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On a Time Symmetric Formulation of Quantum Mechanics

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Abstract

We explore further the suggestion to describe a pre- and post-selected system by a two-state, which is determined by two conditions. Starting with a formal definition of a two-state Hilbert space and basic operations, we systematically recast the basics of quantum mechanics - dynamics, observables, and measurement theory - in terms of two-states as the elementary quantities. We find a simple and suggestive formulation, that "unifies" two complementary observables: probabilistic observables and non-probabilistic 'weak' observables. Probabilities are relevant for measurements in the 'strong coupling regime'. They are given by the absolute square of a two-amplitude (a projection of a two-state). Non-probabilistic observables are observed in sufficiently 'weak' measurements, and are given by linear combinations of the two-amplitude. As a sub-class they include the 'weak values' of hermitian operators. We show that in the intermediate regime, one may observe a mixing of probabilities and weak values.

A consequence of the suggested formalism and measurement theory, is that the problem of non-locality and Lorentz non-covariance, of the usual prescription with a 'reduction', may be eliminated. We exemplify this point for the EPR experiment and for a system under successive observations.

1 Introduction

Initial and final conditions play significantly different roles in quantum mechanics and classical mechanics. In classical mechanics the exact state of a system $\mathcal{S}$ at any time t is determined by a single condition; i.e. by feeding the equations of motion with appropriate initial conditions on a Cauchy surface and working out the evolution of the system in the future or past. Traditionally, quantum mechanics is formulated in a similar manner. A measurement of a complete set of commuting observables determines a state $|\psi_1(t_i)\rangle$ of $\mathcal{S}$; this provides the initial condition at $t = t_i$. To derive probabilities for various possible measurements at $t' > t_i$ the Schrödinger equation is fed with $|\psi_1(t_i)\rangle$, and $|\psi_1(t')\rangle$ is computed. Now suppose we perform at $t = t_f > t'$ another set of measurements which also determine the state of $\mathcal{S}$. While classically, this second measurement is trivial, in quantum theory the second result ($|\psi_2(t_f)\rangle$) is usually not determined from the initial condition, i.e., in general $|\psi_2(t_f)\rangle \neq |\psi_1(t_f)\rangle$. Should we regard $|\psi_2(t_f)\rangle$ as a second condition for the system at intermediate times $t_f > t > t_i$? After all the dynamical laws of motion either the Schrödinger or Hiesenberg equations are time symmetric. Indeed in quantum mechanics we are free to select ensembles using two (almost) independent initial and final conditions.

In 1964 Aharonov Bergman and Lebowitz [1] where the first to recognize the non-triviality of such circumstances. They have derived the basic expressions for probability distributions when the physical system under observation is determined by a pre- and a post-selection. More recently the formalism was re-discovered independently by Griffiths [2], Unruh [3], and Gell-Mann and Hartle [4].[‡] A main

[‡]The relation between the approach developed in this article, and the decoherent histories approach is studied elsewhere [5].

elementary observation of these investigations, which we would like to emphasize, is that in most situations, a pre- and post-selected system *can not be reduced* to an equivalent system with only one condition, that is, $\mathcal{S}$ can not be described by a wave function. This observation has been amplified in Ref. [6, 7]. It turns out that for rare situations, the outcomes of *ordinary* measurements can yield very strange and unusual results. It should be emphasized however, that these results are derived by using *standard* quantum mechanics. The 'strangeness' of the results is only due to the very special conditions which where imposed on $\mathcal{S}$.

Nonetheless, the discovery of such new phenomena, was deeply motivated by a new physical picture, which was implicitly used already in Ref. [6]. In this picture, the evolution of the wave function in a pre- and post-selected systems is conceived in a time symmetric fashion. The two conditions determine two wave functions and both are used to describe the system at intermediate times. In fact, the concept of the 'weak value' [7, 8], A_w, of a Hermitian operator $\hat{A}$, was discovered while attempting to grasp this additional information between two conditions. In such a weak measurement, instead of getting one of the eigenvalues of A, one observes a complex number: $A_w = \langle\psi_f|A|\psi_i\rangle/\langle\psi_f|\psi_i\rangle$. Weak values have been found useful in studying various problems [9, 10, 11, 12].

However, several basic questions remained. Since in general the total information on a pre- and post-selected system $\mathcal{S}$ can not be stored in single wave function, what is the proper language to describe $\mathcal{S}$ under such conditions? In particular, does this mean that we loss any notion of a state at each time slice, or, does it call for an extension of some of the basic notions of quantum mechanics?

Indeed, it has been suggested in Ref. [13, 14, 15], that the usual notion of a state should be generalized. Generalized states which are determined by two

conditions where defined and studied[15]. In this work we shall study in more details the structure and the implications of such a possible extension. We shall call the extension of the usual state ψ, a *two-state*, and denote it by $\hat{\varrho}$. Two states, are elements of an extended Hilbert space which is equipped with the standard set of operations: an inner product, expansion in terms of basis vectors, and a projection which yields a two-amplitude. This two-state Hilbert space is also further generalized to the case of successive N conditions.

We then systematically recast the basics of quantum mechanics - dynamics, observables, and measurement theory - in terms of two-states as the elementary quantities. What we find is a simple and suggestive formulation that is particularly suitable to describe systems in a state of pre- and post-selection, or a sub-system which is coupled to a pre- and post-selected environment [16]. Although our formalism is entirely equivalent to ordinary quantum mechanics, it suggests new insights.

Two basic types of observables arise naturally in this formalism. In the limit of strong coupling between the measurement device ($\mathcal{MD}$) and $\mathcal{S}$, one measures eigenvalues of Hermitian operators, but with a probability proportional to $|\varrho|^2$, *the absolute square of the two-amplitude*, instead of $|\psi|^2$. On the other, in the limit of a vanishing interaction between $\mathcal{MD}$ and $\mathcal{S}$, one generally measures the weak value A_w, which is expressed as a complex valued linear combinations, $\sum a_k \varrho_k$, of the two-amplitude ϱ. This implies that the weak value should not be given a probabilistic interpretation[17], but rather should be understood as a direct reflection, and hence as a non-demolition *observation of the two-state amplitude of the system*. In fact, we show that weak values of Hermitian operators, are only *a sub-class* of amplitude-like quantities that can be measured. For example, we show how the two-amplitude itself, which is not a weak value of a Hermitian operator, can still be observed by a

suitable weak measurement.

What happens when the coupling strength between the observed system and the measuring device is not one of the latter two limiting cases, but falls in some intermediate regime? In such a regime, the 'reading' obtained by the measuring device can not be explained in terms of probabilities nor by weak values alone.[18] We shall show that in some cases one measure *mixed* quantities, which are determined by probabilities and by weak values. The observable is then given by an average of various weak values with a probability distribution of some set of eigenvalues.

Finally, we argue that our approach has also some conceptual advantages. A major conceptual difficulty in the standard interpretation is the issue of the 'reduction of the wave function'. We argue that this difficulty may be avoided in this suggested approach. (See also the discussion in [19]). We exemplify this point by showing that the EPR experiment and the evolution of a general system under successive observations, can be described by a two-state without appealing to a non-local procedure of reduction. The non-local collapse is 'replaced' by *local* conditions. The Lorentz covariance of our description is obtained by including the possibility of correlations between different times.

The article continues as follows. In the next section of we define the basic notion of a two-state Hilbert space and its further extension to the case of several conditions, and show how the two classes of observables discussed above are expressed in terms of two-states. In Section 3, we study measurement theory in terms of our formalism. The two limiting cases, of a weak and a strong measurement, are discussed. We also show that in the intermediate regime, a new mixing of probabilities and weak values is observed. In Section 4, we study the implications to conceptual problems, such as the EPR experiment and to the situation of successive observations. Finally, in

the appendix we show how non-generic two- states, which correspond to correlation between initial and final conditions, can be obtained for an open system.

2 Time Symmetric Quantum Mechanics

We start this section by providing the definition of a two state and constructing a Hilbert space of two-states. Then, we study the basic operations between two-states and in Section 2.2 we show how to handle situation with more than two conditions by using multiple-states. The generalized Schrödinger equation for a two-state is presented in Section 2.3, and in Sections 2.4 and 2.5 we express the basic observables in terms of two-states or multiple-states.

2.1 Two-States

Consider a closed system $\mathcal{S}$ with a given Hamiltonian H, and two given conditions, say $|\psi(t_2)\rangle = |\psi_2\rangle$ and $|\psi(t_1)\rangle = |\psi_1\rangle$, $(t_2 > t_1)$. A mild restriction on these conditions is that

$$\langle\psi_2|U(t_2 - t_1)|\psi_1\rangle \neq 0, \tag{1}$$

where $U = \exp(-i\int H dt')$ is the evolution operator, must be satisfied. At any intermediate time $t_2 > t > t_1$, we have both 'retarded' and 'advanced' states, $|\psi_1(t)\rangle = U(t - t_1)|\psi_1(t_1)\rangle$ and $|\psi_2(t)\rangle = U(t_2 - t)|\psi_2(t_2)\rangle$, respectively. We now combine the total information on the state of $\mathcal{S}$ at time t, and define a two-state $\hat{\varrho}(t)$ by

$$\hat{\varrho}(t) \equiv |\psi_1(t)\rangle\langle\psi_2(t)|. \tag{2}$$

The two-state is formally an operator and is similar to the density matrix operator.[§]

[§]A closely related object called a 'multiple-time state'

However, $\hat{\varrho}(t)$ is in general not a Hermitian operator. It coincides with the density operator only for two trivial conditions. We shall call a two-state which can be expressed in the form (2), of a direct product of a *ket* and a *bra*, a *generic two-state*. In the general case, any two-state is an element of a linear space, $\mathcal{H}_{II}$, of two-states which we define as follows.

Definition

Given by a Hilbert space of states $\mathcal{H}_I = \{|\alpha\rangle\}$, a linear space of two-states $\mathcal{H}_{II}$ is defined by all the linear combination of generic two-states $\{|\alpha\rangle\langle\beta|\}$, where $|\alpha\rangle$ and $|\beta\rangle$ are any two elements of $\mathcal{H}_I$.

The most general expression for a two-state $\hat{\varrho} \in \mathcal{H}_{II}$ is that of a superposition of generic two-states:

$$\hat{\varrho} \;\; = \;\; \sum_{\alpha \;\beta} C_{\alpha\beta} \; |\alpha\rangle\langle\beta|. \tag{3}$$

The space $\mathcal{H}_{II}$ is a Hilbert space with the *inner product* operation [15*b*] defined between $\hat{\varrho}_1, \;\; \hat{\varrho}_2 \in \mathcal{H}_{II}$ by

$$\langle \hat{\varrho}_1, \hat{\varrho}_2 \rangle \;\; \equiv \;\; \mathrm{tr}(\hat{\varrho}_1^{\dagger} \;\; \hat{\varrho}_2). \tag{4}$$

The trace in Eq. (4) is over a complete set of states in $\mathcal{H}_I$.

Due to the restriction (1) of non-orthogonality of the conditions, not all the two-states in $\mathcal{H}_{II}$ correspond to physical states. We define a subspace of physical states, $\mathcal{H}_{phys} \subset \mathcal{H}_{II}$, as the collection of states that satisfy $\mathrm{tr}\hat{\varrho} = \langle 1, \hat{\varrho} \rangle \neq 0$. A *normalized two-state* will be defined by the condition $\langle 1, \hat{\varrho} \rangle = 1$.

was introduced first in [13, 14]. The physical meaning of the two-state we use is identical to the 'generalized state' defined in Ref. [15]. However, in our notation the two-state is formally an operator, and therefore simpler to use.

A normalized two-state basis of $\mathcal{H}_{phys}$ may then be constructed as follows. Given by two different orthonormal basis $S_1 = \{|\alpha\rangle\}$ and $S_2 = \{|\beta\rangle\}$ of $\mathcal{H}_I$ with non-orthogonal elements ($\langle\alpha|\beta\rangle \neq 0, \quad \forall \quad |\alpha\rangle \in S_1, \ |\beta\rangle \in S_2$), the collection of all the two-states $\{\hat{\varrho}_{\alpha\beta}\}$ defined by

$$\hat{\varrho}_{\alpha\beta} \ \equiv \ \frac{|\alpha\rangle\langle\beta|}{\langle\beta|\alpha\rangle} \ \in \ \mathcal{H}_{phys}, \tag{5}$$

forms a normalized two-state basis of $\mathcal{H}_{phys}$.

Contrary to the usual case, not all the linear combinations of basis elements remain in $\mathcal{H}_{phys}$. However, if $dim(\mathcal{H}_{II}) = N^2$, then only a $N^2 - 2$ dimensional hypersurface in this space is not in $\mathcal{H}_{phys}$. Therefore, $\mathcal{H}_{phys}$ is a closed sub-space up to a set of points of measure zero.

We also note, that this construction of a normalized basis is limited to the case of a discrete Hilbert space. We can use the basis $\{|\alpha\rangle\langle\beta|\}$, which has also the advantage of simplifying Eq. (6) and (10) bellow, and is somewhat more convenient for computations. However, as we shall see in Section 2.4, the advantage of using the normalized basis (5) is that it displays more simply and directly probabilities in terms of two-states.

The inner product, of two normalized basis elements satisfies the orthogonality relation

$$\langle\hat{\varrho}_{\alpha\beta}, \hat{\varrho}_{\alpha'\beta'}\rangle = \frac{1}{|\langle\alpha|\beta\rangle|^2}\delta_{\alpha\alpha'}\delta_{\beta\beta'}. \tag{6}$$

Next we define the *two-state amplitude* $\varrho(a, b)$, which will play the a role analogue to $\psi(a)$, by the projection

$$\varrho(a, b) \ \equiv \ \frac{\langle\hat{\varrho}_{ab}, \hat{\varrho}\rangle}{\langle\hat{\varrho}_{ab}, \hat{\varrho}_{ab}\rangle} \ = \ \langle a|\hat{\varrho}|b\rangle\langle a|b\rangle. \tag{7}$$

For example in the case of a generic normalized two-state, $\hat{\varrho}_{12} = \frac{|\psi_1\rangle\langle\psi_2|}{\langle\psi_2|\psi_1\rangle}$, the two-

amplitude is given by

$$\hat{\varrho}_{12}(a\ b)\ =\ \frac{\psi_2^*(b)\langle b|a\rangle\psi_1(a)}{\langle\psi_2|\psi_1\rangle}. \tag{8}$$

In terms of the two-amplitude, any two-state $\hat{\varrho}$ can be written as

$$\hat{\varrho}\ =\ \int dadb\ \varrho(a,b)\hat{\varrho}_{ab}, \tag{9}$$

and the product between $\hat{\varrho}_1, \hat{\varrho}_2 \in \mathcal{H}_{phys}$ as

$$\langle\hat{\varrho}_1, \hat{\varrho}_2\rangle\ =\ \int dadb\langle\hat{\varrho}_{ab}, \hat{\varrho}_{ab}\rangle\ \varrho_1^*(a,b)\varrho_2(a,b). \tag{10}$$

Note that by simple operations we obtain a sub-space of $\mathcal{H}_{phys}$ that can be mapped back to $\mathcal{H}_I$. Given by $\hat{\varrho} \in \mathcal{H}_{phys}$, say $\hat{\varrho} = |\psi_1\rangle\langle\psi_2|$, we can define an '*in* ' and an '*out*' density matrix by

$$\rho_{in} = \frac{\hat{\varrho}\hat{\varrho}^\dagger}{\langle\hat{\varrho}, \hat{\varrho}\rangle} = |\psi_1\rangle\langle\psi_1| \tag{11}$$

and

$$\rho_{out} = \frac{\hat{\varrho}^\dagger\hat{\varrho}}{\langle\hat{\varrho}, \hat{\varrho}\rangle} = |\psi_2\rangle\langle\psi_2|. \tag{12}$$

This property can be used to extract from a given two-state the corresponding set of conditions. However, notice that only in the case that $\hat{\varrho}$ is a generic two-state, (i.e. of the form $\hat{\varrho} = |\psi_1\rangle\langle\psi_2|$) the conditions (11) and (12) can be represented as pure states. In general, ρ_{in} and ρ_{out} have the form of a mixed states.

Indeed the Hilbert space $\mathcal{H}_{II}$ can be classify to two basic groups; of generic two-states or of non-generic two-states, i.e. two-states that can not be transformed to the generic form. Generic two-states always satisfy the equation

$$\mathrm{tr}(\hat{\varrho}^2)\ \ = (\mathrm{tr}\hat{\varrho})^2. \tag{13}$$

The physical significance of these two classes can be understood as follows. A generic two-state describes a system $\mathcal{S}$ that is pre and post selected and possibly

observed at some intermediate time by an "external" observer as discussed above. Non-generic two-state, on the other hand, describe an open system $\mathcal{S}'$, which may be defined by some division of $\mathcal{S}$ into a sub-system and environment, e.g. $\mathcal{S} = \mathcal{S}_{environment} + \mathcal{S}'$. If the total system $\mathcal{S}$ is pre and post-selected but only observables in $\mathcal{S}'$ are of interested, then this open system can be described by a "reduced" two-state: $\hat{\varrho}_{eff} = \mathrm{tr}_{environment}\hat{\varrho}$. In general $\hat{\varrho}_{eff}$ is a non-generic two-state. As is shown in the appendix, non-generic two-states can be obtained even when there is no direct interaction between the sub-system and the environment. In this case the correlations between the system and the environment are generated by the act of pre and post selecting measurements. The more general case of a direct interaction between the subsystem and an environment is discussed in Ref. [16].

2.2 N sequential conditions and multiple-states

In the general case, an arbitrary number of successive conditions may be imposed on a single quantum system. These conditions may be independent (up to the restriction of non-orthogonality), or can be inherently correlated. Let us impose on the system $N+1$

sequential conditions at the times $t = \tau_1, \tau_2 .... \tau_{N+1}$. We have already constructed a Hilbert space of two-states for the case of only two conditions. Let us consider only such two sequential conditions, at τ_i and τ_{i+1}, and for a moment ignore all the other conditions. At this i'th time interval, we can construct as before a two-state $\hat{\varrho}^{(i)}(t)$, where $t_i \in (\tau_i, \tau_{i+1})$, which

is an element of the Hilbert space $\mathcal{H}^{(i)}_{phys}$ defined above.

A 'generic' multiple-state $\hat{\varrho}_{abc...z}$ that describes the system in the interval $t \in$

(t_1, t_N) is defined as an element of a Hilbert space formed by the direct product

$$\hat{\varrho}_{abc...z} \ \in \ \mathcal{H}^{(1)}_{phys} \otimes \mathcal{H}^{(2)}_{phys} \otimes \cdots \otimes \mathcal{H}^{(N)}_{phys} \tag{14}$$

or expressed in terms of normalized basis elements:

$$\hat{\varrho}_{abc...z}(t_1, t_2, ..., t_N) \ = \ \hat{\varrho}^{(1)}_{ab}(t_1) \otimes \hat{\varrho}^{(2)}_{bc}(t_2) \otimes \cdots \otimes \hat{\varrho}^{(N)}_{yz}(t_N). \tag{15}$$

The most general multiple-state may also describe correlations between various conditions. Therefore, in general

$$\hat{\varrho}(t_1, t_2, ..., t_N) \ = \ \sum_{abc...z} C_{abc...z} \hat{\varrho}_{abc...z}(t_1, t_2, ..., t_N). \tag{16}$$

Therefore, in the case of $N+1$ conditions, the most general multiple state is an element of the Hilbert space which is defined by: $\mathcal{H}_{N+1} = \{\hat{\varrho}_{abc...z}\}$, i.e. by all the linear combinations of generic multiple states. When the conditions are not correlated, as in the case of $N+1$ independent measurements, the expression for the multiple state $\hat{\varrho}$ has the form of the generic state in (15).

The generalizations of the inner product and of the projection of the multiple-state to a multiple-amplitudes are straightforward. The inner product between generic multiple-states is generalized to

$$\langle \hat{\varrho}_{a,b,c...,z}, \hat{\varrho}_{a',b',c'...,z'} \rangle \ = \ \frac{1}{|\langle a|b\rangle\langle b|c\rangle \cdots \langle y|z\rangle|^2} \delta_{aa'}\delta_{bb'}\delta_{cc'} \cdots \delta_{zz'} \tag{17}$$

and for any to multiple states

$$\langle \hat{\varrho}_1, \hat{\varrho}_2 \rangle \ = \ \sum_{aa'bb'cc'...zz'} C^*_{1abc...z} C_{2a'b'c'...z'} \langle \hat{\varrho}_{abc...z}, \hat{\varrho}_{a'b'c'...z'} \rangle. \tag{18}$$

We define the multiple-state amplitude according to equation (7) as

$$\varrho(a, b, c, ..., z; \ t_1, t_2, ..., t_N) \ = \ \frac{\langle \hat{\varrho}_{abc...z}, \hat{\varrho}(t_1, t_2, ..., t_N) \rangle}{\langle \hat{\varrho}_{abc...z}, \hat{\varrho}_{abc...z} \rangle}. \tag{19}$$

When the multiple-amplitude is expended in term of the normalized basis, the expansion coefficients are given by the multiple-amplitude:

$$\hat{\varrho}(t_1, t_2, ..., t_N) \;=\; \int dadb...dz\varrho(a, b, ..., z;\; t_1, t_2, ..., t_N)\hat{\varrho}_{abc...z}(t_1, t_2, ..., t_N). \quad (20)$$

The inner product generalizes to

$$\langle\hat{\varrho}_1, \hat{\varrho}_2\rangle \;=\; \int dadb...dz\langle\hat{\varrho}_{ab...z}, \hat{\varrho}_{ab...z}\rangle\varrho_1^*(a, b, ..., z)\varrho_2(a, b, ..., z). \quad (21)$$

As in the case of two-states, multiple states also be classified according to Eq. (13) to generic and non-generic states. The latter case corresponds to correlations between the conditions at various times.

2.3 Dynamics

Two states satisfy the Liouville equation

$$i\hbar\partial_t\hat{\varrho}(t) \;=\; [H, \hat{\varrho}(t)]. \quad (22)$$

Expanding in terms of the two-amplitude we can obtain a Schrödinger-like equation. For example, if $H = \hat{p}^2/2m + V(\hat{x})$, the two-amplitude in the coordinate basis, $\varrho(x', x'', t) = \langle x'|\hat{\varrho}(t)|x''\rangle$, satisfies the equation

$$\begin{aligned} i\hbar\partial_t\varrho(x', x'', t) \;&=\; -\frac{\hbar^2}{2m}\Big(\partial_{x'} - \partial_{x''}\Big)\varrho(x', x'', t) + \Big(V(x') - V(x'')\Big)\varrho(x', x'', t) \quad (23)\\ &= \Big(H(x', p') - H(x'', p'')\Big)\varrho(x', x'', t). \end{aligned}$$

The evolution operator is therefore given by

$$U(t) = \exp\left\{-\frac{i}{\hbar}\int dt\Big(H(x', p') - H(x'', p'')\Big)\right\}. \quad (24)$$

Clearly, for any solution of (22) or (23) we can construct appropriate conditions, and vice versa. We also note that the scalar product $\langle\hat{\varrho}_1, \hat{\varrho}_2\rangle$

is conserved under the evolution. Therefore U is a unitary operator in the Hilbert space $\mathcal{H}_{phys}$.

From (23) we can derive the (generalized) continuity equation

$$\partial_t(\varrho_1^*\varrho_2) + \partial_{x'}\mathcal{J}' - \partial_{x''}\mathcal{J}'' = 0, \tag{25}$$

where the two-current $\mathcal{J}'$ is given by

$$\mathcal{J}'(x'\ x''\ t) \ = \ \frac{\hbar}{2im}\Big(\varrho_1^*(x'\ x''t)\partial_{x'}\varrho_2(x'\ x''\ t) - c.c.\Big), \tag{26}$$

and $\mathcal{J}''$ by a corresponding equation.

To get the equation of motion for the multiple-state case, we simply need to replace (22) by an N-times generalization:

$$i\hbar\Big(\partial_{t_1} + \partial_{t_2} + ... + \partial_{t_N}\Big)\varrho(t_1, t_2, ..., t_N) = [H, \varrho(t_1, t_2, ..., t_N)]. \tag{27}$$

The multiple-states defined in Section 2.2 are solutions of (27) and are determined by $N + 1$ conditions.

2.4 Probabilistic observables

Given an ensemble of n different particles, all in the same two-state, we may perform a measurement of an observable A. To this end, n different measurement devices are couple to each of the components of the two-state of the ensemble

$$\hat{\varrho}_{ensemble} = \hat{\varrho}(1) \otimes \hat{\varrho}(2) \otimes \cdots \otimes \hat{\varrho}(n). \tag{28}$$

Each of the measurements will yield as an outcome one of the eigenvalues a of the Hermitian operator A with a probability $\mathcal{P}rob(a)$. This probability was evaluated first in Ref. [1]. In our notation we find

$$\mathcal{P}rob(a) \ = \ \frac{|\mathrm{tr}(\hat{\varrho}_{aa}\hat{\varrho})|^2}{\int da|\mathrm{tr}(\hat{\varrho}_{aa}\hat{\varrho})|^2} = \frac{|\langle\hat{\varrho}_{aa}, \hat{\varrho}\rangle|^2}{\int da|\langle\hat{\varrho}_{aa}, \hat{\varrho}\rangle|^2}, \tag{29}$$

or in terms of the two-amplitude $\varrho(a,a)$

$$\mathcal{P}rob(a) \;=\; \frac{|\varrho(a,a)|^2}{\int da|\varrho(a,a)|^2}. \tag{30}$$

The last expression for the probability is of particular interest. We see that the projection of the two-state $\varrho(a,a)$ behaves as an amplitude. The absolute square of the two-amplitude yields the probability. The expression for the average value of the observable A is simply

$$\langle A\rangle \;=\; \frac{\int da\ a|\varrho(a,a)|^2}{\int da|\varrho(a,a)|^2}. \tag{31}$$

Does $\varrho(a,b)$, the non-diagonal element of the two-state, correspond to a physical amplitude? Remember that the two-state $\hat{\varrho}$ may be written as a linear superposition of two-states $\hat{\varrho}_{ab}$ with a (complex) amplitude $\varrho(a,b)$:

$$\hat{\varrho} \;=\; \int dadb\ \varrho(a,b)\hat{\varrho}_{ab} \tag{32}$$

A straightforward computation confirms that the absolute square of $\varrho(a,b)$ yields the probability to find the generic two-state $\hat{\varrho}_{ab}$. In other words, if we would measure first the operator A at time t and then the operator B at time $t+\epsilon$, then (when $\epsilon \to 0$) the probability to find he eigenvalues a and b is given by

$$\mathcal{P}rob(a,b) = \frac{|\varrho(a,b)|^2}{\int dadb\ |\varrho(a,b)|^2} \tag{33}$$

Equation (30) above corresponds to the special case of a two-state $\hat{\varrho}_{ab} = \hat{\varrho}_{aa}$.

Comparing to the ordinary expressions when only a pre-selection is involved, we notice that the normalization $\int dadb\ |\varrho(a,b)|^2$ above, or in Eq. (30), is not a constant of motion. It is also interesting to note that the two-amplitude is generally a product of two wave functions. For example, if $\psi_1(x)$ is pre-selected and later $\psi_2(x)$ is post-selected, then the (non-normalized) two-amplitude in this case is

$$\varrho(x,x,t) = \psi_2^*(x)U^\dagger(t-t_2)U(t-t_1)\psi_1(x) \tag{34}$$

It is amusing, that when $H = 0$, and the same state is pre- and post-selected, the two-amplitude $\varrho = |\psi|^2$ plays also the role of a measurable probability. In the next section we shall see that this probability can also be re-written as a weak value.

All the expressions above are generalized directly to the case of a multiple-state. Given by an ensemble of system with the same multiple-state, we can measure various Hermitian operators at any of the N time intervals. Let us denote these operators by $A^{(1)}, B^{(2)}, ..., Z^{(N)}$ and their eigenvalues by $a, b, ..., z$. The latter operators act on elements of the two-state Hilbert spaces $\mathcal{H}^{(1)}_{phys}, \mathcal{H}^{(2)}_{phys}, ..., \mathcal{H}^{(N)}_{phys}$, respectively. The probability to obtain the values $a, b, c, ..., z$ for N measurements, one at each interval, is given by

$$\mathcal{P}rob(a^{(1)}, b^{(2)}, ..., z^{(N)}) \;=\; \frac{|\varrho(a, a, b, b, ..., z, z)|^2}{\int da'db'...dz'|\varrho(a', a', b', b', ..., z'z')|^2}. \tag{35}$$

When two measurement are performed at each interval, say $A^{(1)}$ and $B^{(1)}$ on the first interval etc., we find

$$\mathcal{P}rob(a^{(1)}, b^{(1)}, ..., y^{(N)}, z^{(N)}) \;=\; \frac{|\varrho(a, b, ..., y, z|^2}{\int da'db'...dz|\varrho(a', b', ..., y', z'|^2}. \tag{36}$$

Therefore, the coefficients in the expansion of the multiple-state in (20) correspond, in this general case as well, to physical amplitudes.

Having spelled out the general expressions, we can easily verify that they are time symmetric. Taking $t \to -t$, corresponds to the transformation $\hat{\varrho} \to \hat{\varrho}^\dagger$ or to replacing the two-amplitude ϱ by ϱ^*. Clearly this transformation does not affect Eq. (33) or (36).

Finally, we would like to show that all the usual probabilistic information in the case of an ensemble with only one condition is contained in our formalism. Given by two conditions, say $|\psi(T)\rangle = |\psi_2\rangle$ and $|\psi((-T)\rangle = |\psi_1\rangle$, the two-state $\hat{\varrho}$ is determined. But now suppose we are given by $\hat{\varrho}$ and we would like to reconstruct

the probabilistic quantities related to an ensemble which is only pre (post) -selected, i.e. with only one given condition $|\psi_1\rangle$ ($|\psi_2\rangle$). In this case the probability $\mathcal{P}rob_I(a)$ to measure the state $|a\rangle$ is given simply by

$$\mathcal{P}rob_I(a) \;=\; |\langle a|\psi_1\rangle|^2 \;=\; \langle \hat{\varrho}_{aa}, \rho_{in}\rangle, \tag{37}$$

(or by $\langle \hat{\varrho}_{aa}, \rho_{out}\rangle$), where ρ_{in} and ρ_{out} where defined in (11,12). (In fact, as shown in Section 4.2, Eq. (37) can be reconstructed directly from Eq. (29).) The expectation value of an hermitian operator for a pre-selected ensemble is simply given by

$$\langle A\rangle_I = \mathrm{tr}A\rho_{in} = \frac{\langle \hat{\varrho}, A\hat{\varrho}\rangle}{\langle \hat{\varrho}, \hat{\varrho}\rangle}. \tag{38}$$

Viewing the two conditions as results of measurements

we can also ask what is the probability to get $|\psi_2\rangle$ given by an ensemble described by $|\psi_1\rangle$. This probability is given by

$$\mathcal{P}rob_I(\psi_1 \to \psi_2) \;=\; |\langle \psi_2|\psi_1\rangle|^2 \;=\; \langle \rho_{out}, \rho_{in}\rangle. \tag{39}$$

2.5 Non-probabilistic observables and 'weak values'

Given by a pre- and post- selected ensemble the weak value of an operator $\hat{A}$ is defined [7] by

$$A_w \;=\; \frac{\langle \psi_2|A|\psi_1\rangle}{\langle \psi_2|\psi_1\rangle}. \tag{40}$$

The weak value is in general a complex quantity. However, *both* the real and the imaginary parts of the weak value are observable quantities[7] (and see Section 3.4). We shall argue that the weak values are only a subclass of the non-probabilistic observables that are available to us.

Let us see how observables of the weak type are expressed in our notation. Given

by a two-state $\hat{\varrho}$, Equation (40) can be written as[¶]

$$A_w \;=\; \frac{\mathrm{tr}(A\hat{\varrho})}{\mathrm{tr}\hat{\varrho}} = \frac{\langle A, \hat{\varrho}\rangle}{\langle 1, \hat{\varrho}\rangle} \tag{41}$$

or in terms of the two-amplitude $\varrho(a,a)$ we have

$$A_w \;=\; \frac{\int da\; a\varrho(a,a)}{\int da\varrho(a,a)}. \tag{42}$$

This expression is correct also for the more general case of non-generic two-states

The last expression for the weak value is of particular interest. Comparing this equation to expression (31) for the expectation value of operator, we note that the weak value is given by an *average* of a two-amplitude rather then the square of the absolute value of a two-amplitude. The weak value is in fact a measure of the two-amplitude itself. Inserting for A a projection operator $\pi_a = \hat{\varrho}_{aa}$, we get

$$(\pi_a)_w = (\hat{\varrho}_{aa})_w \;=\; \varrho(a,a). \tag{43}$$

Therefore the weak value of a Hermitian operator is simply a superposition of the diagonal elements of the two-amplitude.

We now see that there is no basic difference between the physical interpretation that should given to the weak value of a Hermitian operator and to the components of a two-state. In fact the two-amplitude, say $\varrho(a,b)$, can also be represented as a weak value of the non-Hermitian operator (two-state) $\hat{\varrho}_{ab}$

$$\varrho(a,b) = \frac{(\hat{\varrho}_{ab})_w}{\langle \hat{\varrho}_{ab}, \hat{\varrho}_{ab}\rangle}. \tag{44}$$

We shall see in the next section that although $\varrho(a,b)$ corresponds to the weak value of a non-Hermitian observable it can still be measured.

[¶]A similar expression for weak values was found also in Ref. [15].

As a consequence of Eq. (41) the weak observables share the linearity property of two-states. Given by the two-states $\hat{\varrho}_1$ and $\hat{\varrho}_2$ we may construct by superposition the two-state $\hat{\varrho} = c_1\hat{\varrho}_1 + c_2\hat{\varrho}_2$. The weak values of an observable A satisfies the same linear relation

$$A_w(\hat{\varrho}) \;=\; c_1 A_w(\hat{\varrho}_1) \;+\; c_2 A_w(\hat{\varrho}_2). \tag{45}$$

Here $A_w(\hat{\varrho})$ stands for the weak value of an observable $\hat{A}$ for a system with a two-state $\hat{\varrho}$. This additivity of weak values can now be understood as a natural consequence of a *superposition principle* for two-states, or two-amplitudes.

Equation (45) can be further generalized. Given by the weak value of an operator A with respect to the two-state $\hat{\varrho}$ we can express this weak value with respect to an arbitrary basis, $\hat{\varrho}_{ab}$ of $\mathcal{H}_{phys}$, by the transformation law

$$A_w(\hat{\varrho}) \;=\; \int dadb\; \varrho(a,b) A_w(\hat{\varrho}_{ab}) \tag{46}$$

Notice that this is exactly the same expression for decomposing a two-state $\hat{\varrho}$ in term of the basis $\hat{\varrho}_{ab}$. Hence, Equation (46) expresses an interesting inner-relation between probabilistic and non-probabilistic quantities. If we could measure strongly $\hat{\varrho}_{ab}$ and simultaneously the weak value of A in the 'branch' $\hat{\varrho}_{ab}$ of $\hat{\varrho}$, we would obtain the value $A_w(\hat{\varrho}_{ab})$ with a probability given by the square of the two-amplitude! It is amusing that such a circumstances does in fact occur, for measurements of intermediate coupling strength. This will be further discussed in Section 3.3.

3 Time Symmetric Description of Measurements

In this section we shall examine the relation between the two classes of observables, which were defined in the last section, to measurements. We first give a time symmetric description of a measurement in a pre- and post- selected ensemble.

Consider a system $\mathcal{S}$ with a given Hamiltonian $H_{\mathcal{S}}(x,p)$ and a measuring device $\mathcal{MD}$ with a Hamiltonian $H_{\mathcal{MD}}(q,\pi)$. The measurement process of an observable $A(x,p)$ is described by coupling $\mathcal{S}$ and $\mathcal{MD}$ via and some interaction term H_I. The prescription of von-Neumann is to take

$$H_I \;=\; g(t)qA \tag{47}$$

and use the canonical variable π as the 'pointer' of the measuring device. For $g(t) = g_0\delta(t)$, the shift in the pointer's location is $\delta\pi = \pi_f - \pi_i = g_0 A$. In this impulsive limit, the free part of H has no effect. Therefore, for simplicity we shall set in the following $H_{\mathcal{MD}} = H_{\mathcal{S}} = 0$.

The Hilbert space of the total system is $\mathcal{H} = \mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{MD}}$. Given by two (consistent) conditions, say $\hat{\varrho}(-T)\hat{\varrho}^\dagger(-T) = \rho_1 = |\psi_1\rangle\langle\psi_1|$ and $\hat{\varrho}^\dagger(+T)\hat{\varrho}(+T) = \rho_2 = |\psi_2\rangle\langle\psi_2|$, we now wish to solve equation (22) and find $\hat{\varrho}(t)$ in the time interval $t \in [-T,+T]$. The consistency of the two conditions is that our solution must satisfy $\mathrm{tr}\hat{\varrho} \neq 0$, or $\langle\rho_1,\rho_2\rangle \neq 0$, which meaning that there is a finite amplitude for the system to evolve the initial to the final condition.

The Schrödinger equation for the (non-normalized) two-amplitude, $\rho(a,a',\pi,\pi',t) = \langle a,\pi|\hat{\varrho}(t)|a',\pi'\rangle$, is

$$i\hbar\partial_t\rho(a,a',\pi,\pi',t) \;=\; -ig(t)\Big(a\frac{\partial}{\partial\pi} - a'\frac{\partial}{\partial\pi}\Big)\rho(a,a',\pi,\pi',t). \tag{48}$$

The two-amplitude may be decomposed as $\rho = \psi_1(a,q,t)\psi_2(a',q',t)$ where ψ_1 and

ψ_2 are the ordinary wave functions with Hamiltonians $H(a,q,t)$ and $-H(a',q',t)$, respectively.

The two-state is therefore given by

$$\hat{\varrho}(t) = |\psi_1(t)\rangle\langle\psi_2(t)| \tag{49}$$

with $|\psi_1(t)\rangle = U(t+T)|\psi_1\rangle$ and $|\psi_2(t)\rangle = U(t-T)|\psi_2\rangle$.

3.1 Measurements with a probabilistic outcome.

Consider a measurement of an observable $\hat{A}$ with discrete eigenvalues which for simplicity we set to be: $a = 0, \pm 1, , , , \pm n...$ In the idealized description (47) of a measurement given above, the accuracy in reading A is given by $\Delta A = \Delta\pi/g_0$, where $\Delta\pi$ is the uncertainty in the initial and final locations of the pointer, i.e. $\Delta\pi \simeq \Delta\pi_i \simeq \Delta\pi_f$. Remembering that the spectrum of A is discrete with intervals of 1, we can now say that for an accurate measurement we must set

$$\frac{\Delta\pi}{g_0} << 1 \tag{50}$$

We now notice that, this conditions also implies that the uncertainty in the interaction term must be very large, that is, $\Delta(H_I) = (g_0/\Delta\pi)A >> A$. We shall call this type of measurements, *strong measurements*, since while the value of A is unchanged ($[A, H_I] = 0$) any other quantity which does not commute with A is disturbed strongly. This of course reflects the consistency of measurement theory with the uncertainty principle. In the next section we shall see what happens if one tries to relax Eq. (50).

Let us consider as an example, a measurement of A with an outcome $\delta\pi = \pi_f - \pi_i = 1$. The measuring device was prepared at the state $|\pi(-T) = 0\rangle$ and was determined in the final state to be in the state $|\pi(+T) = 1\rangle$. Let us also assume

that the initial and final states of the observed system were $|\psi_1(-T)\rangle = \sum_n C_n|n\rangle$ and $|\psi_2(T)\rangle = \sum_m C'_m|m\rangle$, respectively. This is a complete specification of two conditions for the total system. The interaction (47) between the measuring device and the system occurred at the instant $t = 0$ and for the rest of the interval there is no evolution, $H_{total} = 0$. Therefore, we can easily derive the two-state of the total system.

$$\hat{\varrho}(t) = N \sum_{nm} C_n C'^{*}_m \Big(|\pi = n\rangle\langle\pi' = 1|\Big) \otimes \Big(|n\rangle\langle m|\Big), \qquad t \in (0, +T) \tag{51}$$

and

$$\hat{\varrho}(t) = N \sum_{nm} C_n C'^{*}_m \Big(|\pi = 0\rangle\langle\pi' = 1 - m|\Big) \otimes \Big(|n\rangle\langle m|\Big), \qquad t \in (-T, 0) \tag{52}$$

A schematic description of the evolution of the wave functions due to the measurement is depicted in Figure 1. In the 'forward' time direction (upwards in Fig. 1), the single component $\pi = 0$ of the measurement device 'splits' at $t = 0$ to discrete branches according to the possible final values of π. The forward moving (retarded) state is a product state, $|\pi = 0\rangle \otimes \sum_a C_a|a\rangle$, before the instant of interaction, and an entangled state, $\sum_n C_n|\pi = n\rangle \otimes |n\rangle$ for $t \in (0, +T)$ (correlated states are depicted by doted arrows). The backwards moving wave behaves symmetrically. The advanced state is given by a direct product for $t \in (0, +T)$, and by an entangled state for $t \in (-T, 0)$. The two-state of the system (51) is a product of the corresponding forward (retarded) state, and backwards (advanced) state.

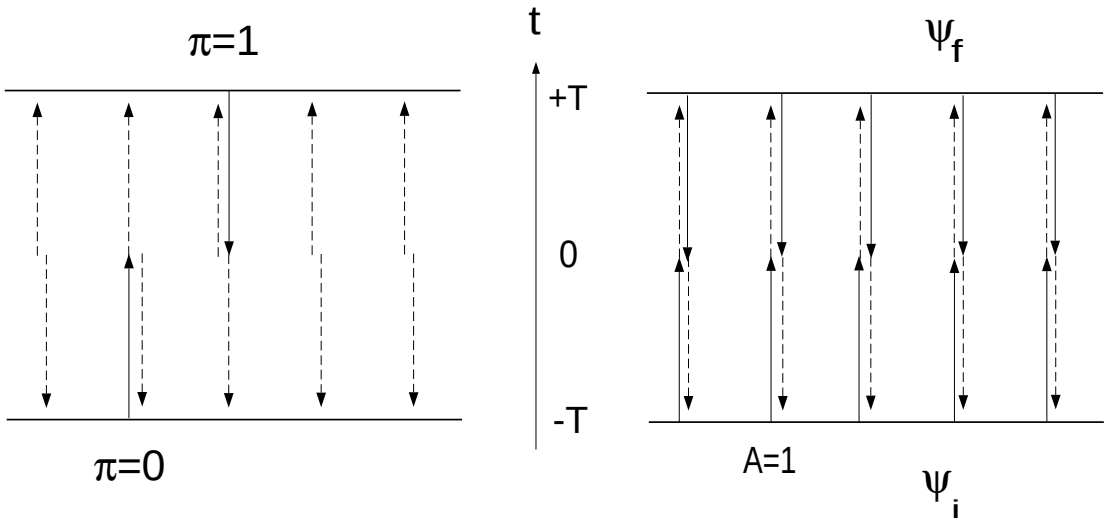

Figure 1. A pictorial description of the two-state $\hat{\varrho}(t)$ give in eq. (51) and (52) of a measuring device $\mathcal{MD}$ and a system $\mathcal{S}$ during a measurement, in the special case that the result $\pi_f - \pi_i = A = 1$ was recorded. The system and the measuring device are pre-selected to the state $|\psi_i\rangle$ and $|\pi_i = 0\rangle$ at $t = -T$, and post selected to $|\psi_f\rangle$ and $|\pi_f = 1\rangle$ at $t = +T$. The interaction between $\mathcal{MD}$ and $\mathcal{S}$ occurs at $t = 0$. Time flows in the upwards direction, while the horizontal axis describes the internal space of $\mathcal{MD}$ (left) and $\mathcal{S}$ (right). Arrows in the up (down) direction represent "ket" ("bra") components of $\hat{\varrho}$ that evolve forward (backward) in time. E.g. for $t \in (-T, 0)$, in the forward time direction, $\hat{\varrho}$ has only one component of $\mathcal{MD}$ with $\pi = 0$. After the interaction, for $t \in (0, T)$ the two-state $\hat{\varrho}(t)$ has several components of $\mathcal{MD}$ that propagate forward in time. These states are entangled with forward evolving states of $\mathcal{S}$. Whenever, such entanglement occurs we use dashed lines. Undashed lines represent the case of a direct product. .

How can we extract the ordinary (only pre-selected) probabilities from this picture? Clearly given by only one pre- and post-selected ensemble we cannot. However, we can consider different ensembles and compute the conditional probability to find

$\pi' = 1$ when $\pi = 0$ and the initial and final states of $\mathcal{S}$ are given. This yields:

$$\mathcal{P}rob(\pi' = 1) \;=\; \frac{\mathcal{P}rob_I(\pi = 0 \to \pi' = 1 | \psi_i(\mathcal{S}), \psi_f(\mathcal{S}))}{\sum_{\pi'=n} \mathcal{P}rob_I(\pi = 0 \to \pi' | \psi_i(\mathcal{S}), \psi_f(\mathcal{S}))} \tag{53}$$

Using Eq. (11,12,39) we get

$$\mathcal{P}rob(\pi' = 1) \;=\; \frac{\langle \rho_{out}(\pi' = 1), \rho_{in} \rangle}{\sum_{\pi'=n} \langle \rho_{out}(\pi'), \rho_{in} \rangle} \;=\; \frac{|C_1 C_1'|^2}{\sum_n |C_n C_n'|^2}, \tag{54}$$

which is of course identical to the probability derived in this case from Eq. (30).

We now observe that in the two-state formulation we do not need to invoke any assumption on a non-local reduction of the wave function of $\mathcal{S}$ due to the (final) determination of the measuring device. The traditional formulation of the measurement process states that after determining the location of the pointer the wave function of the pointer *and of the system* are reduced instantly to one of the components $|\pi = 1\rangle |A = a\rangle$. This reduction, is frequently a non-local process. For example, we could make the final measurement of the location of the pointer (coupling to a external macroscopic environment) after separating $\mathcal{S}$ and $\mathcal{MD}$ to a large distance from each other. Contrary to the usual description in this symmetric formulation of quantum mechanics we need to invoked only two *local* conditions on the system and the measuring device to fully determine the two-state. Thus the determination of the final location of the pointer reduces only the location of the pointer, but does not affect (via a collapse) the system.

To exemplify this point let us return to the measurement above but view the process in two different Lorentz frames $\mathcal{O}_1$ and $\mathcal{O}_2$ with velocities $\vec{v}_1 = v\hat{x}$ and $\vec{v}_2 = -v\hat{x}$, respectively. To make the argument clearer let us assume that the measurement process described above takes place in the following way. $\mathcal{MD}$ and $\mathcal{S}$ are post selected (prepared) at $t = -T$ at two different locations, say $x_{\mathcal{MD}} = -L$ and $x_{\mathcal{S}} = +L$. $\mathcal{MD}$ and $\mathcal{S}$ are then transported to one location, say $x = 0$, and

interact at $t = 0$ via a von-Neumann coupling (47). They are then transported back to $x_{\mathcal{MD}}$ and x_s, and at $t = +T$ they are post selected, i.e. coupled to a macroscopic device that determines the final states $|\pi_f\rangle$ and $|\psi_f\rangle$ of $\mathcal{MD}$ and $\mathcal{S}$, respectively. We assume that the variables π and A are internal local degrees of freedom. Therefore the process of pre and post selection and the interaction can taken as local. In the original (stationary) frame the evolution in this internal space is depicted in Fig. 1.

Clearly, as the preparation (or post selection) of $\mathcal{MD}$ and $\mathcal{S}$ take place in space-like separated locations, the temporal order of the events is different in $\mathcal{O}_1$ and $\mathcal{O}_2$. In $\mathcal{O}_1$, an observer sees the post-selecting of $\pi = 1$ occur *before* the post-selection of $\mathcal{S}$. On the other hand, in $\mathcal{O}_2$ the post-selection of $\mathcal{MD}$ seem to take place *after* the post-selection of $\mathcal{S}$. Nevertheless, both observers calculated the same probability distributions for the spectrum of A. Probabilities are Lorentz invariant. However, suppose we now ask observers in $\mathcal{O}_1$ and $\mathcal{O}_2$ to describe the evolution of the state of the system during a particular measurement. The standard interpretation, yield two totally different descriptions. According to the description given in frame $\mathcal{O}_1$, the determination of the condition $\pi_f = 1$ of the $\mathcal{MD}$, induces a non-local reduction of the wave function of § *before* the condition ψ_f has occurred (Figure 2). On the other hand, a second equally valid description give by $\mathcal{O}_2$ is that the determination of ψ_f occurs before, and hence causes a non-local collapse of the pointer *before* the event that recorded $\pi = 1$ occurred. Obviously, the reduction invalidates any possibility of providing a Lorentz covariant description in terms of wave functions.

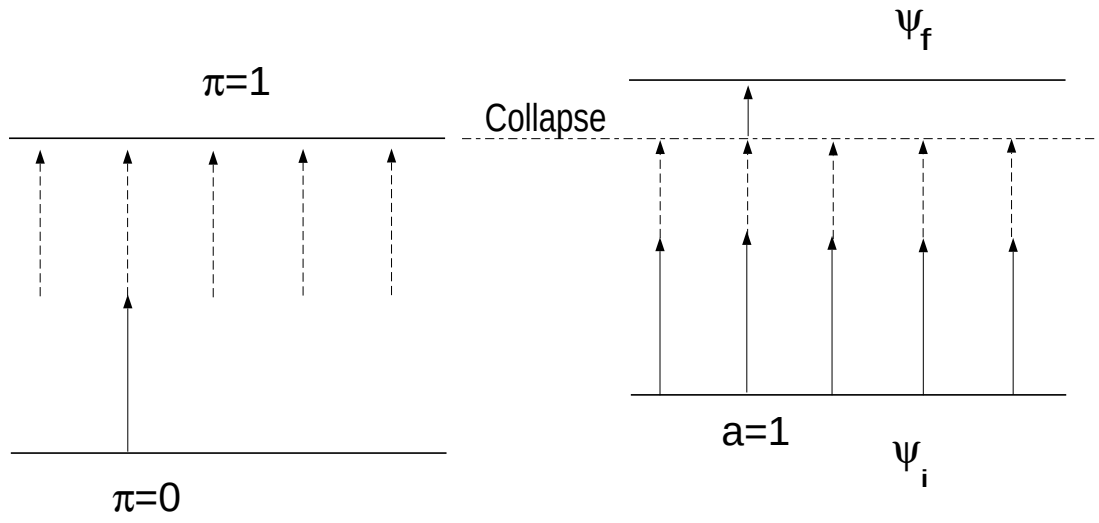

Figure 2. The evolution of the wave function in reference frame $\mathcal{O}_1$ according to the traditional interpretation. Since the final post selection of $\mathcal{MD}$ and $\mathcal{S}$ takes place in two space-like related locations, an observer in $\mathcal{O}_1$ sees the recording o f $\pi = 1$ take place before the final post selection of $\mathcal{S}$.

In the two-state formulation, there is no collapse in non of the Lorentz frames $\mathcal{O}_1$ or $\mathcal{O}_2$ described above. In both cases we continue to describe the evolution by using the *non-collapsed states*. The schematic description given by $\mathcal{O}_1$ in this case is depicted in Figure 3. Notice that the two-state of $\mathcal{S}$ after the post-selection of $\mathcal{MD}$ is *still correlated* with the two-state of the $\mathcal{MD}$ *before* the post-selection. In a general Lorentz frame the total system, $\mathcal{S} + \mathcal{MD}$ is most naturally described in terms of the multiple states discussed in section 2.2. All the Lorentz frames will use the same multiple-state, up to the time ordering of *local* conditions at space-like separated regions. Therefore, multiple-states can provide a Lorentz Covariant description.

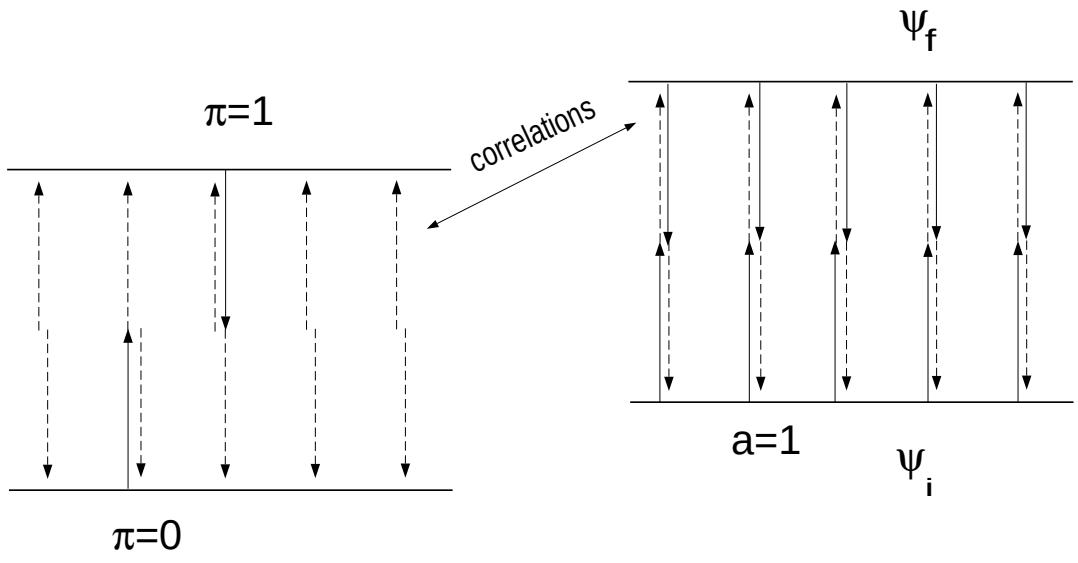

Figure 3. The evolution of the two-state during the measurement in reference frame $\mathcal{O}_1$. There is no reduction. Instead there are additional time like correlation.

describe $\mathcal{S}$ and calculate the probabilities symmetric formulation at determined by mathematical

3.2 Measurements of non-probabilistic observables

In Section 2.3 we have presented a class of complex-valued amplitude-like quantities which we have said are non-probabilistic observables. The 'weak values' of Hermitian operators, which can be expressed as $\sum_a C_a \varrho(a,a)$, is a subclass of these observables. We shall now discuss measurements of weak values and of other amplitude-like observables. We shall show how non-diagonal elements of the two-state, i.e. $\varrho(a,b)$, which generally can be expressed as 'weak values' of non-Hermitian operators, can be measured as well.

A consequence of the condition (50) for an accurate and hence 'strong measurement', is that the conjugate variable q is strongly fluctuating and the coupling

between S and MD (see H_I in Eq. (47)) is large. Therefore, any observable that does not commute with the measured observable A is strongly disturbed. If we try to weaken H_I by making $g_0\Delta q$ small, we indeed disturb less the system $\mathcal{S}$. However, since $\Delta\pi$ becomes large we obtain a less accurate measurement of $\hat{A}$. In other words, by making the location of the pointer uncertain, we can not say if the distribution of the results we have obtained is due to the uncertainty $\Delta\pi$ in the location of the pointer, or due to the probability distribution of $\hat{A}$ which is obtained in a "good" measurement. In the limit $\frac{\Delta\pi}{g_0} \to \infty, \quad g_0\Delta q \to 0$ the system S is *undisturbed* at all, that is $H_I|\psi\rangle \to 0$. At first, it may seem that this limit is uninteresting since we can not extract any information on the system. However, as long as we do not set $\Delta q = 0$ identically, we can still observe the changes in the wave function of the pointer while causing the smallest disturbance we wish to the system. Indeed, since there is a large uncertainty in the location of the pointer we shall need a large number of measurements to find the modification of the pointer's wave function. However, in this limit the uncertainty is a *property of the measurement device and not of the system under observation.* In this weak interaction limit, the evolution of the state $|\mathcal{MD}\rangle$, takes a simple and universal form:

$$|\mathcal{MD}(t)\rangle = \lim_{g_0\Delta q\to 0}\langle\psi_f(\mathcal{S})|U|\psi_i(\mathcal{S})\rangle = N(t)\exp\left(-i/\hbar\int(H_I)_w dt\right)|\mathcal{MD}(t=0)\rangle \tag{55}$$

For the special case H_I that corresponds to a von-Neumann coupling (47), this yields

$$\psi_{\mathcal{MD}}(\pi,t) = N(t)\psi_{\mathcal{MD}}(\pi - A_w, t=0). \tag{56}$$

The initial wave function of the $\mathcal{MD}$ is shifted by the real part of A_w. The imaginary part of the weak value can also be measured. For example, when the initial wave function of the $\mathcal{MD}$ is a gaussian, the imaginary part of A_w affects the 'velocity' of

the pointer, which in our case is represented by the q-coordinate. Notice that the wave function of *all* the measurement devices in the ensemble are modified in the *same* way. In principle this can be confirmed by projecting the final state of the pointer on the computed projection operator $|\mathcal{MD}(t)\rangle\langle\mathcal{MD}(t)|$. In the usual case, one determines the final state of the pointer in π-space. Therefore, an ensemble of measurement devices is needed only to eliminate the (known) uncertainty in π.

We now consider an alternative measurement set up which can be used to measure the two-amplitude $\varrho(a,b)$. Since $\varrho(a,b) = (\hat{\varrho}_{ab})_w / \langle \hat{\varrho}_{ab}, \hat{\varrho}_{ab} \rangle$, we need actually to measure weakly the non-Hermitian operator $A_{ab} \equiv (\langle b|a\rangle)|a\rangle\langle b|$. This can be achieved by the following modification of the usual procedure. We add a *third* device, which is large spin $L = N$, and pre and post-select the rare states $L_z = N$ and $L_x = N$, respectively. At an intermediate time we set the interaction

$$H_I = \frac{g(t)}{\sqrt{2}N} q (A_{ab}^{\dagger} L_+ + A_{ab} L_-) \tag{57}$$

We find that the evolution of the $\mathcal{MD}$ is given by

$$\psi_{\mathcal{MD}}(\pi, t) = C(t) \psi_{\mathcal{MD}}(\pi - \varrho(a,b), t = 0) + O(g_0 \Delta q / N) \tag{58}$$

The idea of this procedure is to achieve an effective coupling with a non-Hermitian operator. Although the total interaction is Hermitian, this specific pre- and post-selection of the large spin, makes the contribution of the term with L_- negligible, while leaving the second terms as the main contribution. When the correction $O(g_0 \Delta q / N)$ is negligible, we obtain a measurement of the two-amplitude $\varrho(a,b)$. Note that we need either a small $g_0 \Delta q$ or a large N. In the first case our coupling yields a 'weak' measurement of A_{ab}. However in the case of large N we can regard our coupling as an ordinary measurement, i.e. for every given finite accuracy $\Delta\pi$ of our measuring device, we use a sufficiently large N such that we *always* measure

$\varrho(a,b)$. Of course, in the latter case we need to work harder in order to prepare our ensemble. The "large-N limit" can of course be used in measuring weak values of Hermitian operators as well.

The common property of the two limits is that in both cases we can regard the effect of the interaction (57) on the observed system $\mathcal{S}$ as very small, i.e. $H_I|\psi\rangle\psi_{\mathcal{S}} \sim O(\Delta q/N)$. Therefore, in the limit, the wave function of the system is unmodified.

3.3 The intermediate regime: mixing of probabilities and weak values

In the previews two sections we have considered measurements that according to the strength of the coupling, could be classified either as strong or as weak measurements. In the first case, the results are described by a probability distribution, while in the second case, they are interpreted as a measure of essentially non-probabilistic two-state amplitudes. What happens when the strength of the coupling correspond to some intermediate regime and the accuracy of the measurement is not sufficient for a strong measurement and too small to be regarded as a weak measurement?

We shall now show ,that at least in some cases, in the intermediate regime, we measure observables which are expressed by a mixing of probabilities and amplitude-like quantities. Suppose that the system under observation is pre- and post-selected to a two-state $\hat{\varrho}_{\mathcal{S}} = |\psi_{in}\rangle\langle\psi_{out}|$, and that the measurement device in is initially in the state $|\mathcal{MD}(0)\rangle$. Then, restoring the corrections previously omitted in equation (55), the final state of the measurement device is given by

$$|\mathcal{MD}(t)\rangle = \left[\exp(-ig_0qA_w) + \sum_{n=2}^{\infty} \frac{(-ig_0q)^n}{n!}\Delta A_w^n\right] |\mathcal{MD}(0)\rangle, \tag{59}$$

where $\Delta A_w^n \equiv (A^n)_w - (A_w)^n$. The 'weak' approximation requires that the 'evo-

lution operator' above is given only by the exponential term. If the sum above is dominated by the first term, then a sufficient condition for a weak measurement is that $g_0^2|\Delta A_w^2|\Delta q^2 << 1$. Now suppose that this condition is not satisfied for our given two-state $\hat{\varrho}_s$, but we can still find a decomposition in terms of normalized two-states $\hat{\varrho}_k$

$$\hat{\varrho}_s = \sum_k a_k \hat{\varrho}_k, \tag{60}$$

such that each of the component $\hat{\varrho}_k$ satisfies

$$g_0^2|\Delta(A_w^2)_k|\Delta q^2 << 1, \tag{61}$$

Here, $(A_w)_k = \mathrm{tr}(A\hat{\varrho}_k)/\mathrm{tr}\hat{\varrho}_k$ is the weak value of A with respect to the k component of the two-state. Although for this *given* coupling strength $g_0^2\Delta q^2$, the 'weak uncertainty' ΔA_w^2 for the two-state $\hat{\varrho}_s$ is not sufficiently small, in each of the components $\hat{\varrho}_k$ the 'weakness' condition is satisfied.

Pictorially we can clarify the meaning of this condition as follows. In order to obtain a weak measurement we need that the uncertainty in the measurement will be larger then the given uncertainty of the observable. If A is distributed in several disconnected areas, say $A \in \Delta_k \ , \quad k = 1, ..l$, then generally the total uncertainty could be larger then the uncertainty in each of the component, i.e. $\Delta A >> \max_k(\Delta_k)$. Due to the existence of these two scales, it is quite possible, that while the accuracy of the measurement is too high to yield a weak measurement of A for the total two-state, (since it can differentiate between the different branches $\hat{\varrho}_k$ of $\hat{\varrho}$), it is sufficiently large for each of the components with smaller uncertainty Δ_k.

We can now rewrite equation (59) as

$$|\mathcal{MD}(t)\rangle \simeq N \sum_k a_k \Big[\exp\Big(-ig_0 q(A_w)_k\Big) + \frac{(-ig_0 q)^2}{n!}\Delta(A_w^2)_k\Big]|\mathcal{MD}(0)\rangle$$

$$\simeq N \sum_k a_k \exp(-ig_0 q (A_w)_k)\ |\mathcal{MD}(0)\rangle \tag{62}$$

or,

$$|\mathcal{MD}(t)\rangle \simeq N \sum_k a_k |\psi_{\mathcal{MD}}(\pi - (A_w)_k))\rangle \tag{63}$$

Since at each of the measurements one of the components is selected with probability $|a_k|^2$, this measurement determines the 'averaged weak value'

$$\frac{1}{\sum_k |a_k|^2} \sum_k |a_k|^2 (A_w)_k \tag{64}$$

This mixed average, can be contrasted with the purely amplitude-like weak value which by equations (46) and (60) is given by

$$\frac{1}{\sum_k a_k} \sum_k a_k (A_w)_k \tag{65}$$

To exemplify this interesting case, consider the system $\mathcal{S}$ to be a large spin with a maximal value $L = N$. Let the system be pre-selected in the state $|\psi_1\rangle = a'|L_x = N\rangle + b'|L_x = -N\rangle$, and post-selected in the state $|\psi_2\rangle = |L_y = N\rangle$. Thus the two-state is given (for $H = 0$) by

$$\hat{\varrho} = a\hat{\varrho}_+ + b\hat{\varrho}_- \tag{66}$$

where $\hat{\varrho}_\pm = |L_x = \pm N\rangle\langle L_y = N|/\langle L_y = N|L_x = \pm N\rangle$ are normalized two-states, $a = \langle L_y = N|L_x = N\rangle a'$, and $b = \langle L_y = N|L_x = -N\rangle b'$. We choose the operator to be observed as

$$A = \frac{1}{\sqrt{2}}(L_x + L_y). \tag{67}$$

The weak value of A is

$$A_w = \frac{1}{\sqrt{2}} N \left(1 + \frac{(L_x)_w}{N}\right) \sim \frac{1}{\sqrt{2}} N \tag{68}$$

In the two branches $\hat{\varrho}_\pm$ we have

$$A_{w+} = \sqrt{2}N, \qquad A_{w-} = 0 \tag{69}$$

The 'weak uncertainty' of A in the two-state $\hat{\varrho}$ is

$$\Delta A_w = \frac{1}{2}\Big(N^2\big[1 - \frac{(L_x)_w}{N}\big] + i(L_z)_w\Big) \simeq \frac{1}{2}(N^2 + iN), \tag{70}$$

while in the two branches

$$\Delta A_{w\pm} = \frac{i}{2}(L_z)_{w\pm} \simeq \frac{i}{2}N \tag{71}$$

Therefore, for a sufficiently large N, we have two scales. For $g_0^2\Delta q^2 << 1/N^2$ we shall obtain the weak value (68), but in the range $1/N^2 << g_0^2\Delta q^2 << 1/N$ we shall measure the mixed quantity

$$\frac{1}{|a|^2 + |b|^2}\Big(|a|^2 A_{w+} + |b|^2 A_{w-}\Big). \tag{72}$$

4 Conceptual implications

In this section we re-examine some possible implications of the two-state formalism to well known conceptual problems in quantum mechanics. We shall suggest that by replacing the wave function by the two-state as the fundamental object, the problem of non-local reduction can be avoided.

4.1 The EPR experiment

To set notations, suppose an observer in the 'rest frame' $\mathcal{O}$ prepares at $t < -T$ two particles with an internal spin 1/2 degree of freedom, in a singlet state. At $t = -T$ the initial state is

$$|\psi(-T)\rangle \;=\; \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2) \tag{73}$$

The indices $1, 2$ stand for the spatial location of the particles at x_1 and $x_2 = x_1 + L$, respectively. The distance L between the particles can be arbitrarily large. Suppose that at $t = +T$, an observer measures $\sigma_1 = \hat{n}_1 \cdot \vec{\sigma}_1$ and at $t = T + \epsilon$ (the spin of 1 in the $\hat{n}_1$ direction) and another observer measures $\sigma_2 \hat{n}_2 \cdot \vec{\sigma}_2$. The usual way to describe the evolution of the state is to say that the wave function (73) should be reduced according to the result of the first measurement. At $t = T + \epsilon$, the correlation between the particles is already washed out and the wave function of particle 2 is given by $\langle\sigma_1|\psi\rangle$. This description involves a non-local reduction of $|\psi\rangle$ which is clearly not covariant. An observer in a moving frame $\mathcal{O}'$ observes the measurement at site 2 take place *first*, hence he will reduce $|\psi\rangle$ according to the observed value of σ_2. From a practical point of view this discrepancy is not a problem. Probabilities are Lorentz invariant quantities.

However, from the conceptual point of view, it presents a deep difficulty. Can

we attribute any reality to the wave function if two observers $\mathcal{O}$ and $\mathcal{O}'$ describe the evolution of the system in two totally different ways?

To this well known criticism we would like now to add the following. We can define or relate to a "physical collapse" the following operational meaning. Consider a measurement described by the von-Neumann coupling

$$H_I \;=\; g_0\Big(\delta(t-\xi)-\delta(t+\xi)\Big)q\sigma_z \tag{74}$$

We imagine the measuring apparatus as another quantum system and read of the result of the measurement by coupling it to a macroscopic large system ('the environment') only after $t=\xi$. Suppose that the measurement device was prepared at $t<-\xi$ and was left undisturbed at $t\in(\xi,-\xi)$. Then, the final reading at $t>\xi$ yields the value $\delta\pi=\pi_f-\pi_i=g_0\Big(\sigma_z(t=\xi)-\sigma_z(t=-\xi)\Big)$. If the evolution of the spin (and the measuring device) in the time interval $t\in(\xi,-\xi)$ was undisturbed, then we can predicted with probability 1 that $\delta\pi=0$. However, if at $t=0$ the value of say σ_x was measured by some other device, or if some other interaction took place, then the evolution in this time interval would be disturbed and the result would generally by given by $\delta\pi\neq 0$! Therefore, we have a physical criteria to identify a reduction of the state.

Returning to the EPR experiment, let us assume that $\mathcal{O}$ measured σ_{1z} and then uses our apparatus (74) to search some discontinuity in the evolution of σ_2. Clearly, he will find $\delta\pi=g_0(\sigma_{2z}(t=T+\xi)-\sigma_{2z}(t=T-\xi))=0$ *always*!. Similarly the observer in the frame $\mathcal{O}'$ may confirm that the collapse for the spin σ_2 did not take place on his hypersurface of simultaneity. Although this argument does not role out the possibility of a non-local reduction, it shows that while we can operationally identify a local reduction, we cannot by the same measurement identify a non-local reduction. This again suggests that non-local reduction of the wave function may

not be a real physical process. Nevertheless it is possible that there exists a *local* physical process of reduction of the wave function.

If we assume that a non-local reduction is not a physical process. How should we then describe the state of the system after observation, and how can we calculate and find the (non-local) correlations in the EPR experiment?

Let us now examine the EPR experiment in the context of the two-state formulation. The state of the system is fully described only when two conditions are determined for both particles. The first condition, $|\psi_1\rangle$ is in this case a singlet state. The second condition is provided by the values of σ_1 and σ_2, i.e. by $|\psi_2\rangle = |\sigma_1\rangle \otimes |\sigma_2\rangle$. Hence, in the case $H_1 = H_2 = 0$, the normalized two-state that corresponds to the EPR experiment is given by

$$\hat{\varrho}_{EPR} \;=\; \frac{1}{2}\Big(|\uparrow_z\rangle_1 \otimes |\downarrow_z\rangle_2 - |\downarrow_z\rangle_1 \otimes |\uparrow_z\rangle_2\Big)\Big(\langle\sigma_1| \otimes \langle\sigma_2|\Big). \tag{75}$$

The EPR two-state is Lorentz covariant since it is completely determined local conditions, which are a result of local observations of the spin. To retain the usual probabilistic information consider for example the case we found $\sigma_{1z} = 1$. The probability to measure $\sigma_{2\hat{n}} = \pm 1$, for the spin of particle 2 in the direction $\hat{n}$ is obtained as a conditional probability which is derived from the two-states $\hat{\varrho}(\sigma_{2\hat{n}} = 1) \equiv \hat{\varrho}(\uparrow_{2\hat{n}})$ and $\hat{\varrho}(\sigma_{2\hat{n}} = -1) \equiv \hat{\varrho}(\downarrow_{2\hat{n}})$. The latter correspond to the two (only) possible final conditions obtained by an observation of the spin of particle 2 in the $\hat{n}$ direction. We first calculate $\rho_{in} = \hat{\varrho}\hat{\varrho}^\dagger/\mathrm{tr}(\hat{\varrho}\hat{\varrho}^\dagger)$ and $\rho_{out} = \hat{\varrho}^\dagger\hat{\varrho}/\mathrm{tr}(\hat{\varrho}^\dagger\hat{\varrho})$. The probability is then expressed by

$$\mathcal{P}rob(\uparrow_{2z}) = \frac{\langle\rho_{out}(\uparrow_{2z}), \rho_{in}\rangle}{\langle\rho_{out}(\uparrow_{2z}), \rho_{in}\rangle + \langle\rho_{out}(\downarrow_{2z}), \rho_{in}\rangle} \tag{76}$$

For $\hat{n} = \hat{z}$ we can form only the two-state $\hat{\varrho}(\downarrow_{2z})$, while for $|S_{2z}\rangle = |\uparrow_{2z}\rangle$ we *do not have* a corresponding two-state $\hat{\varrho}(\uparrow_{2z}) \in \mathcal{H}_{phys}$. In this case $\langle\psi_2|\psi_1\rangle = 0$ and

we can not form a normalized ($\mathrm{tr}\hat{\varrho} = 1$) two-state. Since we have only one possible two-state, the conditional probability equals 1.

To summarize, our description of an EPR experiment by means two-state in equation (75) is Lorentz covariant. There is no element of non-local reduction since the information on the final results is coded in the final *local* conditions. Finally, probability distributions my be restored by constructing conditional probabilities as in equation (76), i.e. by comparing different two-state ensembles.

4.2 Repeated measurements without reduction

In the usual description of repeated measurements, the state of the observed system $\mathcal{S}$ is viewed as changing discontinuously after each observation. For example, consider successive measurement of $x, p, x, ..$,or any other two non-commuting observables. These discontinuities generally correspond to non-local reductions of the wave function.

We now argue that in the two-state formulation, the evolution of the system $\mathcal{S}$ *is continuous* and the only (possible) local-reduction takes place at the measurement device. Let us consider a system $\mathcal{S}$ and two measurement devices $\mathcal{MD}_1$ and $\mathcal{MD}_2$, with the initial conditions $|\psi_1\rangle = |\pi_1 = 0\rangle \otimes |\pi_2 = 0\rangle \otimes \sum C_n |A = n\rangle$ at $t = 0$. The interaction Hamiltonian given by

$$H_I = g_0\Big(\delta(t - t_1)q_1 A + \delta(t - t_2)q_2 B\Big). \tag{77}$$

At $t = t_1$, $\mathcal{MD}_1$ interacts with $\mathcal{S}$ and at $t = T_1 = t_1 + \Delta$ the result $\pi_1 = a$ is recorded on some macroscopic body. Latter, at $t = t_2$, $\mathcal{MD}_2$ interacts too with $\mathcal{S}$, and the result $\pi_2 = b$ is recorded on a macroscopic body at time $t = T_2 = t_2 + \Delta$. The time interval, Δ, between the interaction and the final reading of π, due to some coupling to an 'external' environment, is finite but otherwise can be arbitrary. A

schematic evolution of the system in the 'forward' and 'backward' directions of time is represented in Figure 4. As long as the final state of $\mathcal{S}$ is unknown we can not fully determine the two-state of the system. The probability distribution for finding $\pi_1 = a$ and $\pi_2 = b$ depends on the final condition $|\psi_f\rangle$ (obtained by post selection) of $\mathcal{S}$ at $t = T$. Therefore, if the observations by $\mathcal{MD}_1$ and $\mathcal{MD}_2$ where performed only on a pre-selected ensemble we must average over all final possible states, i.e. consider conditional probabilities of different two-time ensembles.

For example let us consider the case of only one (known) measurement. Suppose that at some time at the future a some Hermitian operator $\hat{K}$ with eigenfunctions $|\psi_k\rangle$ is measured. Therefore *one* of the two-states $\hat{\varrho}_k$ has been determined but is unknown to us. Therefore, the probability $\mathcal{P}rob_I(a)$ to measure a is given by

$$\mathcal{P}rob_I(a) = \sum_k \mathcal{P}rob(a; \hat{\varrho}_k)\mathcal{P}rob(\psi_k), \tag{78}$$

where $\mathcal{P}rob(a; \hat{\varrho}_k)$ and $\mathcal{P}rob(\psi_k; \hat{\varrho}_k)$ are the probability to find $A = a$ (given that the final state is ψ_k), and the probability to find ψ_k, respectively. A straightforward substitution yields $\mathcal{P}rob_I(a) = |\langle a|\psi(initial)\rangle|^2$ as expected. Notice that this result does not depend on what observable is actually measured in the future. In a similar way one can reconstruct the probability to find $B = b$ at the second measurement. Therefore, as before all the usual probabilistic information may be obtained.

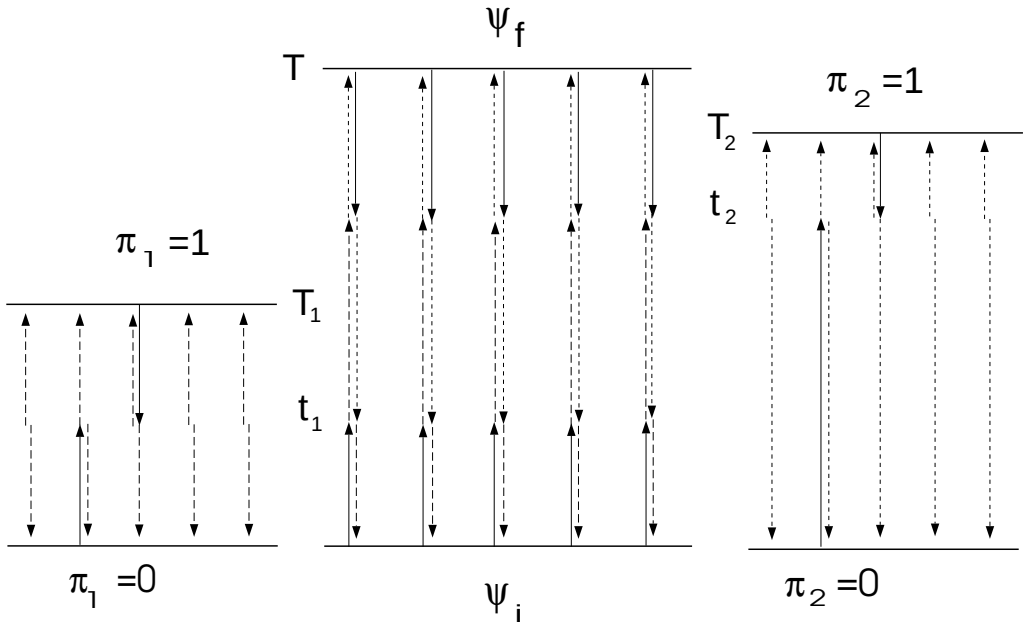

Figure 4: Pictorial description of the two-state (79) of a system under two successive observations, in the special case of a successive measurement of the same observable with the result $A = 1$. At $t = t_1$, $\mathcal{MD}_1$ (on the left) interacts with $\mathcal{S}$ and at $t = T_1$ $\mathcal{MD}_1$ is post-selected to a final state with $\pi_1 = 1$. At $t = t_2$, a second measuring device $\mathcal{MD}_2$ (on the right) interacts with $\mathcal{S}$, and post-selected to a final state with $\pi_2 = b$ at $t = T_2$. Finally at $t = T$ the system is post-selected to a final state ψ_f. Correlations between $\mathcal{MD}_1$ and $\mathcal{S}$ are denoted by dashed lines, and with $\mathcal{MD}_2$ by dotted lines. The two measuring devices must yield the same result with probability one because for any other result $\mathrm{tr}\hat{\varrho} = 0$.

Only in the special case, when the same observable is measured twice, i.e. $A = B$, we find that *for every* final state we must have $\pi_1 = \pi_2 = a$. When this condition is not satisfied we find that for every initial and final state of $\mathcal{S}$, the initial state of the total system can not evolve to the final state, i.e., $\mathrm{tr}\hat{\varrho} = \langle\psi_1|\psi_2\rangle = 0$. Therefore, in

this special case, the two measurements must yield the same result with probability one.

Therefore, let us assume that the final state of $\mathcal{S}$ has been determined and consider the evolution of the two-state in the case of repeated measurements. Since the two-state is determined only by the local conditions the state of $\mathcal{S}$ is not reduced after the coupling with $\mathcal{MD}_1$ and $\mathcal{MD}_2$. However we do pay a prize for avoiding the reduction, which is the necessity of including in our description of the total system time-like correlations. As depicted in Figure 4, the forward evolving state of $\mathcal{MD}_1$ at $t \in (t_1, T_1)$ remains correlated to the state of $\mathcal{S}$ at $t > T_1$. Similarly, the forward evolving state of $\mathcal{MD}_2$ at $t \in (t_2, T_2)$ becomes correlated with $\mathcal{S}$ and hence also with $\mathcal{MD}_1$. These time-like correlations are natural from the point of view of our formalism. The multiple-state of the total system is generally given by:

$$\hat{\varrho}(t_1, t_2, t_3) = \sum C_{ijklmn}(t_1, t_2, t_3) \hat{\varrho}_{\mathcal{MD}1ij}(t_1) \otimes \hat{\varrho}_{\mathcal{MD}2kl}(t_2) \otimes \hat{\varrho}_{\mathcal{S}mn}(t_3), \tag{79}$$

where $t_1 \in (0, T_1)$, $t_2 \in (0, T_2)$ and $t_3 \in (0, T)$.

5 Discussion

The first part of this article was devoted to a formal construction of the two-state formalism. We have seen that this formalism incorporate in a natural way two basic classes of observables. Probabilistic observables which arise whenever a system is observed by means of a (strong) demolition experiment, and complex amplitude-like observables which are measured in any non-demolition (weak) experiment. These amplitude-like observables include as a subclass, the weak values of hermitian operators. The second class of observables is also related to the recent proposal for a "measurement of the wave function"[20]. To see the connection, consider a sys-

tem, with $H = 0$, which is pre- and post-selected in the same wave function $\psi(x)$. In such circumstances, the weak value of the projection operator, $\int_\Delta dx|x\rangle\langle x|$, is given by the average value of $|\psi(x)|^2$ in the domain $x \in \Delta$. However, by Eq. (34), $|\psi(x)|^2 = \varrho(x,x)$, i.e. it is the diagonal element of the two-amplitude. Therefore, the same quantity, which is being measured in Ref. [20] by means of an adiabatic process, can be obtained also by a weak measurement. A way to measure the two-state is suggested also in Ref. [21]. We have also discovered that in the intermediate regime between strong and weak measurements, there can exist an amusing mixing of probabilities and weak values.

We have shown that the two-state formalism has also conceptual advantages. By recasting measurement theory in terms of two states as elementary objects, it seems that we came closer to formulating a sensible consistent interpretation of the measurement process. We did not eliminate completely the element of reduction, but instead we used conditions. However, by avoiding the non-local reduction, we opened the possibility of incorporating consistent local physics. Another possibility is that there is no local physical process of reduction, and that the solution may be found by handling the conditions of a closed system in a dynamical way. In this program one would like to eliminate some 'special' initial and final conditions which yield a consistency of the total history.

6 Appendix

In this appendix we shall show that non-generic two- states can describe sub-systems. For further discussion see ref. [15, 16]). Consider two non-interacting systems $\tilde{\mathcal{S}}$ and $\mathcal{S}$ that are pre- and post selected in the following states:

$$|\Psi_{in}(t=0)\rangle = \sum_{nm} a_{nm} |\tilde{\phi_n}\rangle \otimes |\psi_m\rangle \tag{80}$$

and

$$|\Psi_{out}(t=T)\rangle = \sum_{ij} b_{ij} |\tilde{\phi_i}\rangle \otimes |\xi_j\rangle \tag{81}$$

$\{|\tilde{\phi_n}\rangle\}$ is an orthonormal basis of the Hilbert space $\tilde{\mathcal{H}}$ of $\tilde{\mathcal{S}}$, $(\langle\tilde{\phi_n}|\tilde{\phi_m}\rangle = \delta_{nm})$. $\{|\psi_i\rangle\}$ and $\{|\xi_j\rangle\}$ are two orthonormal basis of the Hilbert space $\mathcal{H}$ of $\mathcal{S}$ but with the property $\langle\psi_i|\xi_j\rangle \neq 0$ for all i, j.

The total system is described by the generic two-state $\hat{\varrho}_{total} = |\Psi_{in}\rangle\langle\Psi_{out}|$. The probability of measure the eigenvalue λ of some general operator acting in $\tilde{\mathcal{H}} \otimes \mathcal{H}$ is

$$\mathcal{P}rob(\lambda) = N|\mathrm{tr}(\pi_\lambda \hat{\varrho}_{total})|^2, \tag{82}$$

where N is the normalization, and $\pi_\lambda = |\lambda\rangle\langle\lambda|$. Now suppose we are interested in measuring observables that are related only to $\mathcal{S}$, i.e. an Hermitian operators that acts in $\mathcal{H}$. In this case, equation (82) can be replaced by

$$\mathcal{P}rob(\lambda) = N|\mathrm{tr}(\pi_\lambda \hat{\varrho}_{eff})|^2 \tag{83}$$

where

$$\hat{\varrho}_{eff} = \sum c_{ij} |\psi_i\rangle\langle\xi_j|, \qquad c_{ij} = \sum_n a_{ni} b^*_{nj}. \tag{84}$$

is the reduced effective two-state. $\hat{\varrho}_{eff}$ is a non-generic two-state. Generic two-states correspond to a complete specification of the initial and final conditions for the system. When the conditions are determined only "partially" the system is

initially and finally in a mixed state. In the context of our formalism this can be interpreted as a situation with correlations between the initial and final conditions.

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