors $y$ in $\mathbb{B}$, homogeneous of degree minus two in $y$.

Note that there is one such transform for each ordered pair from the set $\{\mathbb{A}, \mathbb{B}, \mathbb{C}\}$, giving *six* such transforms in all (three of these initially take values in self-dual forms and the other three take values in anti-self-dual forms) (figure 3).

The following results hold (Sparling 2006*a*).

— The transforms $\Xi_1$, $\Xi_2$ and $\Xi_3$ *coincide, mutatis mutandis*. This means that one can prove results for one of the transforms and deduce analogous results for the others, which might be harder to get at directly. For the transform $\Xi_2$, introduce the Casimir operator $\Box_2 = C_+ - C_-$, where each of $C_+$ and $C_-$ is the standard Casimir operator of $\mathbb{SU}(2, \mathbb{C})$ acting on the first and second factors of the product $\mathbb{SU}(2, \mathbb{C}) \times \mathbb{SU}(2, \mathbb{C})$, respectively. So $\Box_2$ is a differential operator of the second order. Then the key result is

$$\Box_2 \circ \varXi_2 = \varXi_2 \circ \Box_2 = 0.$$

— Equivalently, the kernel of $\varXi_2$ contains the image of $\Box_2$ and vice-versa. It is probably true that the kernel of $\varXi_2$ *exactly matches* the image of $\Box_2$ and vice-versa, but at the time of writing, this has only been proved fully under the restriction that the functions involved are finite sums of spherical harmonics. Note that $\Box_2$ is the (ultra-hyperbolic) wave operator in six dimensions for the natural metric on $\mathbb{S}^3 \times \mathbb{S}^3$ of signature (3, 3).

— Using the translation principle, one can reformulate these results at the level of the other operators $\varXi_1$ and $\varXi_3$. For $\varXi_3$, the space of all null non-zero vectors $y$ in $\mathbb{B}$, where $y$ is identified with $ty$, for $t > 0$, is a space of topology $\mathbb{S}^3 \times \mathbb{S}^3$, which now has only a (natural) conformal structure, rather than a fixed metric structure, as in the case of $\varXi_2$. At this point the work of Graham *et al.* (1992) comes into play: it shows (among other results) that there is a natural conformally invariant second-order wave operator, denoted $\Box_3$, for conformally flat geometries in six dimensions, which maps functions of conformal weight minus two to functions of conformal weight minus four. These are the correct weights for the transform $\varXi_3$ and we have:

$$\Box_3 \circ \varXi_3 = \varXi_3 \circ \Box_3 = 0.$$

— Finally, for the spin bundle, the appropriate operator can be defined as follows. Let $f(x, \pi)$ be a given twistor function of degree minus two. Then, using abstract spinor and vector indices, since $f$ is constant along the null geodesic spray, its gradient, $\partial_a f$, with respect to the variable $x$, may be expressed as $\partial_a f = \pi_{A'} \bar{f}_A + \bar{\pi}_A f_{A'}$, where $f_{A'}$ is of degree minus three. Denote by $\partial^{A'}$ the (complex) gradient with respect to the spinor $\pi_{A'}$. Then write $\Box_1 f = \mathrm{i}(\partial^{B'} f_{B'} - \bar{\partial}^B \bar{f}_B)$. Then it can be shown that $\Box_1 f$ is a twistor function of degree minus four. Now the basic result is

$$\Box_1 \circ \varXi_1 = \varXi_1 \circ \Box_1 = 0.$$

What can be said in curved space–time? Recall the stumbling blocks that prevented progress from the direction of twistor theory, in the past. Twistor theory worked beautifully in the cases of self-dual gauge fields and self-dual gravity and related equations (Mason & Woodhouse 1996). It seems clear, in retrospect, that the reason for this success is that these systems of equations were integrable; the methods of the theory always used this fact implicitly or explicitly. Nevertheless, for ordinary non-self-dual gravity, there were significant successes: the $\mathcal{H}$-space (self-dual) theory of Newman and Penrose, although valid only for analytic space–times, arises out of the gravitational radiation data of a real non-self-dual space–time (Newman 1976; Penrose 1976). Also the equations of Frederick Ernst for stationary axisymmetric space–times were shown by Ward (1983) to admit a twistor interpretation. These include the basic physical metrics of Schwarzschild (1916) and Kerr (1963). However, it is believed that neither the source-free, gauge-field equations, nor the vacuum equations of gravity are integrable in general, so ordinary twistor theory is inapplicable.

In particular, chaos, perhaps the antithesis of integrability, seems to appear near a generic future singularity according to the work of Belinski *et al.* (1970). For the first time, the $\Xi$-transform appears likely to give a precise criterion sorting out the more tractable space–times from the rest, according to the nature of its image. One says that the space–time is *coherent*, if and only if the image of the $\Xi$-transform obeys a *pseudo-differential* equation. If not, one says that the space–time is chaotic. Then, in this language, the results described above show that conformally flat space–time is coherent. One would conjecture that all the real space–times, that have proved to be amenable to twistor-type treatments in the past, are coherent.

As of the time of writing, the author has been able to show by direct calculation that the prototypical space–times of Kapadia & Sparling (2000) are coherent, at least for complex homogeneous input functions, giving the first known example of a curved space–time that is such. Note that this proposed classification is inherently *non-perturbative* and potentially gives a *coordinate-independent* definition of dynamical chaos (Cvitanović *et al.* 2004).

## 3. Going up to six dimensions

At the level of space–time, there is a disparity of dimensions: space–time is four dimensional, whereas the twistor spaces depend on functions of six real variables. Is there a realm in physics in which the triality is more manifest? This would require enlarging space–time from four to six real dimensions. This should be done in a natural way building directly from the conventional space–time. Remarkably, it emerges that this is do-able. There are two clues: first consider the conformally flat case. The basic triality spaces have the symmetry group $\mathbb{O}(4, 4)$. This group is too big for the twistor spaces to relate to ordinary physics to reproduce the standard successful quantization of massless particles, using (holomorphic) sheaf cohomology, due to Hughston, Penrose and the author. One has to implement the standard twistor commutation relations, which form the algebra of Werner Heisenberg: $[Z^a, Z^b] = \mathrm{i}\omega^{ab}$; here the indices run from 1 to 8 (Penrose & Rindler 1986). This entails a symplectic form $\omega^{ab}$; to recover the standard theory, $\omega^{ab}$ must give a complex-structure for the eight-dimensional vector space, such that its symmetry group is reduced from $\mathbb{O}(4, 4)$ to the group $\mathbb{U}(2, 2)$. A similar story applies to the other twistor space. However, for the space–time triality space, space–time needs to separate out: this entails reducing the symmetry group from $\mathbb{O}(4, 4)$ to $\mathbb{O}(4, 2)$. This is essentially distinct from the reductions for the twistor spaces: although the groups $\mathbb{SO}(4, 2)$ and $\mathbb{SU}(2, 2)$ are locally isomorphic, the latter being a double cover of the former, they sit inside the group $\mathbb{O}(4, 4)$ in *different* places.

Surprisingly, it emerges that *a single technique* does the job simultaneously for all three triality spaces. For the case of the triality space that one wants to be space–time, say the space $\mathbb{A}$, one simply selects an oriented two-dimensional subspace $\mathbb{J}$ of the eight-dimensional vector space with a positive definite induced metric. Then the orthogonal subspace is six dimensional, which intersects the null cone of the triality space in a five-dimensional space, whose real projectivization gives the four-dimensional space–time, conformally compactified, with a natural conformal structure and the correct conformal symmetry group. Let $j_1$ and $j_2$ be unit orthogonal elements in the subspace $\mathbb{J}$, such that $\{j_1, j_2\}$ is an oriented basis for $\mathbb{J}$. For $b \in \mathbb{B}$ and for $c \in \mathbb{C}$, denoted by $J(b)$ in $\mathbb{B}$ and $K(c)$ in $\mathbb{C}$, the quantities $J(b) = (j_1(j_2 b))$ and $K(c) = (j_2(j_1 c))$, respectively. Then it is easy to see from the properties of the triality that $J$ and $K$ are complex structures for the spaces $\mathbb{B}$ and $\mathbb{C}$, giving these spaces the desired reduction from $\mathbb{O}(4, 4)$ to $\mathbb{U}(2, 2)$. Also the structures $J$ and $K$ are invariant under rotations of the basis $\{j_1, j_2\}$.

The second clue comes from the structural spin tensor of the spin-bundle of space–time. This takes the form $\mathsf{F} = \mathrm{i}\theta^a \otimes (\bar{\pi}_A d\pi_{A'} - \pi_{A'} d\bar{\pi}_A)$. It has three fundamental properties, which encode precise details of the space–time: first, its skew part gives the two-form used by Witten (1981*a*,*b*) in his argument for positive energy; second, properties of the exterior derivative of the skew part can be used to analyse the Einstein field equations; third, its symmetric part, when restricted to any hypersurface, gives the conformal structure of the type of Fefferman (1974) for the twistor theory of that hypersurface, as shown by the author (Sparling 2001). In particular, it provides the *central fact of twistor theory*, from which everything else follows. In moving to a higher-dimensional framework, one would like to extend this tensor to maintain that same control over the field equations and over the twistor theory.

Remarkably, it emerges that in extending to six dimensions, with a conformal structure of signature $(3, 3)$, the tensor $\mathsf{F}$ has a beautiful, completely natural extension, which actually looks better than the original: it is the tensor, still called $\mathsf{F}$, given by the formula $\mathsf{F} = \theta^{\alpha\beta} \otimes \pi_\alpha d\pi_\beta$; note that complex numbers do not appear. Here, $d$ represents the spin connection in six dimensions and one exploits the fact that the spin group for the group $(3, 3)$ is the group $\mathbb{SL}(4, \mathbb{R})$. The basic spinor $\pi_\alpha$ is then *four real dimensional*, carrying the fundamental (dual) representation of $\mathbb{SL}(4, \mathbb{R})$. This means that the spinors *restrict naturally*, without any loss of information, to four-dimensional submanifolds: the correspondence with the spinors of Brauer & Weyl (1935) is just $\pi_\alpha \to (\pi_{A'}, \bar{\pi}_A)$.

The canonical one-form $\theta^{\alpha\beta}$ is skew, so has the required six degrees of freedom. Decomposing into the spinors of relativity, gives a quartet $(\theta^{AB}, \theta_1^{AB'}, \theta_2^{A'B}, \theta^{A'B'})$. Here $\theta^{AB} = \theta\epsilon^{AB}$ may be construed as giving a kind of complex 'dilaton' field and has $\theta^{A'B'} = \bar{\theta}_\epsilon^{A'B'}$ as its complex conjugate. To recover the standard four-dimensional metric one would want the one-form $\theta$ to vanish on restriction to the four-manifold. Then for the rest of the canonical one-form $\theta^{\alpha\beta}$, one has the relations $\overline{\theta_1^{AA'}} = \theta_2^{A'A}$ and $\theta_1^{AA'} = -\theta_2^{A'A}$. These relations, taken together, mean that $\theta_1^{AA'} = \mathrm{i}\theta^{AA'} = -\theta_2^{A'A}$, where $\theta^{AA'}$ is self-conjugate, giving, on restriction, the required real canonical one-form of relativity. Then $\mathsf{F} = \theta^{\alpha\beta} \otimes \pi_\alpha d\pi_\beta$ restricts to $\mathrm{i}\theta^{AA'} \otimes (\bar{\pi}_A d\pi_{A'} - \pi_{A'} d\bar{\pi}_A)$, exactly the Fefferman tensor, the necessary factors of i emerging naturally, even though the spinors of the ambient space–time are entirely real.

However, there is a subtle catch, from where the two clues need to be brought to bear simultaneously: when the ambient spin connection is restricted to the four-dimensional space–time submanifold, there is no reason that it should preserve the complex structure of the space–time spinors. From the ambient viewpoint, if $D$ is the space–time spin connection, which *does* preserve the complex structure, the restricted spin connection, $d$, reads $d\pi_{A'} = D\pi_{A'} + \Gamma_{A'}^{A}\bar{\pi}_A$. This is a disaster, since the field $\Gamma^{A}_{A'}$ is a vector-valued one-form, so has spin-two components, giving gravity *extra spin-two degrees of freedom*, that are probably unphysical.

The resolution is beautifully simple: one postulates that the conformal geometry has a conformal Killing vector, or if the actual metric is specified that it has a Killing vector. Recall that if a metric $g_{ab}$ has a Killing vector $t^a$, then its tensor covariant derivative $\partial_a t_b = \mathsf{F}_{ab}$ is skew. Here, indices are abstract and $\partial_a$ is the Levi-Civita connection of $g_{ab}$ (Penrose & Rindler 1984). A standard formula gives the full covariant derivative of the tensor $\mathsf{F}_{ab}$: $\partial_a \mathsf{F}_{bc} = 2\mathsf{R}_{bca}{}^{d} t_d$. Here, $\mathsf{R}_{bca}{}^{d}$ is the Riemann tensor of $\partial_a$. Now, if the Killing vector vanishes on space–time, then the restriction of $\mathsf{F}_{bc}$ is covariantly constant, so becomes part of the space–time structure. Thus, here one demands that the metric in six dimensions have a Killing symmetry, whose orbits are circles, such that the space–time is the set of fixed points of the symmetry (one thinks of the symmetry as a rotation in the 'two-plane' perpendicular to the four-dimensional 'axis'). Reviewing the construction given above in the conformally flat case, one sees that it is exactly what one has: the rotation is simply the ordinary rotation in the space $\mathbb{J}$, keeping the orthogonal space fixed: this 'axis' then provides the space–time. The derivative of the Killing field provides the invariant complex structure needed for the spinors and twistors in the space–time.

Thus, it is suggested that space–time extends naturally and conformally into six dimensions, where it is the set of fixed points of an appropriate conformal Killing vector field. The signature of the six dimensions is, quite unambiguously, (3, 3). So the extra dimensions are quite definitely time-like. Here, a perilous philosophical principle has been invoked, that in the context of physics, could perhaps be attributed to Dirac (1928*a*,*b*): if it is elegant, then it must be right! This approach has three pay-offs: first, space–time seems to be a kind of 'brane', allowing the ideas of Polchinski (1995) to enter. Second, there is a natural place for arguments of the type given by Randall & Sundrum (1999*a*,*b*), who make the case that the extra dimensions can compensate for the apparent weakness of gravity. Third, on factoring out by the Killing field, the signature becomes (2, 3), giving a suitable arena to apply the ideas of Maldacena (1997).

## 4. The concinnity

Finally, consider the proposed concinnity. Here, there is not yet a definitive theory. However, there are some constraints which are as follows.

- It must provide an arena for the fundamental fermionic quantum liquid of Zhang & Hu (Zhang & Hu 2001; Zhang 2002; Sparling 2002). Indeed, the desire to construct such an arena which is applicable to general non-analytic space–times is one of the main motivations behind the present work.
- It must be geometrical, analytical and algebraic (Hopf).
- It must encode the concepts of sheaves and sheaf cohomology that are critical in twistor theory (Penrose & Rindler 1984, 1986; Penrose 2005) and in string theory (Candelas *et al.* 1985).
- It must unify quantum mechanics and geometry.

— It would be desirable that it include the main ideas of current physics, apart from those already mentioned.

For the last, some ideas will be proffered. The famous Calabi-Yau theory of Candelas *et al.* (1985), as discussed in earlier work, seems to find a home in the null hypersurface twistor spaces, where the hypersurface has no vertex, but terminates in a singularity (Calabi 1954; Yau 1977; Candelas *et al.* 1985; Sparling 2000). Thus, essentially it would appear that their theory classifies the structure of certain space–time singularities. Similarly, the counting of black hole states works with horizons, which are null hypersurfaces (Horowitz & Strominger 1996). The manifolds of Joyce (2000) are more problematic, probably living outside the usual space–time arena and in the ambient six-dimensional space, the Calabi–Yau theory for space–time being a limiting case. Support here comes from the breakthrough work of Nurowksi (2005) and the author on the structure of third-order differential equations.

The structure needed is so powerful that it must involve deep mathematics. One conjectures that it is a *coherent topos* (a generalized approach to sheaf theory due to William Lawvere and Myles Tierney), with a triangulated structure of the type developed by Jean Louis Verdier and Alexandre Grothendieck to provide the cohomology (Lawvere 1971).

Hints of such a triangulated structure appear in the cohomological character of the $\Xi$-transform for conformally flat space–time found above. Where can one begin to look for this structure in space–time? Conjecturally, it lies in the ensemble of all conformally invariant hyperbolic differential or even pseudo-differential operators on the space–time, together with the 'modules' that they act on. These somehow express the *non-analytic essence of hyperbolicity*, which is the key new feature introduced into physics by James Clerk Maxwell and Albert Einstein.

The full structure should be a non-commutative geometry, probably derived from string theory, as in the work of Connes (1994). It should have the property that looked at ('observed!') one way, involving 'going to the boundary', one recovers the basic quantum twistor structure, describing massless particles, whereas looked at another such way, one recovers the relevant space–time phase space (the null cotangent bundle). Note that the very fact that there are twistor and space–time-based descriptions of the same basic physical reality, that of massless particles, hints at a common ontology. The structure would not be in itself dynamical, owing to the lack of a preferred time concept, but would create the required dynamics at the 'edge', following Zhang & Hu (2001).

Then the fundamental 'seat of pants' picture of string theory could be recovered as a generalization of the $\Xi$-transform: a method of transferring information between the various edges. However, unlike conventional string theory, where the strings at the boundary of the pants are much of a muchness, here the three boundary strings belong to *three different spaces* (Green *et al.* 1987).

The concepts presented here should have analogues in other areas. There may be a direct application in the context of superfluid helium three, which has a natural $\mathbb{SU}(2, \mathbb{C}) \times \mathbb{SU}(2, \mathbb{C})$ structure; if this materializes, one may be able to probe the present theory using superfluids and incorporate some of the ideas of Volovik (Volovik 2003; Bain 2006). Finally, there should be a close analogue for the theory of solitons, extending the deep recent work of Le Brun & Mason (2007) and linking it with the ideas of Bondal & Orlov (2002).

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## References

Allanach, B., Kosuke, O., Palmer, M., Parker, M., Sabetfakhri, A. & Webber, B. 2002 Exploring small extra dimensions at the Large Hadron Collider. *J. H. Ener. Phys.* **0212**, 039–060. (doi:10.1088/1126-6708/2002/12/039)

Antoniadis, I. 1990 A possible new dimension at a few TeV. *Phys. Lett. B* **246**, 377–384. (doi:10.1016/0370-2693(90)90617-F)

Antoniadis, I., Arkani-Hamed, N., Dimopoulos, S. & Dvali, G. 1998 New dimensions at a millimeter to a fermi and superstrings at a TeV. *Phys. Lett. B* **436**, 257–263. (doi:10.1016/S0370-2693(98)00860-0)

Arkani-Hamed, N., Dimopoulos, S. & Dvali, G. 1998 The hierarchy problem and new dimensions at a millimeter. *Phys. Lett. B* **429**, 263–272. (doi:10.1016/S0370-2693(98)00466-3)

Baez, J. 2002 The octonions. *Bull. Am. Math. Soc.* **39**, 145–205. (doi:10.1090/S0273-0979-01-00934-X)

Bain J. 2006 *Condensed matter physics and the nature of space–time*, pp. 35. New York, NY: Polytechnic University.

Belinski, V., Khalatnikov, I. & Lifshitz, E. 1970 Oscillatory approach to a singular point in the relativistic cosmology. *Adv. Phys.* **19**, 525–573. (doi:10.1080/00018737000101171)

Benslama, K. 2005 Search for extra dimensions with ATLAS at LHC. In *Fundamental interactions: Proc. Nineteenth Lake Louise Winter Institute—Lake Louise, Alberta, Canada, 15–21 February 2004* (eds A. Astbury, B. Campbell, F. Khanna & M. Vincter), p. 346. New York, NY: World Scientific Publishing Company.

Bondal A. & Orlov D. 2002 Derived categories of coherent sheaves. In *Proc. Int. Cong. Mathematicians*, vol. II, pp. 47–56. Beijing, China: Higher Education Press.

Brauer, R. & Weyl, H. 1935 Spinors in $n$ dimensions. *Am. J. Math.* **57**, 425–449. (doi:10.2307/2371218)

Calabi, E. 1954 The space of Kähler metrics. In *Proc. Int. Cong. Mathematicians*, vol. 2, pp. 206–207. Amsterdam, The Netherlands: North Holland.

Candelas, P., Horowitz, G., Strominger, A. & Witten, E. 1985 Vacuum configurations for superstrings. *Nucl. Phys. B* **258**, 46–74. (doi:10.1016/0550-3213(85)90602-9)

Cartan, E. 1925 Le principe de dualité et la théorie des groupes simple et semi-simples. *Bull. Sci. Math.* **49**, 361–374.

Connes, A. 1994 *Noncommutative geometry*, pp. 661. San Diego, CA: Academic Press.

Cvitanović, P., Artuso, R., Dahlqvist, P., Mainieri, R., Vattay, G., Whelan, N. & Wirzba, A. 2004 *Chaos: classical and quantum*, pp. 906. Copenhagen, Denmark: Niels Bohr Institute.

Dienes, K., Dudas, E. & Gherghetta, T. 1998 Extra spacetime dimensions and unification. *Phys. Lett. B* **436**, 55–65. (doi:10.1016/S0370-2693(98)00977-0)

Dirac, P. 1928*a* The quantum theory of the electron. Part I. *Proc. R. Soc. A* **117**, 610–624. (doi:10.1098/rspa.1928.0023)

Dirac, P. 1928*b* The quantum theory of the electron. Part II. *Proc. R. Soc. A* **118**, 351–361. (doi:10.1098/rspa.1928.0056)

Einstein, A. 1916 Die Grundlage der allgemeinen Relativitaetstheorie. *Ann. d. Phys.* **49**, 769–822.

Euclid of Alexandria 300BC *Euclidus Opera Omnia.* (eds I. Heiberg & H. Menge), Teubner 1883–1916.

Fefferman, C. 1974 The Bergman kernel and biholomorphic mappings of pseudoconvex domains. *Invent. Math.* **26**, 1–65. (doi:10.1007/BF01406845)

Freudenthal, H. 1985 Ausnahmegruppen und Oktavengeometrie. *Geom. Ded.* **19**, 7–63.

Galilei G. 1638 *Discorsi e Dimostrazioni Matematiche, intorno a due nuoue scienze Leida*, Appresso gli Elsevirii.

Graham, C., Jenne, R., Mason, L. & Sparling, G. 1992 Conformally invariant powers of the Laplacian. *J. Lond. Math. Soc.* **46**, 557–565. (doi:10.1112/jlms/s2-46.3.557)

Green, M., Schwartz, J. & Witten, E. 1987 *Superstring theory, volume 1, introduction.* Cambridge Monographs on Mathematical Physics, pp. 484. Cambridge, UK: Cambridge University Press.

Green, M., Schwartz, J. & Witten, E. 1988 *Superstring theory, volume 2, loop amplitudes, anomalies and phenomenology.* Cambridge Monographs on Mathematical Physics, pp. 608. Cambridge, UK: Cambridge University Press.

Hilbert, D. 1916 Die Grundlagen der Physik, Erste Mitteilung, vorgelegt in der Sitzung vom 20. November 1915. *König. Gesell. d. Wissens. zu Göttingen, Nachrichten Mathematische-Physikalisch Klasse* **3**, 395–407.

Horowitz, G. & Strominger, A. 1996 Counting States of near-extremal black holes. *Phys. Rev. Lett.* **77**, 2368–2371. (doi:10.1103/PhysRevLett.77.2368)

Hughston, L. 1990 Applications of the geometry of SO(8) spinors to Laplace's equation in six dimensions. In *Further advances in twistor theory*, vol. 1 (eds L. Mason & L. Hughston), Research notes in mathematics 231, pp. 400. New York, NY: Longman.

Jordan, P., von Neumann, J. & Wigner, E. 1934 On an algebraic generalization of the quantum mechanical formalism. *Ann. Math.* **35**, 29–64. (doi:10.2307/1968117)

Joyce, D. 2000 *Compact manifolds with special holonomy.* Oxford Mathematical Monographs series, pp. 436. Oxford, UK: Oxford University Press.

von Kaluza, T. 1921 Zum Unitätsproblem der Physik. *Sitzungsb. d. Preuss. Akad. d. Wissens. zu Berlin, Physikalisch-Mathematische Klasse* **1**, 966–972.

Kant, I. 1783 *Prolegomena zu einer jeden kŸnftigen Metaphysik die als Wissenschaft wird auftreten kšnnen.* Riga, Latvia: Johann Friedrich Hartknoch.

Kapadia, D. & Sparling, G. 2000 A class of conformally Einstein metrics. *Class. Quant. Grav.* **24**, 4765–4776. (doi:10.1088/0264-9381/17/22/314)

Kerr, R. 1963 Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.* **11**, 237–238. (doi:10.1103/PhysRevLett.11.237)

Klein, O. 1926 Quantentheorie und fünfdimensionale Relativitätstheorie. *Zeits. Phys.* **37**, 895–906. (doi:10.1007/BF01397481)

Lawvere, F. 1971 Quantifiers and sheaves. In *Proc. Int. Cong. Mathematics (Nice 1970)*, pp. 329–334. Paris, France: Gauthier-Villars.

Le Brun, C. & Mason, L. 2007 Nonlinear gravitons, null geodesics, and holomorphic disks. *Duke Math. J.* **136**, 205–273. (doi:10.1215/S0012-7094-07-13621-4)

Malchus, P. 1886 *Vita Pythagorae.* Leipzig, Germany: Teubner.

Maldacena, J. 1997 The large $N$ limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **2**, 231–252.

Mason, L. & Woodhouse, N. 1996 *Integrability, self-duality, and twistor theory.* London Mathematical Society Monographs, New Series 15, pp. 376. Oxford, UK: Clarendon Press.

Minkowksi, H. 1908 Raum und Zeit, Vorträge von der 80. Naturforcherversammlung zu Köln. *Phys. Zeits* **10**, 104–111.

Moreschi, O. & Sparling, G. 1986 On the formulation of the positive-energy theorem in Kaluza–Klein theories. *J. Math. Phys.* **27**, 2402–2408. (doi:10.1063/1.527011)

Newman, E. 1976 Heaven and its properties. *Gen. Rel. Grav.* **7**, 107–127. (doi:10.1007/BF00762018)

Newton, I. 1687 *Philosophiae naturalis principia mathematica*, 1st edn. London, UK: S. Pepys.

Nordström, G. 1914 Uber die Möglichkeit, das elektromagnetische Feld und das Gravitationsfeld zu vereinigen. *Phys. Zeits.* **15**, 504–506.

Nurowksi, P. 2005 Differential equations and conformal structures. *J. Geom. Phys.* **55**, 19–49. (doi:10.1016/j.geomphys.2004.11.006)

Ockhamus, G. 1495 *Quaestiones et decisiones in quattuor libros Sententiarum Petri Lombardi*, Editione Lugdunensi, i, dist. 27, qu. 2, K.

Penrose, R. 1967 Twistor algebra. *J. Math. Phys.* **8**–**2**, 345–366. (doi:10.1063/1.1705200)

Penrose, R. 1975 Twistor theory: its aims and achievements. In *Quantum gravity: an Oxford symposium* (eds C. Isham, R. Penrose & D. Sciama). Oxford, UK: Clarendon Press.

Penrose, R. 1976 Non-linear gravitons and curved twistor theory. *Gen. Rel. Grav.* **7**, 31–52. (doi:10.1007/BF00762011)

Penrose, R. 2005 *The road to reality: a complete guide to the laws of the universe*, pp. 1136. New York, NY: Alfred A. Knopf.

Penrose, R. & MacCallum, M. 1970 Twistor theory: an approach to the quantization of fields and space–time. *Phys. Rep., Phys. Lett. C* **6**, 241–315.

Penrose, R. & Rindler, W. 1984 *Spinors and space–time. vol. 1: two-spinor calculus and relativistic fields*, pp. 472. Cambridge, UK: Cambridge University Press.

Penrose, R. & Rindler, W. 1986 *Spinors and space–time. vol. 2: spinor and twistor methods in space–time geometry*, pp. 512. Cambridge, UK: Cambridge University Press.

Polchinski, J. 1995 Dirichlet Branes and Ramond–Ramond charges. *Phys. Rev. Lett.* **75**, 4724–4727. (doi:10.1103/PhysRevLett.75.4724)

Randall, L. & Sundrum, R. 1999*a* A large mass hierarchy from a small extra dimension. *Phys. Rev. Lett.* **83**, 3370–3373. (doi:10.1103/PhysRevLett.83.3370)

Randall, L. & Sundrum, R. 1999*b* An alternative to compactification. *Phys. Rev. Lett.* **83**, 4690–4693. (doi:10.1103/PhysRevLett.83.4690)

Schwarzschild, K. 1916 Uber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. *Sitzungs. d. Kšnig. Preuss. Akad. d. Wissens* **1**, 189–196.

Sparling, G. 2000 Zitterbewegung. *Séminaires et Congrès, Soc. Math. Fr.* **4**, 275–303.

Sparling, G. 2001 The twistor theory of hypersurfaces in space–time. In *Further advances in twistor theory, volume III* (eds L. Mason, L. Hughston, P. Kobak & K. Pulverer). London, UK: Pitman.

Sparling, G. 2002 Twistor theory and the four-dimensional quantum hall effect of Zhang and Hu, pp. 31. cond-mat/0211679.

Sparling, G. 2006*a* Schizzo d'una sintesi, Seminar at the University of Urbino.

Sparling, G. 2006*b* The $\Xi$-transform for conformally flat space–time, pp. 51. gr-qc/0612006.

Sparling, G. & Tillman, P. 2004 A primordial theory, pp. 55. cond-mat/0401015.

Volovik, G. 2003 *Universe in a helium droplet*, pp. 507. Oxford, UK: Oxford University Press.

Ward, R. 1983 Stationary axisymmetric space–times: a new approach. *Gen. Rel. Grav.* **15**, 105–109. (doi:10.1007/BF00762470)

Weyl, H. 1918 Gravitation und Elektrizität, *König. Preuss. Akad. d. Wissen.* Sitzungsberichte 465–480.

Witten, E. 1981*a* A new proof of the positive energy theorem. *Commun. Math. Phys.* **80**, 381–402. (doi:10.1007/BF01208277)

Witten, E. 1981*b* Search for a realistic Kaluza–Klein theory. *Nucl. Phys. B* **186**, 412–428. (doi:10.1016/0550-3213(81)90021-3)

Yau, S.-T. 1977 Calabi's conjecture and some new results in algebraic geometry. *Proc. Natl Acad. Sci. U S A* **74**, 1798–1799. (doi:10.1073/pnas.74.5.1798)

Zhang, S.-C. 2002 To see a world in a grain of sand, pp. 14. hep-th/0210162.

Zhang, S.-C. & Hu, J. 2001 A four-dimensional generalization of the quantum hall effect. *Science* **294**, 823–828. (doi:10.1126/science.294.5543.823)