A Wheeler-DeWitt Equation with Time

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Abstract

The equation for canonical gravity produced by Wheeler and DeWitt in the late 1960s still appears to present insurmountable issues both in terms of matematrhical solutions and physical insight, one being the explicit absence of time. In this short note we introduce one possible way to make time appear again in this equation by going back to the classical equation that inspired Wheeler and DeWitt, namely the Hamilton-Jacobi-Einstein equation, and introduce a classically meaningful geometrodynamical notion of time before quantization. While straightforward solutions could not be found, we provide a simpler version of the equation that applies to the minisuperspace DeSitter universe of interest to quantum cosmology.

1 Introduction

One traditional avenue open to the quantization of gravity is that of geometrodynamics, represented by the infamous Wheeler-DeWitt (WDW) equation [1, 2, 3]. Expected to describe the quantum evolution of spatial geometry, its solution and interpretation is a long-standing problem. Inherent mathematical problems are in particular the indefiniteness of the DeWitt metric (the metric of the configuration space of the theory, also called superspace, as it is the space inhabited by all possible spatial geometries at all points of space, up to diffeomorphism) and the divergence due to the presence of a double functional derivative, to which adds the unspecified ordering of the operators.

All these technical problems occur in the "kinetic term," i.e. the term that we would expect to play such role if the analogy with the standard Schrodinger equation held. There are reasons, however, to consider this analogy not all too useful; for one, when it is interpreted as a Schroedinger equation for gravity, the WDW equation appears to be describing a stationary state. The difficulty in understanding how time evolution may formally emerge from the equation has traditionally been referred to as the problem of time. The absence itself of time from the equation is a consequence of the fact that the Hamiltonian constraint, of which the WDW equation is the quantization, specifically enforces the time diffeomorphism of General Relativity. When we observe the Hamilton-Jacobi-Einstein (HJE) equation developed by Peres [4], equivalent to the 00 component of the Einstein field equations in the Hamilton-Jacobi formalism, it is clear that also in that case time is absent where it would have been expected to be, even though the theory is classical. As the Hamilton-Jacobi-Einstein equation does not describe a timeless geometry, the true problem is not that time evolution has vanished in the quantum case, but how time evolution can be extracted as we do in the classical case.

Study of the problem of time in quantum gravity has been extensive (see [5] for a thorough review), with diverse attempts at recovering time, either at the classical or the quantum level. In the semiclassical limit of the WDW equation, it has been shown that when we include quantum matter fields in the picture, the standard Schroedinger equation for these fields can be recovered by expanding the wave functional in inverse squares of the Planck mass. The operation has analogies with the WKB and Born-Oppenheimer approximations [6, 7]. In order to retrieve the standard time evolution, one defines a functional time $\tau = \tau[x]$.

It is not clear, however, what becomes of this time beyond the semiclassical level, when gravity does not act as a stage for matter fields but rather partakes in the quantum dance. One may observe that functional time can be applied to the aforementioned HJE equation which describes the pure energetic component of gravity, and make classical time explicit while still describing the spatial geometry of General Relativity. One then recalls that this equation was the starting point of Wheeler and DeWitt.

> One day in 1965, John Wheeler had a two hours stopover between flights at the Raleigh-Durham airport in North Carolina. He called Bryce DeWitt, then at the University of North Carolina in Chapel Hill, proposing to meet at the airport during the wait. Bryce showed up with the Hamilton-Jacobi equation of general relativity, published by Asher Peres shortly earlier [...] Bryce mumbled the idea of repeating what Schroedinger did for the hydrogen atom: getting a wave equation by replacing the square of derivatives with (i times) a second derivative—a manner for undoing the optical approximation. [...] Wheeler got tremendously excited (he was often enthusiastic) and declared on the spot that the equation for quantum gravity had been found. [8, 9]

In this short note, we consider the following question: what if DeWitt had presented Wheeler with the HJE equation where functional time was made explicit rather than not? The resulting equation not only presents a notion of time (the viability for quantum gravity will still have to be assessed), albeit a functional one, but altogether eliminates the troublesome kinetic term from the WDW equation. Furthermore, by construction it still reduces to the functional Schroedinger equation in the semiclassical limit.

In section 2, we briefly review the definition of functional time. In section 3, we apply the definition of functional time to the HJE equation and quantize it. In particular, in subsection 3.1, we see how the equation simplifies in the minisupersapce of the DeSitter universe.

2 Definition of Functional Time

Let us start from the Wheeler-deWitt equation in the form

$$\int d^3x \left[-\frac{1}{2M} \left(G_{AB} \frac{\delta}{\delta h_A} \frac{\delta}{\delta h_B} \right) + M\mathcal{V}(h_A) + \mathcal{H}_\phi(h_A, \phi) \right] \Psi[h_A, \phi] = 0\,. \tag{1}$$

Here the capital indeces are pairs of the indeces, $A = \{ij\}$, $i, j \in (1, 2, 3)$, of the spatial metric $h_{ij} = g_{\mu\nu}, \mu, \nu \in \{1, 2, 3\}$, and G_AB (symmetric in these indices) is the DeWitt metric

$$G_{AB} = G_{ijkl} = \frac{1}{2\sqrt{h}} \left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl} \right)\,. \tag{2}$$

The physical scale (we have set $\hbar = c = 1$) of quantum gravity is fixed by the geometrodynamical mass M, which is proportional to the square of the Planck mass m_P

$$M = (m_P/2)^2\,, \quad m_P = (8\pi G)^{-1/2}\,. \tag{3}$$

The geometrodynamical potential density $\mathcal{V}$ is

$$\mathcal{V} = -2\sqrt{h}(R - 2\Lambda)\,, \tag{4}$$

h and R being the determinant and the Ricci scalar of the spatial metric, respectively. The Hamiltonian density operator $\mathcal{H}_\phi$ is taken to describe bosonic matter. We ignore the operator ordering problem in the geometrodynamical "kinetic" operator.

In the Born-Oppenheimer approximation, one makes the ansatz

$$\Psi[h_A, \phi] = \chi[h_A]\psi[\phi; h_A] \tag{5}$$

and considers an expansion in inverse powers of M [6, 7]. The wave functionals χ and ψ describe the "heavy" (i.e., the spatial metric components) and "light" degrees of freedom (i.e., matter), respectively.

(Notice that ψ depends on the geometry only parametrically.) One substitutes the expansion

$$\Psi[h_A, \phi] = \exp\left(iM \sum M^{-n} S_n[h_A, \phi]\right) \tag{6}$$

in the WDW equation and equates contributions to equal powers of M.

To order M^2, one has $S_0 = S_0[h_A]$, i.e. the leading contribution is purely geometrodynamical.

To order M^1, one has the Hamilton-Jacobi-Einstein equation in vacuum

$$\int d^3x \left[\frac{1}{2} G_{AB} \frac{\delta S_0}{\delta h_A} \frac{\delta S_0}{\delta h_B} + \mathcal{V}\right] = 0\,. \tag{7}$$

To order M^0, one obtains the functional Schroedinger equation for matter

$$\int d^3x \left[i\frac{\delta}{\delta\tau} - \mathcal{H}_\phi\right] \psi[\phi; h_A] = 0 \tag{8}$$

upon requiring conservation of the current associated with χ and implicitly defining WKB time via the derivative operator

$$\frac{\delta}{\delta\tau[x]} := G_{AB} \frac{\delta S_0}{\delta h_A} \frac{\delta}{\delta h_B}\,. \tag{9}$$

The definition of WKB time (9) is adopted in order to obtain the functional Schroedinger equation. However, it is a sensible definition, in that it is analogous to the material derivative in the absence of explicit dependency on time

$$\frac{d}{dt} = \boldsymbol{u} \cdot \nabla\,, \tag{10}$$

$\boldsymbol{u}$ being the velocity field (see [10]). Although we are not concerned at this level to explicitly select a foliation of Cauchy hypersurfaces by choosing $\tau(x)$, notice that from (9) follows that necessarily $h_{ij} = h_{ij}[\tau(x)]$. With this definition, the functional time derivative indeed becomes equivalent to the partial derivative when applied to the matter wave functional $\psi[\phi(x); h_A(\tau(x))]$, as it acts only on the time-dependency due to the background metric and not on the matter field itself.

3 WDW Equation

We can apply the newly acquired definition of time to rewrite the HJE equation (7) as

$$\int d^3x \left[\frac{1}{2N}\frac{\delta S_0}{\delta\tau} + \mathcal{V}\right] = 0\,. \tag{11}$$

Using this form of the HJE equation we therefore attempt quantization along the lines of Schroedinger, Wheeler, and DeWitt, and thus obtain

$$\int d^3x \frac{i}{2N}\frac{\delta}{\delta\tau}\Psi[h_{ij},\phi] = \int d^3x \left[M\mathcal{V}(h_{ij}) + \mathcal{H}_\phi(h_{ij},\phi)\right]\Psi[h_{ij},\phi]\,. \tag{12}$$

Taking into account the operator ordering problem, which we have neglected, would only contribute to the multiplying factor of the geometrodynamical momentum. In the WDW equation obtained by this quantization, the second order functional derivatives with respect to the spatial metric components are exchanged for a first order functional time derivative. One crucial point in the passage from classical to quantum is that, as we mentioned, the spatial metric is to be seen as a function of time $\tau(x)$, and not vice versa, as imposing the definition of τ might induce to think. Therefore, there should be no worries in promoting the spatial components of the metric to operators, and in treating $\tau(x)$ as a general functional definition of time that survives the passage from classical to quantum.

3.1 DeSitter Universe

While the absence of the second order functional derivative operator in (12) casts the WDW equation in a form that seems much better to deal with, a general solution far from being easily obtainable. Formally it is an equation at least as hard to handle as the Tomonaga-Schwinger equation [12]. We will not attempt a general solution or a physical interpretation of $\tau(x)$ in the general quantum case. In this section we will just show how the equation can be further simplified by considering a minisuperspace model. We will constrain ourselves the vacuum DeSitter universe, with line element

$$ds^2 = dt^2 - a(t)^2\delta_{ij}dx^i dx^j\,. \tag{13}$$

In this model we take the square root of the determinant of the spatial metric, $\gamma := \sqrt{h} = a^3$, as the only effective dynamical variable. The minisuperspace metric with respect to the variable γ is

$$G_{11} = -\frac{3}{8}\gamma\,. \tag{14}$$

Integrating through the spatial volume (which we set to unity), the HJE equation reads

$$\frac{3}{16}\gamma\left(\frac{\delta S_0}{\delta\gamma}\right)^2 = 4\Lambda\gamma \quad \rightarrow \quad S_0 = -8\sqrt{\Lambda/3}\int \gamma dt\,. \tag{15}$$

The negative sign solution corresponding to the forward time direction. We obtain in fact that the definition of functional time gives

$$\frac{\delta\gamma}{\delta\tau} = -\frac{3}{8}\gamma\frac{\delta S_0}{\delta\gamma} = \sqrt{3\Lambda}\gamma\,, \tag{16}$$

for any position on the spatial hypersurface. (This can also be equivalently obtained from the condition (**??**).) The relation (16) runs parallel to

$$\frac{d\gamma}{dt} = \sqrt{3\Lambda}\gamma\,, \tag{17}$$

which is satisfied by the classical solution $\gamma(t) = a(t)^3 = \exp(3Ht)$, where $H = \sqrt{\Lambda/3}$ is the Hubble parameter for the DeSitter universe. While t is coordinate time, however, τ is to be treated as a function, albeit a spatially constant one. Observe, in particular, that the presence of the square root in (16) is consistent with the new form of the HJE equation (11): treating time τ as a functional, the differentiation of the action correctly gives

$$\begin{aligned} S_0[\gamma[\tau+\delta\tau]] &= -8\int H\,\gamma[\tau+\delta\tau]dt \\ &= -8\int H\left(\exp(3H\tau)(1+3H\delta\tau+\dots)\right)dt \\ \rightarrow \frac{\delta S_0}{\delta\tau} &= -24H^2\gamma = -8\Lambda\gamma = -2\mathcal{V}\,. \end{aligned} \tag{18}$$

Upon quantization, the Wheeler-deWitt equation (12) in this minisuperspace simplifies to

$$i\frac{\delta}{\delta\tau}\Psi[\gamma(\tau)] = 8M\Lambda\gamma\Psi[\gamma(\tau)]\,. \tag{19}$$

Although the (bounded) "Hamiltonian" acts as a multiplicative operator, and as such admits dirac distributions as eigenfunctions (with a γ-dependent prefactor to ensure normalization with respect to the measure $\sqrt{-G_{11}}$) general time-dependent solutions to (19) are not straightforward, as we cannot treat τ as a simple scalar and the functional derivative as an ordinary derivative.

4 Discussion and Conclusions

Following DeWitt's intuition, but starting from a HJE equation, (11), rewritten so as to show the presence of a classically meaningul time, we have rewriteen the WDW equation in a way, (12), that makes time explicit in functional form while also eliminating, at least apparently, the problems associated with the kinetic term. Nevertheless, and as we could expect, simple analytical solutions are not at hand. The equation is in form analogous the Tomonaga-Schwinger equation, which presents its own problems, both mathematical and interpretative in nature [13]. In the minisuperspace DeSitter model the equation reduces to a particularly simple expression, (19), that presents, however, the same difficulties. Due to its simplicity and its cosmological interest, the DeSitter universe minisuperspace might provide the best setting to try and solve these difficulties.

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