

# Multiple Time Dimensions\*

Steven Weinstein<sup>†</sup>

University of Waterloo Dept. of Philosophy  
 200 University Ave W, Waterloo, ON N2L 3G1, Canada  
 Perimeter Institute for Theoretical Physics  
 31 Caroline St N, Waterloo, ON N2L 2Y5, Canada

## Abstract

The possibility of physics in multiple time dimensions is investigated. Drawing on recent work by Walter Craig and myself [3], I show that, contrary to conventional wisdom, there is a well-posed initial value problem—deterministic, stable evolution—for theories in multiple time dimensions. Though similar in many ways to ordinary, single-time theories, multi-time theories have some rather intriguing properties which suggest new directions for the understanding of fundamental physics.

## 1 Introduction

The theoretical framework of physics has evolved enormously since the time of Newton, but one notable invariant, so pervasive as to be effectively invisible, is the one-dimensionality of time. While time and space have been amalgamated into a composite known as *spacetime* in the wake of relativity theory, and while modern superstring theories follow the Kaluza-Klein theory in postulating more than three *space* dimensions, time itself has remained one-dimensional.

Indeed, very little work has been devoted to the study of multiple time dimensions.<sup>1</sup> Yet one might like to know more about physics with multiple times for at least two reasons:

- It's not at all clear we can be confident that our world has a single time dimension unless we know what a world with multiple times looks like. Kant thought such a world was inconceivable. But Kant also thought that space must be three-dimensional and Euclidean [8].
- Problems connected with the interpretation of quantum mechanics, the construction and interpretation of a quantum theory of gravity, and the origin of cosmological time asymmetry all suggest the need for a new conceptual framework.

These questions motivate recent work of Walter Craig and myself [3], work which explores, from a mathematical perspective, the features one might expect in a theory with multiple time dimensions. The results are surprising, undermining as they do the conventional wisdom that such theories are plagued by instabilities [4] or are hopelessly unpredictable [9]. As I'll show, theories in multiple time dimensions allow a meaningful sense of determinism, while giving rise to intriguingly nonlocal constraints on initial data. Furthermore, the way in which these sorts of constraints arise, from a reconception of the structure of the spacetime background, suggests heretofore unexplored ways of extending physical theory.

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<sup>†</sup>Email: sw@uwaterloo.ca; sweinstein@perimeterinstitute.ca.

<sup>1</sup>A notable exception is the recent work of Bars [1], which however treats the single extra time dimension as “gauge”, thus unphysical.

## 2 Physics in one time and many

### 2.1 One time

The special theory of relativity brought with it the relativity of simultaneity, which in turn prompted a reimagining of space and time as *spacetime*, a four-dimensional manifold of points  $M$  equipped with a metric  $g$ , the latter giving the distance between pairs of nearby points. If the square of the distance is positive, then the distance is spatial, and if it's negative, the distance is temporal. We say that the *signature* of the metric on a 4-dimensional manifold  $M$  metric is  $(-, +, +, +)$  if three of the directions are spatial and one is temporal.<sup>2</sup> The signature of a 5-dimensional metric with four space dimensions and one time dimension is thus  $(-, +, +, +, +)$ , whereas a metric with three space and two time dimensions has signature  $(-, -, +, +, +)$ . Thus it is straightforward to characterize spacetimes with any given number of space and time dimensions.

Matter generally takes the form of either particles or fields, though extended objects such as strings and membranes are also possible. The focus here will be on fields, in particular the massless scalar field  $\phi = \phi(x_1, x_2, x_3, t)$  described by the ‘wave equation’

$$\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) \phi = \frac{\partial^2}{\partial t^2} \phi. \quad (1)$$

This is a simple equation in three space dimensions and one time dimension which describes many phenomena of interest, most notably the propagation of the components of the electromagnetic field (thus the behavior of light).

There is a ‘well-posed’ initial value problem for this equation. What this means is that if we are given sufficient information about the field at a given time, a stable solution of the equation exists, and it is unique. In other words, the initial data completely determine the data at all other times, and do so in such a way that small errors in the specification of the initial data do not lead to uncontrollable errors in the solution.

In the usual case, in which the initial data lies on a hypersurface of codimension one (meaning a hypersurface of dimension one less than the total dimension of spacetime), the initial value problem is called the Cauchy problem. Because the equation contains only second derivatives, the appropriate initial data for the Cauchy problem consist of the field and its first normal derivative at each point. (The ‘normal’ derivative is the derivative perpendicular to the hypersurface, which is the derivative in the time direction.) This is given by the functions

$$\begin{aligned} f(x) &= \phi(x, 0) \\ g(x) &= \frac{\partial}{\partial t} \phi(x, 0) \end{aligned} \quad (2)$$

(where  $x$  stands for  $(x_1, x_2, x_3)$ ). The statement that the Cauchy problem for the wave equation is well-posed means that, given appropriately differentiable functions  $f$  and  $g$  representing the relevant properties of the field at some time, a unique, stable solution exists for all times.

### 2.2 Many times

The generalization of the wave equation to a spacetime with  $n$  space dimensions and  $m$  time dimensions is the ‘ultrahyperbolic’ equation

$$\left( \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right) \phi(x, t) = \left( \frac{\partial^2}{\partial t_1^2} + \dots + \frac{\partial^2}{\partial t_m^2} \right) \phi(x, t) \quad (3)$$

where  $x = (x_1, \dots, x_n)$  and  $t = (t_1, \dots, t_m)$  and where both  $n$  and  $m$  are greater than 1. Let's now investigate the status of the initial value *problems* for this equation, where I say “problems” rather

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<sup>2</sup>Note that the choice of sign is a matter of convention; it could just as well be  $(-, -, -, +)$ .

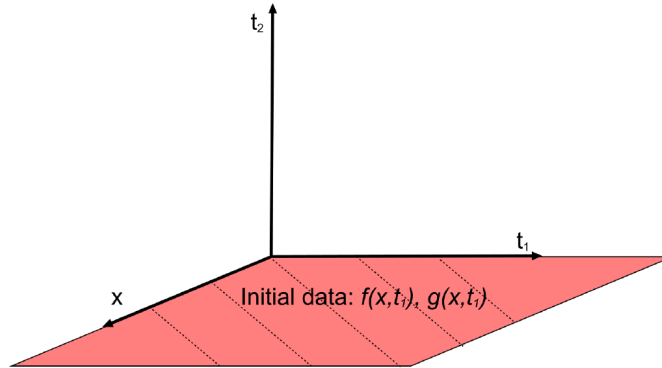
than “problem” in recognition of the fact that the meaning of “initial” is up for grabs in the presence of more than one time dimension.

### 2.2.1 $t_1 = 0$ : The Cauchy problem

It has long been known (to those who know it) that the ordinary Cauchy problem for equation (3)—the initial value problem on a surface of codimension one, i.e. dimension  $n + m - 1$ — is not well-posed. It was shown by Courant [2], using the mean-value theorem of Asgeirsson, that solutions of the equation do not exist for arbitrary choices of initial data

$$\begin{aligned} f(x, t') &= \phi(x, t') \\ g(x, t') &= \frac{\partial}{\partial t_1} \phi(x, t'), \end{aligned} \tag{4}$$

where  $t' = (t_2, \dots, t_m)$  and  $x = (x_1, \dots, x_n)$  as before. This is perhaps unsurprising, given that the initial hypersurface is a ‘mixed’ hypersurface<sup>3</sup>, extended not only in  $n$  space dimensions but also in  $m - 1$  time dimensions. Therefore, it is traversed by lightlike lines, so-called ‘characteristics’ along which one expects disturbances in the field to propagate.

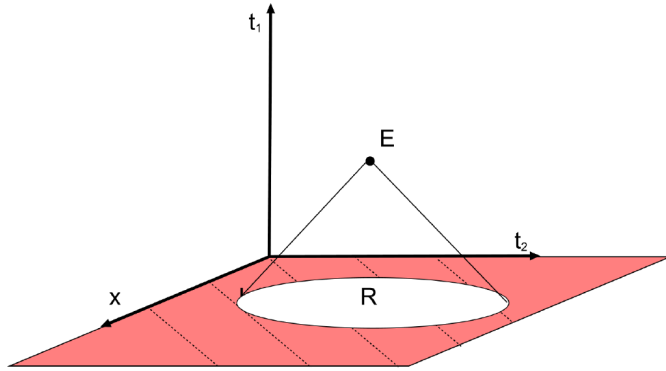


Initial data on a mixed hypersurface (dashed lines represent lightlike lines).

It might be thought that there is an additional obstacle to well-posedness, that just as (global) solutions do not exist at all for some initial data, other data are consistent with *multiple* solutions, conflicting with the uniqueness requirement. However, this is not the case: the Holmgren-John uniqueness theorem guarantees that Cauchy data on our mixed hypersurface uniquely determine the solution everywhere, as long as that initial data is consistent with *some* solution.<sup>4</sup> Indeed, it tells us that domains of dependence and influence are compact, so that we only need to know the solution on a compact region  $R$  of the Cauchy surface in order to determine the solution at a given point  $E$  off the surface.

<sup>3</sup>This is also misleadingly called a ‘timelike’ hypersurface in [6] and a ‘non-space-like’ hypersurface in [2].

<sup>4</sup>The original theorem is due to Holmgren, but is given a more general treatment in [6].



Data in  $R$  on a mixed hypersurface determines data out to  $E$ .

This holds true even for mixed surfaces in ordinary spacetime (replace  $x$  with  $x_2$ ,  $t_1$  with  $x_1$ , and  $t_2$  with  $t$  in the figure above).

The absence of a well-posed initial value problem is tantamount to a lack of any sort of practical predictability, and it has been argued that observers *qua* information processors could not even exist in such a universe, as they would be unable to engage in any meaningful action in response to information gleaned about their environment. Thus it has been argued that universes with more than one time dimension are not meaningful possibilities [9]. However, recent work of Walter Craig and myself [3] shows that this judgment is too hasty. Imposition of a constraint on the initial data yields a well-posed Cauchy problem after all.

For those unfamiliar with the notion of a constraint, consider Maxwell's theory of electromagnetism. The values of the electric and magnetic fields at a given time uniquely determine the evolution of the field. But given arbitrary initial data, which is to say arbitrary electric and magnetic fields at some time, a solution to the Maxwell equations of motion will not, in general, exist. Fortunately, Maxwell's theory comes with two constraints, Gauss's law for electricity  $\nabla \cdot E = 0$  and for magnetism  $\nabla \cdot B = 0$ . Initial  $E$  (electric) and  $B$  (magnetic) fields satisfying these constraints *do* give rise to unique global solutions. With the imposition of the constraints, the initial value problem is indeed well-posed.

The situation turns out to be similar for the ultrahyperbolic equation. There is a constraint on the initial data such that all and only data satisfying the constraint lead to a (stable) solution. The constraint is most straightforwardly specified in terms of the Fourier transforms of the initial data  $\hat{f}(k, \omega') = \mathcal{F}(f(x, t'))$  and  $\hat{g}(k, \omega') = \mathcal{F}(g(x, t'))$ . It consists simply of the requirement that the domains of  $\hat{f}$  and  $\hat{g}$  be restricted to the region

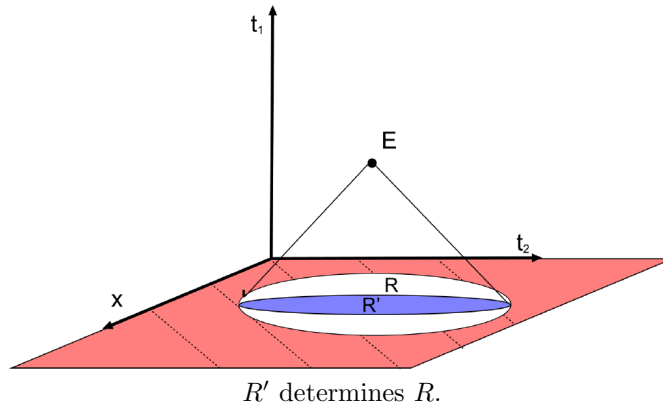
$$|k|^2 - |\omega'|^2 \geq 0. \quad (5)$$

The inverse Fourier transforms

$$\begin{aligned} f(x, t') &= \mathcal{F}^{-1}(\hat{f}(k, \omega')) \\ g(x, t') &= \mathcal{F}^{-1}(\hat{g}(k, \omega')) \end{aligned}$$

of such functions then correspond to the allowable sets of initial data. With the imposition of the constraint (5), the problem is well-posed.

The constraint (5) has an interesting property: it is nonlocal, in that it establishes nontrivial correlations between the values of the field at different points on the hypersurface. Below we have an illustration of this on a surface spanned by one space and one time dimension: Courant [2] shows that the field in  $R'$  uniquely determines the field in  $R$ .



The nonlocality here is causally benign, since there is no sense in which changes in one region bring about instantaneous changes in the larger region; it is a version of what I refer to in [10] as “nonlocality without nonlocality.” I’ll have more to say about the potential physical significance of such nonlocal constraints toward the conclusion of this essay.

We’ve seen, then, that many of the features one physics with one time dimension remain in the transition to multiple time dimensions, when the initial value problem is understood as a Cauchy problem.

- The initial value problem is well-posed, albeit in the company of a novel, nonlocal constraint.
- The domains of dependence are compact, as shown by the Holmgren-John uniqueness theorem.
- Furthermore, there is a well-defined energy functional—a Hamiltonian—which is conserved with respect to the chosen time.<sup>5</sup>

Let’s now move on and look at other versions of the initial value problem, corresponding to other notions of “initial” in the presence of multiple times.

### 2.2.2 $t = 0$ : Initial data of higher codimension

In ordinary physics with a single time dimension, determinism means that the state of the system at one time determines the state at other times. For a field theory, the initial data are naturally given on a hypersurface of codimension one, meaning one fewer dimension than the entire spacetime. The construction of the Cauchy problem in the previous section simply carries this over to a spacetime with multiple times, giving data again on a hypersurface of codimension one, the difference being that the hypersurface is mixed, rather than purely spacelike. But one might suppose that a more natural way to give initial data in a theory with multiple times is to give it on a surface of codimension  $m$  (where  $m$ , again, corresponds to the number of time dimensions.) In other words, instead of giving data at  $t_1 = 0$ , we give it on a purely spacelike hypersurface  $t = 0$  (which stands for  $t_1 = t_2 = \dots t_m = 0$ ).

Now on the one hand, one might think that the higher codimension problem is intractable, because there is simply too little information on a surface of higher codimension to uniquely determine the evolution of the field. After all, in ordinary physics, we do not expect that giving the values of a field on a high-codimension surface—e.g. the  $x, y$  plane (i.e., the  $z = 0$  plane)—will suffice to determine the evolution for all times  $t$ . Not even close! On the other hand, one might think that the presence of a constraint could help, since one cannot fill out the rest of the codimension one hypersurface arbitrarily: one must satisfy the constraint.

It turns out that the constraint is not sufficient to give a unique solution. Craig and I [3] show that the extension of data on  $(x, 0) = (x, t' = 0, t_1 = 0)$  to  $(x, t', t_1 = 0)$  and then to general  $(x, t)$  is

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<sup>5</sup>See Theorem 2 in [3].

highly nonunique, no matter how much initial data is given on the initial hypersurface. Even if one gives not just the value of the field and its first normal derivatives (in the various time directions), but an arbitrary number of additional normal derivatives, the solution is highly nonunique. So the higher codimension problem is, in fact, intractable.

### 2.2.3 $t < \varepsilon$ : An *almost*-initial value problem

Suppose that instead of giving initial data *on* the hypersurface  $t = 0$ , one gives it both on the hypersurface and in an arbitrarily small timelike neighborhood  $t < \varepsilon$  of the hypersurface. The result of Courant [2] discussed earlier has as a consequence that if the data in this neighborhood are compatible with a solution – if a solution exists for these data – then the solution is uniquely determined in the entire spacetime. Courant remarks:

[W]e are dealing with the remarkable phenomenon of functions which are not necessarily analytic, yet whose values in an arbitrarily small region determine the function in a substantially bigger domain. ([2], 760)

So moving from data *on* the higher-codimension hypersurface to data in the immediate region of the hypersurface changes the nature of the problem in an essential way. For data on  $t = 0$ , a solution always exists, but is highly underdetermined, while for data in an arbitrarily small neighborhood  $t < \varepsilon$  of that point, a solution may not exist, but if it does, it is *unique*. This “almost-initial-value” problem is not well-posed, since arbitrarily small changes may take initial data which are compatible with a solution to initial data which are not compatible. Nevertheless we have a strong form of determinism.

Note that this phenomenon is not limited to multiple time scenarios. Just as one can start with almost-initial data on all of space (all points  $x$ ) in the immediate neighborhood of  $t = 0$ , one can start with data on all of time (all points  $t$ ) in the neighborhood of some point in space  $x = 0$ . Thus in ordinary spacetime, with a single time dimension, the field in an arbitrary small volume of space  $x < \varepsilon$  specified at all times  $t$  determines the field *everywhere* in the entire spacetime. Again, one cannot pose arbitrary data on such a timelike worldtube. But if a solution does exist for the data given, then it is uniquely determined by the data in the worldtube. An observer sitting at one point in space for all time would in a sense have information about the entire, infinite spacetime.

## 3 Implications

We’ve looked at three sorts of initial value problem for a single, multi-time theory. The initial value problem on hypersurfaces of higher codimension is such that one has neither existence nor uniqueness of solution for arbitrary data, whereas a slight thickening of the higher codimension hypersurfaces *does* give something closer to a well-posed problem, since solutions are unique, if they exist at all. When we consider the Cauchy problem, though, where our initial data is specified on mixed (space and time) hypersurfaces of codimension one, we finally get something which is remarkably close to ordinary physics. Here we have a well-posed initial value problem, compact domains of dependence, and a conserved energy.

Let us focus, then, on the codimension one problem, the Cauchy problem. Certainly one might wonder about what it means to specify initial data on a hypersurface which is extended in space and time. But there is no reason *a priori* why an observer would not just treat the time dimension or dimensions on the surface as additional spatial dimensions, rewriting for example

$$\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \phi(x, t) = \left( \frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} \right) \phi(x, t). \quad (6)$$

as

$$\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial t_2^2} \right) \phi(x, t) = \frac{\partial^2}{\partial t_1^2} \phi(x, t). \quad (7)$$

Leaving aside the question of where and how the observer obtains her time orientation, let us assume that to some observer,  $t_1$  looks like time, whereas  $t_2$  looks like just another space dimension, albeit one which enters with a different sign in the equations of motion. From this perspective, the primary difference between this and the ordinary wave equation is just the anisotropy of “space”, where the scare-quotes indicate that we are referring to the mixed hypersurface coordinatized by  $x_1, x_2$ , and  $t_2$ . A further difference is that the initial value problem is well-posed only for data satisfying the constraint (5), as we’ve already noted.

From these two features, the presence of a negative sign in the equations of motion and the consequent requirement of a constraint to ensure the existence of global (nonsingular) solutions, the observer might infer one of two things:

1. The world may have begun a finite time ago, as a singularity may lie in the past, or the end of the world may be nigh, as a singularity may lie in the future.
2. The additional, nonlocal constraint is a feature of the laws of nature, a feature which guarantees that the evolution is nonsingular, having no beginning or end.

I say “may” for (1) since, just as with the wave equation in ordinary spacetime, the observer cannot predict arbitrarily far into the future or the past without access to data on the entire Cauchy surface. On the other hand, the observer might take a different attitude toward the constraint, and settle on (3) in the belief that nature abhors a singularity, or in the (related) belief that a finite universe is absurd. This observer takes the view that the constraint is an additional law of nature, one which guarantees meaningful nonsingular, global evolution.

One can imagine an argument amongst theorists in this world as to whether the constraint amounts to an *ad hoc* addition to their laws. One group looks at the laws and notices that they lead to singularities for arbitrary data and concludes that there must be an additional law constraining the data. The other group looks at the laws and from the same evidence concludes that there may be a singularity in the past or the future: the laws break down.

This is an interesting and suggestive scenario from the perspective of present-day physics, since the laws that govern the evolution of spacetime do lead to singularities for arbitrary initial data [5]. At the same time, there is evidence of nonlocality in the large and in the small. In the large, cosmology is dotted with disturbingly *ad hoc* constraints on the states of the universe, particularly the low entropy and near-homogeneity of the early universe. (These are addressed but not resolved by inflation, which requires its own set of fine-tuned parameters [7].) In the small, quantum mechanics predicts nonlocal entanglement between the properties of a given field at various locations in space. It would certainly be worthwhile to explore whether nonlocal constraints might explain any of these phenomena, perhaps in conjunction with a modification of the dynamical laws. The sort of constraint explored in this essay, one arising from the presence of extra time dimensions, exhibits one sort of nonlocality, but there are other sorts as well, given by constraints with different functional forms. What they have in common is that they embody what I have called “nonlocality without nonlocality”, meaning nonlocal correlations without nonlocal causation.<sup>6</sup>

The study of multiple time dimensions here is rather preliminary. I have not discussed gauge fields or other massive fields, and on a conceptual level I have not tackled what may be the most difficult question of all, how to characterize observers and observation in such a theory. What I have shown, I hope, is that theories with multiple time dimensions are a live conceptual possibility, and that if nothing else, they serve to stretch our minds as to what may be physically possible.

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<sup>6</sup>A more extensive treatment of the relevance of such constraints to quantum theory may be found in [10].

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