Appendix A
Time-Reversal Symmetry

The reversal in time of a state $|\psi\rangle$ changes it into a state $|\psi'\rangle$ that develops in accordance with the opposite sense of progression of time. For the new state the signs of all linear and angular momenta are reversed but other quantities are unchanged. Time reversal is effected by a time-independent operator T :

$$T|\psi\rangle = |\psi'\rangle \quad .$$

Time-reversal invariance means that if $|\psi_i\rangle$ is an energy eigenstate:

$$H|\psi_i\rangle = E_i|\psi_i\rangle \quad ,$$

so also $|\psi'\rangle$ is an eigenstate of H with eigenvalue E_i.

Consider now a pair of evolution operations that we can perform on $|\psi_i\rangle$ that are expected to lead to the same physical state. In case A we allow the state to propagate to time t and then let $t \to -t$. In case B we reverse time at $t = 0$ and then allow the reversed state to propagate with the opposite sense of progression of time, that is, to time $-t$.

In operation A we allow the state $|\psi_i\rangle$ to first evolve forward in time:

$$e^{-iHt/\hbar}|\psi_i\rangle = e^{-iE_it/\hbar}|\psi_i\rangle \tag{A.1}$$

and then we apply the time-reversal operator:

$$T\, e^{-iE_it/\hbar}|\psi_i\rangle = |\psi_i\rangle_A \quad . \tag{A.2}$$

For operation B, we first apply T and then let the system evolve backward in time:

$$e^{iE_it/\hbar}T|\psi_i\rangle = |\psi_i\rangle_B \quad . \tag{A.3}$$

We require that these operations prepare the system in the same quantum state:

$$|\psi_i\rangle_A = |\psi_i\rangle_B \quad , \tag{A.4}$$

so:

$$Te^{-iE_it/\hbar}|\psi_i\rangle = e^{iE_it/\hbar}T|\psi_i\rangle \quad . \tag{A.5}$$

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We see from this relation that T is *not* a linear operator. The time-reversal operator is: *antilinear*

$$T(a_1|\psi_1\rangle + a_2|\psi_2\rangle) = a_1^* T|\psi_1\rangle + a_2^* T|\psi_2\rangle \quad . \tag{A.6}$$

Suppose we construct a complete and orthonormal set of energy eigenstates $|n\rangle$,

$$H|n\rangle = E_n|n\rangle \tag{A.7}$$

$$\langle m|n\rangle = \delta_{m,n} \tag{A.8}$$

$$\sum_n |n\rangle\langle n| = 1 \tag{A.9}$$

and we assume that these states remain orthonormal after time reversal:

$$|n'\rangle = T|n\rangle \tag{A.10}$$

$$\langle m'|n'\rangle = \delta_{m,n} \quad . \tag{A.11}$$

We can then expand two states $|\psi\rangle$ and $|\phi\rangle$ as:

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle \quad , \tag{A.12}$$

and:

$$|\phi\rangle = \sum_n |n\rangle\langle n|\phi\rangle \quad . \tag{A.13}$$

When we apply T to Eq. (A.12) we obtain:

$$\begin{aligned} |\psi'\rangle &= T|\psi\rangle = \sum_n \langle n|\psi\rangle^* \, T|n\rangle \\ &= \sum_n \langle\psi|n\rangle T|n\rangle = \sum_n \langle\psi|n\rangle|n'\rangle \quad . \end{aligned} \tag{A.14}$$

Similarly:

$$|\phi'\rangle = T|\phi\rangle = \sum_m \langle\phi|m\rangle|m'\rangle \quad . \tag{A.15}$$

Taking the inner product:

$$\langle\phi'|\psi'\rangle = \sum_{m,n} \langle m|\phi\rangle\langle m'|n'\rangle\langle\psi|n\rangle$$

$$
\begin{aligned}
&= \sum_{m,n}\langle m|\phi\rangle\delta_{m,n}\langle\psi|n\rangle \\
&= \sum_{n}\langle\psi|n\rangle\langle n|\phi\rangle \\
\langle\phi'|\psi'\rangle &= \langle\psi|\phi\rangle \quad .
\end{aligned}
\tag{A.16}
$$

The two properties:

$$T(a_1|\psi_1\rangle + a_2|\psi_2\rangle) = a_1^*T|\psi_1\rangle + a_2^*T|\psi_2\rangle \tag{A.17}$$

$$\langle\phi'|\psi'\rangle = \langle\psi|\phi\rangle = \langle\phi|\psi\rangle^* \quad , \tag{A.18}$$

where $|\psi'\rangle = T|\psi\rangle$, $|\phi'\rangle = T|\phi\rangle$, specify that T is an *antiunitary operator*.

We can also derive the appropriate transformation law for matrix elements of operators. Consider the states $|\alpha\rangle$ and $|\beta\rangle = \hat{B}|\mu\rangle$, where $\hat{B}$ is some operator. We have in general from Eq. (A.18) that:

$$\langle\alpha|\beta\rangle = \langle\beta'|\alpha'\rangle \quad , \tag{A.19}$$

where:

$$|\alpha'\rangle = T|\alpha\rangle \tag{A.20}$$

and:

$$
\begin{aligned}
|\beta'\rangle &= T\hat{B}|\mu\rangle \\
&= T\hat{B}T^{-1}T|\mu\rangle \\
&= \hat{B}'|\mu'\rangle \quad ,
\end{aligned}
\tag{A.21}
$$

where we see that operators transform as $\hat{B}' = T\hat{B}T^{-1}$. We have then the relationship between matrix elements:

$$
\begin{aligned}
\langle\alpha|\beta\rangle &= \langle\alpha|B|\mu\rangle \\
&= \langle\mu'|(B')^\dagger|\alpha'\rangle \\
&= \langle\mu'|(TBT^{-1})^\dagger|\alpha'\rangle \quad .
\end{aligned}
\tag{A.22}
$$

If we look at an average of an operator $\hat{A}$ over some probability operator, $\hat{\rho}$,

$$\langle\hat{A}\rangle = Tr\hat{\rho}\hat{A} \quad , \tag{A.23}$$

then we can write:

$$\langle\hat{A}\rangle = \sum_i\langle i|\hat{\rho}\hat{A}|i\rangle \quad . \tag{A.24}$$

Using Eq. (A.22) for the matrix element gives:

$$\langle \hat{A} \rangle = \sum_i \langle i' | (T\hat{\rho}\hat{A}T^{-1})^\dagger | i' \rangle \quad , \tag{A.25}$$

where $|i'\rangle = T|i\rangle$. Since the set $|i'\rangle$ is also complete,

$$\langle \hat{A} \rangle = Tr\ (T\hat{\rho}T^{-1}T\hat{A}T^{-1})^\dagger \quad . \tag{A.26}$$

If the probability operator is invariant under time reversal:

$$T\hat{\rho}T^{-1} = \hat{\rho} \quad , \tag{A.27}$$

we have:

$$\langle \hat{A} \rangle = Tr\ (\hat{\rho}T\hat{A}T^{-1})^\dagger \quad . \tag{A.28}$$

For two operators $\hat{A}$ and $\hat{B}$, $\left(\hat{A}\hat{B}\right)^\dagger = \hat{B}^\dagger \hat{A}^\dagger$, and:

$$\langle \hat{A} \rangle = Tr\ (T\hat{A}T^{-1})^\dagger \hat{\rho}^\dagger \quad . \tag{A.29}$$

Since the probability operator is hermitian, $\hat{\rho}^\dagger = \hat{\rho}$, we can use the cyclic invariance of the trace to obtain the invariance principle:

$$\langle \hat{A} \rangle = \left\langle \left(T\hat{A}T^{-1}\right)^\dagger \right\rangle \quad . \tag{A.30}$$

We can build up the properties of various quantities under time reversal using the fundamental relations: if $\mathbf{p}$ is a momentum operator,

$$T\mathbf{p}(t)T^{-1} = -\mathbf{p}(-t) \quad , \tag{A.31}$$

while if $\mathbf{x}$ is a position operator:

$$T\mathbf{x}(t)T^{-1} = \mathbf{x}(-t) \quad . \tag{A.32}$$

Clearly, from this one has, if $\mathbf{L}$ is an angular momentum operator:

$$T\mathbf{L}(t)T^{-1} = -\mathbf{L}(-t) \tag{A.33}$$

We see then that Hamiltonians that depend on $\mathbf{p}^2$ and $\mathbf{x}$ are time-reversal invariant:

$$THT^{-1} = H \quad . \tag{A.34}$$

Consider Maxwell's equations, :

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e \tag{A.35}$$

$$c\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \quad , \tag{A.36}$$

where ρ_e is the electric charge density and $\mathbf{J}$ is the charge-current density. Since ρ_e is even under time reversal and $\mathbf{J}$ is odd under time reversal, the electric and magnetic fields transform as:

$$T\mathbf{E}(\mathbf{x}, t)T^{-1} = \mathbf{E}(\mathbf{x}, -t) \tag{A.37}$$

while:

$$T\mathbf{B}(\mathbf{x}, t)T^{-1} = -\mathbf{B}(\mathbf{x}, -t) \quad . \tag{A.38}$$

Observables $A(\mathbf{x}, t)$ typically have a definite signature ε_A under time reversal:

$$A'(\mathbf{x}, t) = TA(\mathbf{x}, t)T^{-1} = \varepsilon_A A(\mathbf{x}, -t) \quad , \tag{A.39}$$

where $\varepsilon = +1$ for positions, electric fields, and $\varepsilon = -1$ for momenta, magnetic fields and angular momenta.