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Relativistic theory for picosecond time transfer in the vicinity of the Earth

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Abstract. The problem of light propagation is treated in a geocentric reference system with the goal of ensuring picosecond accuracy for time transfer techniques using electromagnetic signals in the vicinity of the Earth. We show that the first post-Newtonian approximation of the metric, as defined by the Resolution A4 of the International Astronomical Union, is sufficient for this purpose. We derive explicit formulae for a one way time transfer, to be applied when the spatial coordinates of the time transfer stations are known in a geocentric reference system rotating with the Earth. These expressions are extended, at the same accuracy level of one picosecond, to the special cases of two way and LASSO time transfers via geostationary satellites.

Key words: relativity – reference systems – time – artificial satellites

1. Introduction

It is well known that in relativity the notion of simultaneity is not defined a priori so that a conventional choice of a definition has to be made. This choice will then lead to a corresponding definition of clock synchronization as synchronized clocks must simultaneously produce the same time markers. A widely used definition is that of coordinate simultaneity and corresponding coordinate synchronization, as given, for example, by Klioner (1992):

"Two events fixed in some reference system by the values of their coordinates (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) are considered to be simultaneous with respect to this reference system, if the values of time coordinate corresponding to them are equal: $t_1 = t_2$. In the following this definition of simultaneity (and corresponding definition of synchronization) we shall call coordinate simultaneity (and coordinate synchronization)."

Clearly, the synchronization of two clocks by this definition is entirely dependent on the chosen reference system and is thus relative in nature, rather than absolute.

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In practice, coordinate synchronization between two distant clocks can be achieved by the exchange of an electromagnetic signal. From the knowledge of the positions of the clocks at emission and reception of the signal in the reference system of synchronization and the laws of light propagation in the same reference system, the coordinate time elapsed during transmission T_t can be calculated.

For the construction and dissemination of international reference time scales, coordinate synchronization in a geocentric, reference system is required. We choose the geocentric non-rotating reference system as defined by the resolution A4 of the IAU (1992), with asymptotically flat spatial coordinates, but with Terrestial Time TT being the coordinate time. TT is an ideal form of the International Atomic Time TAI, which is the basis of the measurement of time on the Earth. By its definition, TT differs from the coordinate time of the IAU by a constant rate.

The clocks that are to be synchronized are usually fixed on the Earth, and have their spatial positions given in a rotating reference frame. Using the metric equation of the non-rotating system and taking into account the displacement of the clocks (in the non-rotating system) resulting from the relative movement of the two systems during signal propagation, the transmission coordinate time T_t can be calculated. This is done explicitly in Sect. 2.

Recently, the precision of clock synchronization between remote clocks on the surface of the Earth has reached the sub-nanosecond level (Hetzel & Soring 1993; Veillet et al. 1992; Veillet & Fridelance 1993) with further improvements expected in the near future. For these applications it seems sensible to develop the theory to the picosecond accuracy level. Recent theoretical studies in this field claim an accuracy of 0.1 ns (Klioner 1992), and in some cases (CCIR 1990, CCDS 1980) the provided formulae are expressed in terms of path-integrals making them more difficult to use than explicit expressions. In this article we provide explicit equations for synchronization in a geocentric system of two clocks that have their positions given in the rotating system. All terms that in the vicinity of the Earth (within a geocentric sphere of 200 000 km radius) are greater than one picosecond are included. Outside this sphere terms due to the potentials of the Moon may amount to more than 1 ps and need to be accounted for separately. We also present formulae (to the same accuracy) for the special cases of two way time transfer (Sect. 3) and LASSO (LAser Synchronization from Stationary Orbit, Sect. 4) time transfers via a geostationary satellite. Here a possible small residual velocity of the satellite ($< 1\,\mathrm{m\,s^{-1}}$) results in further terms contributing some tens of picoseconds for two way- and LASSO time transfers.

We will assume that all clocks are rate corrected for the gravitational potential at their positions, and their velocities in the reference system of synchronization, and hence run at the rate of TT. An examination of the effects of this and other factors on the limits of practical applicability of the given formulae is presented in Sect. 5.

Finally, we apply the analytical formula obtained for the two way time transfer to particular situations and we compare the results to those obtained using a more exact numerical analysis (Sect. 6).

2. Derivation of the formula for a one way transfer

We consider a rotating frame $(t, \overline{x}_\mathrm{r})$ which rotates at a constant angular velocity ω with respect to a fixed star oriented one $(t, \overline{x})$ with two clocks a and b, at $\overline{x}_{\mathrm{r}a}$ and $\overline{x}_{\mathrm{r}b}$ at time t_0 when the two frames coincide. The two clocks are to be coordinate synchronized by the transmission of an electromagnetic signal from a (emission at t_0) to b (reception at t_1).

To this end the coordinate time interval $T_\mathrm{t} = t_1 - t_0$ elapsed between emission and reception of the signal needs to be calculated.

The metric of a geocentric non-rotating system in the first post-Newtonian approximation with TT as coordinate time and asymptotically flat spatial coordinates (for $r \longrightarrow \infty$ the components of the spatial metric $g_{ij} = \delta_{ij}$) is:

$$\mathrm{d}s^2 = -(1 - 2U/c^2)(1 + L_\mathrm{g})^2 c^2 \mathrm{d}t^2 + (1 + 2U/c^2)(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2) \quad (1)$$

where:
$\mathrm{d}s$ is the relativistic line element.
t is the coordinate time TT.
r, θ (colatitude), ϕ (longitude) are spherical coordinates in a nonrotating geocentric coordinate system.
U is the gravitational potential of the Earth (positive sign).
$L_\mathrm{g} = 6.969291\ 10^{-10}$.

The scaling factor $(1 + L_\mathrm{g})$ results from the choice of TT as coordinate time. L_g is equal to U_g/c^2, where U_g is the value of the gravitational potential on the geoid including the centrifugal potential due to the rotation of the Earth.

In the general expression of (1), the potential U should include, in addition to the gravitational potential of the Earth, the tidal potential of external bodies. However, these terms are not needed in the framework of this paper. They have an effect of less than 1 ps on time transfer within a geocentric sphere of 200 000 km radius.

Introduction of post-post-Newtonian terms in the metric leads to a correction to the propagation time of a light signal in the vicinity of the Earth of order c^{-4} which is much less than one picosecond. It is hence sufficient to use the post-Newtonian metric (1) (for more detail see Sect. 5.1).

Transforming to the rotating frame by

$$\mathrm{d}\phi = \omega \mathrm{d}t + \mathrm{d}\phi_\mathrm{r} \quad (2)$$

and substituting into the metric (1) we find:

$$\begin{aligned}\mathrm{d}s^2 = &-(1 - 2U/c^2)(1 + L_\mathrm{g})^2 c^2 \mathrm{d}t^2 \\ &+ (1 + 2U/c^2)[\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \\ &+ r^2 \sin^2\theta(\omega^2 \mathrm{d}t^2 + 2\omega \mathrm{d}t \mathrm{d}\phi_\mathrm{r} + \mathrm{d}\phi_\mathrm{r}^2)]\,. \end{aligned} \quad (3)$$

Setting $\mathrm{d}s^2 = 0$ for a light signal and solving the resulting quadratic for $\mathrm{d}t$ provides an expression for the transmission coordinate time T_t:

$$\begin{aligned}T_\mathrm{t} = \int &\{\mathrm{d}u/c - U_\mathrm{g}\mathrm{d}u/c^3 + \omega r^2 \sin^2\theta \mathrm{d}\phi_\mathrm{r}/c^2 \\ &+ [1 + r^2 \sin^2\theta(\mathrm{d}\phi_\mathrm{r}/\mathrm{d}u)^2]\omega^2 r^2 \sin^2\theta \mathrm{d}u/2c^3 \\ &+ 2U\mathrm{d}u/c^3\} + O(c^{-4})\,,\end{aligned} \quad (4)$$

where $\mathrm{d}u$ is the increment of coordinate length along the transmission path and the integral is to be taken from a to b along the transmission path.

Terms of order c^{-4} are less than one picosecond in the vicinity of the Earth and can be neglected (see Sect. 5.1).

Note that, to the required order, geometrical terms (the first four terms of (4)) can be separated from gravitational ones (the last term of (4)). Coupled terms only appear at the c^{-4} order. Hence, to the required accuracy, one can write

$$T_\mathrm{t} = T + T_\mathrm{g} \quad (5)$$

where T and T_g are the transmission times due to the geometrical and gravitational terms respectively.

Instead of integrating along the path in the rotating frame, which is given by the geodesic equations in this system and can be quite complex, we prefer to calculate the positions of the stations at emission and reception in the nonrotating frame which yields the path in this frame (a straight line) and hence the transmission time. The deviation of the path from a straight line in the non-rotating frame due to the gravitational field and atmospheric refraction results in a correction to the transmission time which is less than one picosecond for all practical purposes (see Sects. 5.4 and 5.5).

Considering at first only the geometry (neglecting all gravitational terms) the transmission time can be written as:

$$T = (1 - U_\mathrm{g}/c^2)|\overline{x}_{\mathrm{r}b} - \overline{x}_{\mathrm{r}a}|/c + s \quad (6)$$

where s represents the time taken for the signal to travel the extra path due to the motion of b in the non-rotating frame during transmission (see Fig. 1). This is usually referred to as the

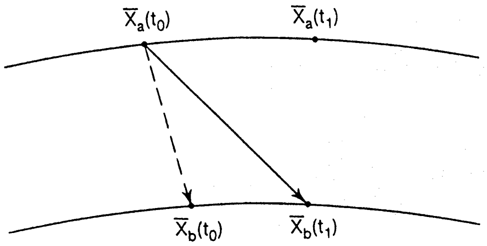

Fig. 1. One way time transfer in the non-rotating frame. The dashed line represents $\overline{x}_{rb} - \overline{x}_{ra}$ while the continuous line shows the path taken by the signal in the non-rotating frame

Sagnac correction, first discovered experimentally by Sagnac (1913).

Defining:

$$\begin{aligned} \overline{R}_0 &= \overline{x}_{rb} - \overline{x}_{ra} \\ \overline{v}_b &= \overline{\omega} \times \overline{x}_{rb} + \overline{v}_{rb} \\ \overline{a}_b &= \overline{\omega} \times (\overline{\omega} \times \overline{x}_{rb})) + \overline{\omega} \times \overline{v}_{rb} + \overline{a}_{rb} \end{aligned} \tag{7}$$

with $\overline{v}_{rb}$ and $\overline{a}_{rb}$ being the velocity and acceleration in the rotating frame, and the two frames coinciding at $t = t_0$.

The path travelled by the signal $\overline{R}(T)$ can be expressed as a series expansion in terms of T in the non-rotating frame:

$$\overline{R}(T) = \overline{R}_0 + \overline{v}_b T + \overline{a}_b T^2/2 + O(T^3) \ , \tag{8}$$

its magnitude $R(T)$ is given by:

$$\begin{aligned} R(T) = R_0[1 &+ (2\overline{R}_0 \cdot \overline{v}_b / R_0^2)T \\ &+ ((v_b^2 + \overline{R}_0 \cdot \overline{a}_b)/R_0^2)T^2 + O(T^3)]^{1/2} \ . \end{aligned} \tag{9}$$

Expanding the square-root and writing $R(T) = cT/(1 - U_g/c^2)$ we find:

$$\begin{aligned} T = [R_0 &+ (\overline{R}_0 \cdot \overline{v}_b / R_0)T \\ &+ ((v_b^2 + \overline{R}_0 \cdot \overline{a}_b - (\overline{R}_0 \cdot \overline{v}_b)^2/R_0^2)/2R_0)T^2 \\ &+ O(T^3)](1 - U_g/c^2)/c \ . \end{aligned} \tag{10}$$

Starting with $T = R_0/c$ and iterating twice yields an expression for the transmission time in terms of the known quantities $\overline{R}_0$, $\overline{v}_b$ and $\overline{a}_b$:

$$\begin{aligned} T = (1 &- U_g/c^2)R_0/c + \overline{R}_0 \cdot \overline{v}_b/c^2 \\ &+ (v_b^2 + \overline{R}_0 \cdot \overline{a}_b + (\overline{R}_0 \cdot \overline{v}_b)^2/R_0^2)R_0/2c^3 \\ &+ O(c^{-4}) \ . \end{aligned} \tag{11}$$

Substituting from (7) for $\overline{R}_0$ and $\overline{v}_b$ in the second term of (11) shows that, when the receiving station is geostationary ($\overline{v}_{rb} = 0$), this term is equivalent to the generally used expression for the Sagnac correction:

$$\overline{R}_0 \cdot \overline{v}_b/c^2 = -\overline{x}_{ra} \cdot (\overline{\omega} \times \overline{x}_{rb})/c^2 = 2\omega A_E/c^2 \tag{12}$$

where A_E is the area of the equatorial projection of the triangle whose vertices are the centre of the Earth and the positions of the clocks in the rotating frame. A_E is positive for signal propagation in the eastward direction and negative otherwise.

The third term is of the next higher order and can amount to ~ 10 ps for a one-way transfer between a geostationary satellite and a station on the surface of the Earth.

The gravitational term of (4) can be integrated along a straight line in the non-rotating frame. Taking as the gravitational potential of the Earth

$$U = GM_E/r \tag{13}$$

with
G = Gravitational constant
M_E = Mass of the Earth
($GM_E = 3.986\ 10^{14}$ is sufficient for our purpose)
and integrating the last term of (4) we find:

$$T_g = 2GM_E \ln\left\{[x_b(t1) + \overline{n} \cdot \overline{x}_b(t1)]/[x_a(t0) + \overline{n} \cdot \overline{x}_a(t0)]\right\}/c^3 \tag{14}$$

where $n = \overline{R}_0/R_0$ is the unit vector along the path of the signal.

This term can amount to about 200 ps for a one way time transfer in the vicinity of the Earth. Higher order terms due to a more realistic expression for the Earth's gravitational potential (including the quadrupole moment) amount to some 10^{-2} ps and can be neglected.

Replacing $\overline{x}_b(t1)$ by $\overline{x}_b(t0)$ in (14) induces an error in T_g of less than 1 ps, hence the total transmission time T_t can be written as:

$$\begin{aligned} T_t = T + T_g = R_0/c + \delta &= R_0/c - U_g R_0/c^3 + \overline{R}_0 \cdot \overline{v}_b/c^2 \\ &+ (v_b^2 + \overline{R}_0 \cdot \overline{a}_b + (\overline{R}_0 \cdot \overline{v}_b)^2/R_0^2)R_0/2c^3 \\ &+ 2GM_E \ln\{[x_{rb} + \overline{n} \cdot \overline{x}_{rb}]/[x_{ra} + \overline{n} \cdot \overline{x}_{ra}]\}/c^3 \end{aligned} \tag{15}$$

where δ is the total relativistic correction.

The above expression provides the coordinate transmission time for a light signal travelling from station a to station b in the vicinity of the Earth (within a geocentric sphere of 200 000 km radius) with the coordinates of the two stations given in an Earth fixed rotating frame. All terms that are greater than one picosecond are included. Note however, that atmospheric delays which can amount to several tens of nanoseconds are not considered and need to be taken into account separately (see also Sect. 5.4).

3. Two way time transfer

We consider a two way time transfer between two stations c and d, fixed on the surface of the Earth, via a geostationary satellite s (as shown in Fig. 2).

Two signals are transmitted in opposite directions leaving c and d at t_0 and $t_0 + \Delta t$ respectively. They reach the satellite at t_1 and t_3, where they are immediately retransmitted, and arrive at the opposite stations at t_2 and t_4. From the clocks two coordinate time intervals are obtained (assuming that the clocks are rate corrected as mentioned in Sect. 1):

$$\begin{aligned} t_c &= t_4 - t_0 \\ t_d &= t_2 - t_0 - \Delta t \ . \end{aligned} \tag{16}$$

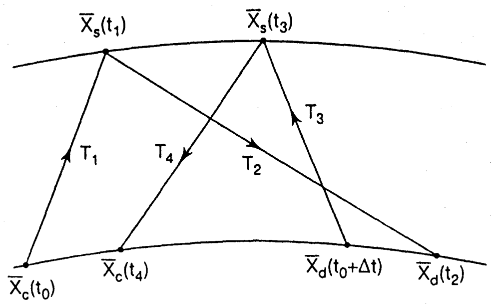

Fig. 2. Two way time transfer in the non-rotating frame. Two signals are transmitted in opposite directions leaving c and d at t_0 and $t_0 + \Delta t$ respectively. They reach the satellite at t_1 and t_3, where they are immediately retransmitted, and arrive at the opposite stations at t_2 and t_4

For synchronization the interval Δt is required. We shall assume that the clocks have been synchronized previously to within 0.1 s, a typical station to satellite transmission time (which can be achieved without difficulty in practice), and that the satellite has a residual velocity v_r smaller than $1\,\mathrm{m\,s^{-1}}$ and a residual acceleration in the rotating frame of less than $10^{-5}\,\mathrm{m\,s^{-1}}$. These values have been chosen as typical after consultation of the EUTELSAT satellite control centre.

The defining equations for the transmission times are:

$$\begin{aligned} T_1 &= t_1 - t_0 \\ T_2 &= t_2 - t_1 \\ T_3 &= t_3 - t_0 - \Delta t \\ T_4 &= t_4 - t_3 \end{aligned} \tag{17}$$

and solving for Δt yields:

$$\begin{aligned} \Delta t &= (t_c - t_d)/2 + \delta \\ \delta &= (T_1 + T_2 - T_3 - T_4)/2 \,. \end{aligned} \tag{18}$$

The relativistic correction δ arises from the motion of the stations and the satellite in the frame of synchronization and the gravitational delays for the individual transmissions T_1 to T_4.

We note that to the required accuracy the gravitational terms cancel in the differences $T_1 - T_4$ and $T_2 - T_3$. Therefore only the geometrical terms need to be considered. For the individual links these are given by (11) when substituting:

for T_1:

$$\begin{aligned} \overline{R}_0 &= \overline{x}_{rs} - \overline{x}_{rc} \\ \overline{v}_b &= \overline{\omega} \times \overline{x}_{rs} + \overline{v}_r \\ \overline{a}_b &= \overline{\omega} \times (\overline{\omega} \times \overline{x}_{rs}) + \overline{\omega} \times \overline{v}_r + \mathrm{d}\overline{v}_r/\mathrm{d}t \end{aligned} \tag{19a}$$

for T_2:

$$\begin{aligned} \overline{R}_0 &= -(\overline{x}_{rs} - \overline{x}_{rd} + \overline{v}_r T_1) \\ \overline{v}_b &= \overline{\omega} \times \overline{x}_{rd} \\ \overline{a}_b &= \overline{\omega} \times (\overline{\omega} \times \overline{x}_{rd}) \end{aligned} \tag{19b}$$

for T_3:

$$\begin{aligned} \overline{R}_0 &= \overline{x}_{rs} - \overline{x}_{rd} + \overline{v}_r \Delta t \\ \overline{v}_b &= \overline{\omega} \times (\overline{x}_{rs} + \overline{v}_r \Delta t) + \overline{v}_r \\ \overline{a}_b &= \overline{\omega} \times [\overline{\omega} \times (\overline{x}_{rs} + \overline{v}_r \Delta t)] + \overline{\omega} \times \overline{v}_r + \mathrm{d}\overline{v}_r/\mathrm{d}t \end{aligned} \tag{19c}$$

for T_4:

$$\begin{aligned} \overline{R}_0 &= -(\overline{x}_{rs} - \overline{x}_{rc} + \overline{v}_r(\Delta t + T_3)) \\ \overline{v}_b &= \overline{\omega} \times \overline{x}_{rc} \\ \overline{a}_b &= \overline{\omega} \times (\overline{\omega} \times \overline{x}_{rc}) \,. \end{aligned} \tag{19d}$$

All positions are given at $t = t_0$ when the two frames coincide.

T_1 and T_3 in (19b) and (19d) can be replaced by their first order approximations (ie. $T_1 \approx |\overline{x}_{rs} - \overline{x}_{rc}|/c$ and $T_3 \approx |\overline{x}_{rs} - \overline{x}_{rd}|/c$), inducing an error of less than 10^{-3} ps for typical cases.

This allows expressions for T_1 to T_4 to be obtained. Substituting these into (18) and neglecting all terms smaller than some 10^{-2} ps (see Sect. 5.1) yields:

$$\begin{aligned} \delta = \{&\overline{R}_{cd} \cdot (\overline{\omega} \times \overline{x}_{rs}) \\ &+ [(R_{cs} - R_{ds} - c\Delta t)(R_{ds}\overline{R}_{cs} + R_{cs}\overline{R}_{ds}) \cdot \overline{v}_r] \\ &/(2R_{cs}R_{ds})\}/c^2 + O((v/c)(v_r/c)\Delta t) \end{aligned} \tag{20}$$

where

$$\begin{aligned} \overline{R}_{cs} &= \overline{x}_{rs} - \overline{x}_{rc} \\ \overline{R}_{ds} &= \overline{x}_{rs} - \overline{x}_{rd} \\ \overline{R}_{cd} &= \overline{x}_{rd} - \overline{x}_{rc} \,. \end{aligned}$$

The first term is equivalent to $2\omega A_E/c^2$ with A_E now being the area of the equatorial projection of the quadrangle whose vertices are the centre of the Earth and the positions of the satellite and the stations in the rotating frame.

Note that there are no terms of order c^{-1} and c^{-3} corresponding to the first and the third term in (11). These terms cancel in the differences $T_1 - T_4$ and $T_2 - T_3$.

The second term of (20) varies with v_r and Δt, and can amount to several hundred picoseconds. If $\Delta t \sim 0$, it can amount to several tens of picoseconds, depending on the residual velocity which is in general not well known. However, one can compensate for it by intentionally introducing a desynchronization in order to drive this term towards zero, which is the case when the two signals arrive at S at about the same time (i.e. $t_1 \approx t_3$).

4. LASSO

In this method laser pulses emitted from the stations c and d at t_0 and $t_0 + \Delta t$ respectively are reflected by the geostationary satellite and return to the stations (as shown in Fig. 3).

The satellite is equipped with a clock which measures the time interval between arrival of the signals. Hence three coordinate time intervals (after rate correction of the clocks) are obtained:

$$\begin{aligned} t_c &= t_2 - t_0 \\ t_d &= t_4 - t_0 - \Delta t \\ t_s &= t_3 - t_1 \,. \end{aligned} \tag{21}$$

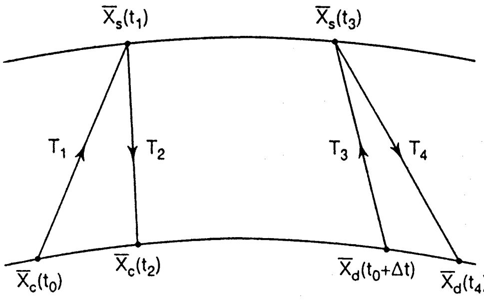

Fig. 3. LASSO time transfer in the non-rotating frame. Laser pulses emitted from the stations c and d at t_0 and $t_0 + \Delta t$ respectively are reflected by the geostationary satellite and return to the stations. A clock on board the satellite measures the time interval between arrival of the pulses

For synchronization Δt is required. Similarly to the two way case, the defining Eqs. (17) for T_1 to T_4 yield:

$$\begin{aligned}\Delta t &= (t_c - t_d)/2 + t_s + \delta \\ \delta &= (T_1 - T_2 - T_3 + T_4)/2 \ .\end{aligned} \tag{22}$$

Again the gravitational terms cancel to the required accuracy in the differences $T_1 - T_2$ and $T_4 - T_3$.

In order to calculate the individual transmission times T_1 to T_4 we substitute into (11):

for T_1:

$$\begin{aligned}\overline{R}_0 &= \overline{x}_{rs} - \overline{x}_{rc} \\ \overline{v}_b &= \overline{\omega} \times \overline{x}_{rs} + \overline{v}_r \\ \overline{a}_b &= \overline{\omega} \times (\overline{\omega} \times \overline{x}_{rs}) + \overline{\omega} \times \overline{v}_r + \mathrm{d}\overline{v}_r/\mathrm{d}t\end{aligned} \tag{23a}$$

for T_2:

$$\begin{aligned}\overline{R}_0 &= -(\overline{x}_{rs} - \overline{x}_{rc} + \overline{v}_r T_1) \\ \overline{v}_b &= \overline{\omega} \times \overline{x}_{rc} \\ \overline{a}_b &= \overline{\omega} \times (\overline{\omega} \times \overline{x}_{rc})\end{aligned} \tag{23b}$$

for T_3:

$$\begin{aligned}\overline{R}_0 &= \overline{x}_{rs} - \overline{x}_{rd} + \overline{v}_r \Delta t \\ \overline{v}_b &= \overline{\omega} \times (\overline{x}_{rs} + \overline{v}_r \Delta t) + \overline{v}_r \\ \overline{a}_b &= \overline{\omega} \times [\overline{\omega} \times (\overline{x}_{rs} + \overline{v}_r \Delta t)] + \overline{\omega} \times \overline{v}_r + \mathrm{d}\overline{v}_r/\mathrm{d}t\end{aligned} \tag{23c}$$

for T_4:

$$\begin{aligned}\overline{R}_0 &= -(\overline{x}_{rs} - \overline{x}_{rd} + \overline{v}_r(\Delta t + T_3)) \\ \overline{v}_b &= \overline{\omega} \times \overline{x}_{rd} \\ \overline{a}_b &= \overline{\omega} \times (\overline{\omega} \times \overline{x}_{rd}) \ .\end{aligned} \tag{23d}$$

All positions are given at $t = t_0$ when the two frames coincide.

As for the two way transfer T_1 and T_3 can be replaced by their first order approximations in (23b) and (23d) inducing an error of some 10^{-3} ps.

Substituting the expressions obtained for T_1 to T_4 into (22) and neglecting terms smaller than 10^{-2} ps (see Sect. 5.1) yields the total relativistic correction for LASSO time transfers:

$$\begin{aligned}\delta &= [\overline{R}_{cd} \cdot (\overline{\omega} \times \overline{x}_{rs}) + \Delta t(\overline{\omega} \times \overline{v}_r) \cdot \overline{x}_{rd}]/c^2 \\ &\quad + O((v/c)(v_r/c)(R_0/c)) \ .\end{aligned} \tag{24}$$

Note that (24) is written with the position of the satellite in the rotating frame at t_0. When considering the position at $t_0 + \Delta t/2$ it can be rewritten in a more symmetric form:

$$\begin{aligned}\delta &= [\overline{R}_{cd} \cdot (\overline{\omega} \times \overline{x}_{rs}) + \Delta t(\overline{\omega} \times \overline{v}_r) \cdot (\overline{x}_{rc} + \overline{x}_{rd})/2]/c^2 \\ &\quad + O((v/c)(v_r/c)(R_0/c)) \ .\end{aligned} \tag{25}$$

As in (20) the first term is equivalent to $2\omega A_E/c^2$.

There are again no terms in c^{-1} and c^{-3} corresponding to the first and the third term in (11). They cancel when the differences $T_1 - T_2$ and $T_4 - T_3$ are formed.

The second term varies with $\overline{v}_r$ and Δt. This term is smaller than 10^{-2} ps for $\overline{v}_r \sim 1\,\mathrm{m\,s^{-1}}$ and $\Delta t \sim 0.1$ s, which is the case for a two way transfer and hence it does not appear in (20). However, for LASSO Δt can amount to several minutes in practice (Veillet et al. 1992; Veillet & Fridelance 1993) and therefore the second term in (24) and (25) can contribute up to 10 ps.

Note also that while the second term of (20) can be minimised by an appropriate choice of Δt, this is not the case in (24) and (25).

The fact that the second terms in (24) and (25) are of higher order than the second term of (20) reflects the only indirect dependence of the relativistic correction for LASSO on the velocity of the satellite. For a two way transfer cancellation takes place when forming the differences $T_1 - T_4$ and $T_2 - T_3$. The magnitudes of these differences are dependent on the movement of the satellite between t_1 and t_3. This is not the case for LASSO, where the differences $T_1 - T_2$ and $T_4 - T_3$ are independent of the motion of the satellite.

5. Limits of applicability in practice

5.1. Neglected terms

Post-post-Newtonian terms, and the g_{0i} components in the metric (1) are of order U^2/c^4 and $G\omega I_E/r^2c^3$ respectively (where I_E is the angular momentum of the Earth and r the distance from the geocentre). The gravitational time delay due to these terms is given by the integral along the path of the signal of terms of order U^2/c^5 and $G\omega I_E/r^2c^4$, the result of which is much less than one picosecond for the situations considered.

The next terms in (4) are coupled terms of order $U\omega r^2/c^4$ which are less than one picosecond for the situations considered.

The next terms in Eq. (20) are of order $(v/c)(v_r/c)\Delta t$ and $(v/c)(v_r/c)T$ and contribute a correction of less than 10^{-2} ps for a transfer with $\Delta t \sim 0.1$ s, via a geostationary satellite with a residual velocity of $1\,\mathrm{m\,s^{-1}}$.

In (24) missing terms are of order $(v/c)(v_r/c)(R_0/c)$ and smaller, and contribute less than 10^{-2} ps for a residual velocity of $1\,\mathrm{m\,s^{-1}}$.

5.2. Transformation from proper to coordinate time

In the derivations all clocks are assumed to be rate corrected for the gravitational potential at their positions, and their velocities in the reference system of synchronization. They thus run at the rate of TT. This can be achieved, in theory, to an accuracy of a few parts in 10^{16} (see for example Allan & Ashby 1986), implying for typical transmission times < 1 s, a time error in the synchronization procedure of less than 10^{-3} ps. For synchronizations at the picosecond level a rate correction at the level of accuracy of a few parts in 10^{13} is sufficient. This can easily be done for clocks on the Earth. For clocks on board satellites this implies a constraint on the knowledge of the position and velocity which is much less severe than the one discussed in Sect. 5.3. The problem of ensuring that the proper time realized by the clocks is accurate to the required level is not considered in this study.

5.3. Computation of the corrections

For picosecond accuracy, the relativistic correction δ contains terms in c^{-2} and in c^{-3} in the case of one-way time transfers (15), and terms in c^{-2} only in the case of two-way (20) and LASSO (24) transfers.

The term in c^{-2} can amount to a few hundred nanoseconds, depending on the relative positions of the transmission and reception points. For example, between the Earth and a geostationary orbit, the maximum value is about 200 ns for the one way- and 400 ns for the two way case. In order to compute this term with picosecond accuracy, it is sufficient for all quantities in the term in c^{-2} to be known with a relative uncertainty of one or two parts in 10^6. This requires coordinates known to within 6–12 m for the Earth stations, including uncertainties in the realization of the reference frame which are below $\sim$ 1 m for e.g. WGS84 and ITRF. This is generally the case for time laboratories. The satellite position should be known to within some tens of metres, depending on its orbit, and this is generally not the case a priori for a satellite without geodesic objectives. In addition the velocity of the satellite should be known to the same relative uncertainty of one or two parts in 10^6, which is also not the case in general. Typically the position of a geostationary satellite is known to an accuracy of $\sim$ 1 km which results in an error in the computation of the c^{-2} term of $\sim$ 10 ps. Similar arguments can be made to set constraints in the case of higher orbits or satellite to satellite time transfers.

If we assume a perfect geostationary orbit, the term in c^{-2} can be computed with picosecond accuracy using the formula $2\omega A_{\mathrm{E}}/c^2$. Indeed, to obtain the required accuracy, the vector $\overline{\omega}$ can be taken colinear to the Z-axis and the true pole coordinates ignored. The effect of this approximation can only marginally reach 1 ps for two way time transfer via a geostationary satellite in very special cases. This can occur when the two stations have the same longitude and are close to the poles, with the value of the longitude depending on the position of the pole. The IAU recommended value of the mean angular velocity of the Earth ω (7.292115 10^{-5} rad s^{-1}) is to be used, and the constraints on the positions deduced in the previous paragraph apply for the computation of A_{E}.

In the real case of a non-perfect geostationary orbit, the constraint on the knowledge of the velocity of the satellite is transferred to the residual velocity v_{r}. For the one way and two way techniques, this constraint is about 1 cm s^{-1} for picosecond accuracy but in the two way technique it can be completely relaxed by an intentional desynchronization of the emission of the signals at the two stations, as mentioned in Sect. 3. For LASSO, the constraint on v_{r} is about 10 cm s^{-1} if one wishes to use laser pulses from the two stations separated by Δt of several minutes. The constraint on v_{r} can be relaxed by severing that on Δt.

With the term in c^{-3} for one way transfers amounting to some tens of picoseconds, the constraints to obtain picosecond accuracy are of a few percent on positions, velocities and accelerations, and do not pose any practical problem.

5.4. Propagation through the atmosphere

When one of the stations is on the Earth, propagation through the atmosphere is the major problem for one way time transfer. It leads to delays that can reach several tens of nanoseconds and can certainly not be calibrated to picosecond accuracy. This problem is not considered in this study. However the effects cancel to the picosecond level in the two way (provided the up and down frequencies are close enough) and LASSO techniques.

For Earth to satellite time transfer the deviation from a straight line of the trajectory of a signal in an inertial frame, due to atmospheric refraction, does not exceed 10^{-4} rad and the resulting terms due to the additional path and area are less than one picosecond for satellite elevations greater than 10°. However they can become significant for satellite to satellite transfer when the signal traverses the atmosphere, but in this case the limiting factor would be the uncertainty in the propagation delay itself. And, of course, traversing the atmosphere for satellite to satellite transfer can be easily avoided in practice.

5.5. Gravitational bending

Deviation from a straight line of the trajectory due to the gravitational field of the Earth is of the order of some 10^{-9} rad, hence its effect on the transmission time of a light signal in the vicinity of the Earth is below one picosecond for all possible paths.

6. Numerical application for a two way transfer

Equation (20) has been applied to two way time transfers within Europe and across intercontinental distances. The results have been compared with those obtained from (12), and from a more exact numerical method.

The numerical method is based on the derivation by Klioner (1992): In this T_2, T_3 and T_4 are calculated as functions of t_c, t_d, t_s and Δt, based on the velocity and acceleration of the satellite and the stations in the inertial frame, and obtained by iterating the expressions. T_1 is calculated using the same iterative procedure as in Sect. 2.

All iterations are continued until the difference between two consecutive terms is less than 10^{-16} s. It is in this sense that the numerical method can be considered more exact than the analytic expression (20).

Table 1. Differences between the results of the numerical calculation of the relativistic correction for two way time transfers and applications of the analytical formulae (12) and (20). The differences between the numerical results and (12) are given for different values of Δt. The last column gives the values of Δt for which the second term in (20) vanishes, which is the case when the two signals arrive at the satellite at the same time. In this case the differences between the numerical results and (12) are of the order of some 10^{-3} ps

Transfer	num.- (20)	num. -(12) (ps)		Δt (ideal)
	(10^{-3} ps)	$\Delta t = 0$ s	$\Delta t = 50$ ms	(ms)
Toulouse (France) – Paris (France) via INMARSAT2	0.02	−5	−172	−1.57
FTZ (Germany) – PTB (Germany)	−0.03	−2	−169	−0.57
Kourou (F. Guyana) – HBK (S-Africa)	−1.6	−11	−177	−3.28
Graz (Austria) – USNO (USA)	1.3	35	−131	10.47

The difference between (20) and the numerical method is typically of the same order as the missing terms mentioned in Sect. 5.1 (some 10^{-3} ps). Hence the numerical application represents a validation of (20) for particular cases at the level of 10^{-2} ps.

The difference between (12) and the numerical results, corresponding to the second term in (20), depends on the initial desynchronization of the stations Δt, the residual velocity $\overline{v}_r$, and the geometry of the particular case. Some of the results are summarised in Table 1.

7. Conclusion

We have explicitly derived the relativistic correction for a one way time transfer between two stations that have their position given in a geocentric reference frame rotating with the Earth (Eq. (15)) including all terms in c^{-3} and larger. For time transfer with a geostationary satellite the terms in c^{-3} can amount to around 10 ps for the Sagnac correction and 80 ps for the gravitational delay. At present, one way time transfers are not accurate enough to necessitate the consideration of these terms. However, with accuracy expected to increase in the near future, and in view of possible satellite to satellite transfers (which would eliminate uncertainties due to atmospheric delays) these terms might well become significant.

Time transfer techniques such as LASSO or two way time transfer provide higher precision than one way techniques. Recently a two way time transfer between PTB (Braunschweig, Germany) and FTZ (Darmstadt, Germany) with a precision of less than 300 ps, and a LASSO time transfer between McDonald (USA) and OCA (France) at a level of precision better than 100 ps were carried out (Hetzel & Soring 1993; Veillet et al. 1992; Veillet & Fridelance 1993).

We have explicitly derived the relativistic corrections that need to be applied to these techniques. We have shown that the main errors in computing these corrections are due to the uncertainties in the position and the residual velocity of the satellite. The uncertainty in the position leads to an error in the computation of $2\omega A_E/c^2$ of the order of 10 ps for both techniques. The uncertainty in the residual velocity affects the two techniques differently. For LASSO the second term in (24) is typically of the order of 10 ps, hence reducing the overall uncertainty for LASSO requires better knowledge of the satellite position as well as consideration of the additional term. For two way time transfers, on the other hand, the second term in (20) can reach 80 ps (for $\Delta t = 0$). Hence reducing this term by an appropriate choice of Δt will improve the overall accuracy of the two way time transfer even in the case where $\overline{v}_r$ is unknown.

In both techniques, the precision of experiments repeated over periods of several weeks could be affected by the variation of the residual velocity of the satellite, if the corresponding terms are not accounted for.

This shows that the time community is rapidly approaching levels of precision and accuracy that will necessitate a more exact development of the theory. We consider the present paper a step in that direction.

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