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Noncommutative Sp(2,R) Gauge Theories
A Field Theory Approach to Two-Time Physics

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abstract

Phase-space and its relativistic extension is a natural space for realizing Sp(2,R) symmetry through canonical transformations. On a D$\times$2 dimensional covariant phase-space, we formulate noncommutative field theories, where Sp(2,R) plays a role as either a global or a gauge symmetry group. In both cases these field theories have potential applications, including certain aspects of string theories, M-theory, as well as quantum field theories. If interpreted as living in lower dimensions, these theories realize Poincaré symmetry linearly in a way consistent with causality and unitarity. In case Sp(2,R) is a gauge symmetry, we show that the spacetime signature is determined dynamically as (D-2,2). The resulting noncommutative Sp(2,R) gauge theory is proposed as a field theoretical formulation of two-time physics: classical field dynamics contains all known results of 'two-time physics', including the reduction of physical spacetime from D to (D-2) dimensions, with the associated 'holography' and 'duality' properties. In particular, we show that the solution space of classical noncommutative field equations put all massless scalar, gauge, gravitational, and higher-spin fields in (D-2) dimensions on equal-footing, reminiscent of string excitations at zero and infinite tension limits.

1 Introduction

In this paper, we construct $D\times 2$-dimensional noncommutative field theories (NCFT) with symmetry group Sp$(2,R)\times$ SO$^*(D)$ and study their properties. We consider Sp$(2,R)$ either as a global or as a gauge symmetry. Denoting $D\times 2$ dimensional coordinates as $X_\mu^{\rm M}$, the group Sp$(2,R)\times$ SO$^*(D)$ acts on the $\mu=1,2$ index as a doublet $(\mathbf{2},\mathbf{1})$ and on the M$=1,2,\cdots,D$ index as a vector $(\mathbf{1},\mathbf{D})$. A variety of choices are possible for the signature η^{MN} in D dimensions or the corresponding real form of the SO$^*(D)\equiv$SO$(D-n,n)$ group, where n is the number of timelike dimensions. Theories with Euclidean signature arise most prominently in the Weyl-Wigner-Moyal formulation [1]-[3] of nonrelativistic quantum mechanics, and in the noncommutative sector of M-theory (when a magnetic background field is present, with a maximum possible symmetry Sp$(2,R)\times$SO(5)). For Lorentzian signature, $n\geq 1$, one can view the coordinates $X_\mu^{\rm M}$'s as labelling a relativistic quantum phase-space, $X_1^{\rm M}=X^{\rm M}$ and $X_2^{\rm M}=P^{\rm M}$, extending the Weyl-Wigner-Moyal formulation of the nonrelativistic quantum mechanics [1][2][3] to relativistic situations. One can also view them as noncommuting spacetime coordinates of two point-particles, where the noncommutativity is induced by the presence of a background field. Because vastly different interpretations are possible, more broadly, we expect the formalism and methods developed for these noncommutative field theories to have a wide range of applications in various physical contexts.

Our main results for these noncommutative field theories are twofold:

- If Sp$(2,R)$ is a gauge symmetry, the spacetime signature is determined dynamically as $(D-2,2)$. The gauge-invariant sector of these theories describes commutative dynamics in $(D-2)$-dimensional spacetime, with $(D-3,1)$ signature, with linearly realized Poincaré and, non-linearly realized, higher spacetime symmetries. They offer an *ab initio* field theory formulation of 'Two-Time Physics' (2T-physics) [4]-[8] in D-dimensional spacetime, a result established by interpreting $X_\mu^{\rm M}$ as coordinates of the $D\times 2$ dimensional covariant phase-space.

- If Sp$(2,R)$ is a global symmetry, the spacetime signature is left arbitrary. Unitarity and causality force an interpretation of these theories as describing dynamics in spacetime of dimensions lower than D.

In investigating this sort of noncommutative field theories, the general question we have posed ourselves is the following. Quantum field theories are traditionally formulated on configuration space. Alternatively, what if one attempts to formulate the theories on corresponding phase-space? In the disguise of non-relativistic quantum mechanics, precisely this sort of an alternative formulation was proposed by Weyl [1], Wigner [2], and studied further by Moyal [3]. It is referred as 'deformation quantization', an alternative to the traditional quantization based on Hilbert space and linear operators therein. In this approach, dynamical equations of quantum mechanics, either the Schrödinger or the Heisenberg equation of motion, are replaced by a sort of evolution equation of distribution functions over phase-space. Mathematically, the Weyl-Wigner-Moyal formalism is equivalent to noncommutative field theories arising as a limit of string theories [10]-[12], and is identifiable as the Euclidean case, $n = 0$, in the present context.

Extension of the deformation quantization to systems with $n \geq 1$ poses several peculiarities, all of which lead to a link with two timelike dimensions. Lorentz covariance $\mathrm{SO}^*(D)$ implies that time and energy (one-particle Hamiltonian) ought to be included, along with spatial coordinates and momenta, as part of covariant phase-space. If so, general canonical transformations consistent with $\mathrm{Sp}(2,R) \times \mathrm{SO}^*(D)$ would require the one-particle Hamiltonian to transform along with coordinates and momenta, thus mapping one system with a given one-particle Hamiltonian to another with a different one-particle Hamiltonian. This is a central feature that, for $n = 2$, 2T-physics has embodied through local $\mathrm{Sp}(2,R)$, leading to the '2T to 1T holography', and 'duality' property among various systems with one physical time [5][4]. Apparently, a relativistic field theory with more than one timelike dimensions introduces ghosts, as is easily seen by expanding a field $\phi(X,P)$ in powers of momenta whose coefficients are local tensors $\phi^{\mathrm{M}_1 \cdots \mathrm{M}_s}(X)$. Timelike components of these tensors could give rise to negative-norm, ghost fields. An approach for eliminating the ghosts is to promote $\mathrm{Sp}(2,R)$ to a gauge symmetry, viz. by demanding equivalence under general canonical transformations. The vanishing of the $\mathrm{Sp}(2,R)$ gauge generators on physical states determines dynamically that spacetime must have two timelike dimensions.

It turns out that the resulting noncommutative $\mathrm{Sp}(2,R)$ gauge theories possess a rich structure, most notably the 'holography' property, in which a D-dimensional system is holographically represented by various $(D-2)$-dimensional systems, each with different dynamics. The signature of the $(D-2)$-dimensional spacetime is $(D-3,1)$, where the timelike direction in each $(D-3,1)$-dimensional system is given by a combination of the two timelike dimensions in the embedding $(D-2,2)$-dimensional spacetime. With one timelike dimension, the $(D-2)$-dimensional systems are causal and have a unitary spectrum of physical states. The $\mathrm{Sp}(2,R)$ gauge symmetry acts as a sort of 'duality' in that all these different dynamical systems are included in the $\mathrm{Sp}(2,R)$ gauge orbit that describes the physical gauge invariant sector.

This paper is organized as follows. In section 2, for later application, we explain conceptual issues in constructing field theories of 2T-physics and recapitulate some results of earlier approaches relevant for later sections. In section 3, we discuss deformation quantization on covariant phase-space and develop a formalism that will be used in later sections. In section 4, we construct examples of noncommutative field theories with global $\mathrm{Sp}(2,R) \times \mathrm{SO}^*(D)$ symmetry for a generic signature of spacetime. Results of these two sections are more general and ought to be applicable to a wide range of physical problems. In section 5, we promote the $\mathrm{Sp}(2,R)$ automorphism to a gauge symmetry and construct noncommutative $\mathrm{Sp}(2,R)$ gauge theories. We show that, the condition for physical states dynamically determines that the signature of spacetime must be

$(D-2,2)$, so inevitably we end up with 2T-physics. We then find a class of nontrivial special classical solutions that reproduce all previously known results of 2T-physics for spinless particles and, most notably, find that the solution space provides a unified description of gauge fields, including the gravitational and high spin gauge fields. Section 6 summarizes various issues left for future work.

2 2T-Physics: Concepts and Field Theory

Part of the motivation for the present work has arisen from the following question: What is the interacting field theory, whose free propagation is given by the first quantized worldline theory of 2T-physics [4]- [8]? Free field equations emerge, in covariant first-quantized description, by imposing constraints on states in configuration space, e.g. worldline reparametrization constraints leads to the Klein-Gordon equation $[\partial^2 - m^2]\,\phi(X) = 0$, worldsheet reparametrization constraints lead to string field equations through the Virasoro constraints $L_n\phi\left(X\left(\sigma\right)\right) = 0$. These constraints are the generators of the underlying gauge symmetries and hence the states obeying them are gauge invariant physical states. In several known situations, the constraint equations can be derived from a field theory when the field interactions are neglected. Field interactions promote the first quantized theory to an interacting field theory which can then be analyzed both with classical and second quantized methods.

In 2T-physics, the fundamental gauge symmetry is $\mathrm{Sp}(2,R)$ and its supersymmetric generalizations. $\mathrm{Sp}(2,R)$ acts as symplectic transformation on coordinates and momenta of a particle's phase-space $\left(X^{\mathrm{M}}, P^{\mathrm{M}}\right) \equiv X_\mu^{\mathrm{M}}$. For a spinless particle, the worldline action with *local* $\mathrm{Sp}(2,R)$ symmetry is given by

$$I = \int d\tau \left[\dot{X}_1^{\mathrm{M}} X_{2\mathrm{M}} - \frac{1}{2} A^{\mu\nu}\left(\tau\right)\, \hat{Q}_{\mu\nu}(X_1, X_2)\right], \tag{1}$$

where, $A_{\mu\nu}\left(\tau\right)$ denotes three $\mathrm{Sp}(2,R)$ gauge fields and the symmetric $\hat{Q}_{\mu\nu} = \hat{Q}_{\nu\mu}$, with $\mu,\nu = 1,2$, are the three $\mathrm{Sp}(2,R)$ generators, whose Poisson brackets obey $\mathrm{sp}(2,R)$ Lie algebra. This action is $\mathrm{Sp}(2,R)$ gauge invariant provided the $\hat{Q}_{\mu\nu}$ satisfy the $\mathrm{Sp}(2,R)$ Lie algebra under Poisson brackets [5][8]. The equations of motion for $A^{\mu\nu}$ lead to three classical constraints, $\hat{Q}_{\mu\nu}\left(X,P\right) = 0$, which become, upon first quantization, differential operator equations,

$$\hat{Q}_{\mu\nu}\left(X_1, \frac{-i\partial}{\partial X_1}\right)\psi(X_1) = 0, \tag{2}$$

with an appropriate operator ordering. The simplest situation occurs for the following form of the generators which we refer to as the "free" case (omitting the hat symbol)

$$Q_{11} = X\cdot X, \quad Q_{12} = X\cdot P, \quad Q_{22} = P\cdot P. \tag{3}$$

where the dot products, $X \cdot P = X^M P^N \eta_{MN}$, are constructed using a flat metric η_{MN} of unknown signature (which later is dynamically determined to be $(D-2,2)$). For the particle in background fields such as Yang-Mills, gravity, higher-spin gauge fields, $\hat{Q}_{\mu\nu}(X,P)$ takes a more general form [6] [8]. With this example, one can see that the $\hat{Q}_{22}$-equation in (2) is nothing but (a generalization of) the massless Klein-Gordon equation. In fact, the (22) components of the $\mathrm{Sp}(2,R)$ gauge fields and generators are associated with the worldline reparametrization invariance. Together with the additional transformations generated by $\hat{Q}_{11}$ and $\hat{Q}_{12}$, worldline reparametrization is promoted to the non-Abelian local $\mathrm{Sp}(2,R)$ symmetry, which may also be regarded as local conformal symmetry on the worldline [5].

One of the goals of this work is to construct interacting field theories with $\mathrm{Sp}(2,R)$ gauge symmetry, which would yield the first quantized 2T-physics physical state equations (2) from the linearized part of the field equations of motion. The obvious benefit of these theories is, of course, to reach a formulation of field interactions from first principles based on gauge symmetry. An interesting outcome of this approach, as will be elaborated in Section 5, viz. noncommutative $\mathrm{Sp}(2,R)$ gauge theories, is that all massless fields in $(D-2)$ dimensions, scalar, gauge, gravity, and higher-spin gauge fields, are all packaged into the noncommutative $\mathrm{Sp}(2,R)$ gauge field $A_{\mu\nu}(X,P)$ in a $D\times 2$-dimensional covariant phase-space! There seems to be an intriguing relationship between this packaging of higher-spin gauge fields and a subset of massless string modes at infinite Regge slope (tensionless string) or zero Regge slope (point-particle) limits [8]. The noncommutative Sp(2, R) gauge theories offers an approach for a 2T-physics description of fields and the formulation of nonlinear interactions among themselves.

The notion of two timelike dimensions raises various technical and conceptual questions and points to deeper physics. Remarkably, the most obvious and troublesome problem concerning causality and unitarity is solvable, and transparently understood in the worldline approach to 2T-physics. The reason for two timelike dimensions is as follows. The $\mathrm{Sp}(2,R)$ gauge invariance imposes constraints, $Q_{\mu\nu} \approx 0$, viz. physical states are defined as gauge singlets. Solving them classically, one finds that nontrivial dynamics is possible only for two or more timelike dimensions. However, the $\mathrm{Sp}(2,R)$ gauge symmetry can remove all the ghosts only if the number of timelike dimensions do not exceed two [1]. Hence, $\mathrm{Sp}(2,R)$ gauge invariance demands two timelike dimensions, no less and no more.

A similar analysis is applicable in the present context and will lead precisely to the same

[1] As an illustration, take the simplest case Eq.(3) wherein the inner products $X \cdot X = X^{\mathrm{M}} X^{\mathrm{N}} \eta_{\mathrm{MN}}$, etc. are defined with a flat metric of unknown signature. For Euclidean metric, the only solution is a trivial one, $X^{\mathrm{M}} = P^{\mathrm{M}} = 0$. For Lorentzian metric with a single timelike dimension, X^{M} and P^{M} ought to be parallel, and is a trivial system since it lacks angular momentum. For Lorentzian metric, with more than two timelike dimensions, the $\mathrm{Sp}(2,R)$ gauge invariance is insufficient to remove all the ghosts. For Lorentzian metric, with precisely two timelike dimensions, $\mathrm{Sp}(2,R)$ gauge invariance is just enough to remove all the ghosts. Furthermore, causality is not violated as, in a unitary gauge, there is only one timelike dimension.

conclusion: noncommutative Sp$(2, R)$ gauge theories on a $D \times 2$-dimensional phase-space with signature $(D-2, 2)$ are unitary (as there are precisely two timelike dimensions) and causal (as, in unitary gauge, the physical spacetime is reduced to $(D-2)$ dimensions with one timelike dimension). In particular, $(D-2)$-dimensional spacetime symmetries – Galilean, Poincaré, conformal, or even general coordinate invariance – arise as symmetries.

If 2T-physics with Sp$(2, R)$ gauge symmetry is equivalent to one-time (1T) physics, what does one gain from the former formulation? By embedding 1T-physics into 2T-physics, one is led to a notion of 'holography' between 2T- and 1T-physics: a given 2T action in D dimensions describes a family of 1T actions in $(D-2)$ dimensions. Examples displaying these properties have been found in worldline and field theory formulations of 2T-physics [4][5][7]. The 'holography' is intimately related to the issue of 'time': which (combination) of the two times corresponds to the causal evolution parameter of the physical 1T systems? In the worldline formulation one can rephrase this question as: which combination of $X^0(\tau), X^{0'}(\tau)$ is identified with the proper time τ? It turns out that Sp$(2, R)$ gauge orbit in the physical sector, $\hat{Q}_{\mu\nu}(X, P) = 0$, encompasses all possible combinations. Furthermore, the Sp$(2, R)$ gauge symmetry thins out the spacetime degrees of freedom from D to $(D-2)$ dimensions, giving rise to the holography property. Thus, different 1T theories in $(D-2)$ dimensions emerge as a result of different choices of the Sp$(2, R)$ gauge fixing, but they all represent the physical sector of D dimensional 2T-theory. This also implies that, within a given 2T-theory, different 1T-theories are related to one another via a sort of 'duality': the Sp$(2, R)$ gauge transformations map a 1T-theory to another, while staying within the physical (gauge-invariant) sector of the same 2T-theory. The 'holography' and 'duality' properties ought to persist in noncommutative Sp$(2, R)$ gauge theories, but now accommodating nonlinear gauge interactions.

We find it compelling to understand the above phenomena in a field-theoretic formulation of 2T-physics, including interactions. An first attempt would be in terms of fields defined on configuration space, as studied in [7]. However, it became clear that a more natural and far reaching approach would result from a phase-space formulation. Naturally, the resulting formulation is in terms of noncommutative Sp$(2, R)$ gauge theories, which as shown below makes contact with the relevant parts of the configuration space approach. Hence it is useful for us to review here the salient aspects of the configuration space formulation [7].

Field equations in configuration space (in the presence of background fields) result from imposing the constraints on physical states as in Eq.(2). For the free case of Eq.(3) these take the form

$$\mathbf{q}_{\mu\nu}\psi(X_1) = 0, \tag{4}$$

where $Q_{\mu\nu} \to \mathbf{q}_{\mu\nu}$ refers to hermitian differential operators

$$\mathbf{q}_{11} = X_1 \cdot X_1, \quad \mathbf{q}_{12} = -\frac{i}{2}\left(X_1 \cdot \frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_1} \cdot X_1\right), \quad \mathbf{q}_{22} = -\frac{\partial}{\partial X_1} \cdot \frac{\partial}{\partial X_1}. \tag{5}$$

The $\mathbf{q}_{11}$ equation is solved by

$$\psi\left(X_1^{\mathrm{M}}\right)=\delta\left(X_1^2\right)\varphi\left(X_1^{\mathrm{M}}\right). \tag{6}$$

The $\mathbf{q}_{12}$ equation becomes

$$\begin{aligned} 0 &= \mathbf{q}_{12}\,\psi\left(X_1^{\mathrm{M}}\right)=-i\left(\frac{D}{2}+X_1\cdot\frac{\partial}{\partial X_1}\right)\psi\left(X_1^{\mathrm{M}}\right) \\ &= -i\delta\left(X_1^2\right)\left[\left(\frac{D}{2}-2+X_1\cdot\frac{\partial}{\partial X_1}\right)\varphi\left(X_1^{\mathrm{M}}\right)\right]_{X_1\cdot X_1=0}, \end{aligned} \tag{7}$$

where we have used an identity $X_1\cdot\frac{\partial}{\partial X_1}\delta\left(X_1^2\right)=2X_1^2\delta'\left(X_1^2\right)=-2\delta\left(X_1^2\right)$ (as a distribution). The $\mathbf{q}_{22}$ equation becomes

$$\left(\frac{1}{2}\mathbf{l}^{\mathrm{MN}}\mathbf{l}_{\mathrm{MN}}\right)\varphi\left(X_1^{\mathrm{M}}\right)\Big|_{X_1\cdot X_1=0}=-\frac{1}{4}D\left(D-4\right)\varphi\left(X_1^{\mathrm{M}}\right)\Big|_{X_1\cdot X_1=0}. \tag{8}$$

Here, $\frac{1}{2}\mathbf{l}^{\mathrm{MN}}\mathbf{l}_{MN}$ is the is the $\mathrm{SO}^*\left(D\right)$ quadratic Casimir operator and $\mathbf{l}^{MN}$ is its generator

$$\mathbf{l}^{\mathrm{MN}}=-i\left(X_1^{\mathrm{M}}\frac{\partial}{\partial X_{1\mathrm{N}}}-X_1^{\mathrm{N}}\frac{\partial}{\partial X_{1\mathrm{M}}}\right). \tag{9}$$

Equation (8) is a rewriting of $\frac{1}{2}\mathbf{q}_{\mu\nu}\mathbf{q}^{\mu\nu}\varphi=\frac{1}{2}\left(\mathbf{q}_{11}\mathbf{q}_{22}+\mathbf{q}_{22}\mathbf{q}_{11}-2\mathbf{q}_{12}\mathbf{q}_{12}\right)\varphi=0$ after using the relation between the $\mathrm{Sp}(2,R)$ and the $\mathrm{SO}(D-2,2)$ Casimirs

$$\frac{1}{2}\mathbf{l}^{\mathrm{MN}}\mathbf{l}_{\mathrm{MN}}=\frac{1}{2}\mathbf{q}_{\mu\nu}\mathbf{q}^{\mu\nu}-\frac{1}{4}D\left(D-4\right), \tag{10}$$

which is derived directly from their representations in Eqs.(5,9). Thus demanding an $\mathrm{Sp}(2,R)$ gauge invariant physical state Eq.(4) implies that such states form an irreducible representation of $\mathrm{SO}^*\left(D\right)$ with a fixed eigenvalue for the quadratic Casimir operator of $\mathrm{SO}^*\left(D\right)$ as given in Eq.(8). The higher Casimir operators for $\mathrm{SO}^*\left(D\right)$ can be computed in the same way, and shown that they are fixed numbers. Hence the physical states in the free case occupy a specific representation of $\mathrm{SO}^*\left(D\right)=\mathrm{SO}(D-2,2)$. This representation is a unitary representation, and are referred as *singleton* or *doubleton* representation, depending on the dimension D.

The above differential equations Eqs.(6-8) have non-trivial solutions only if there are two timelike dimensions. Moreover, the particular $\mathrm{SO}^*\left(D\right)$ representation emerging in this way is unitary provided there are again two timelike dimensions. This implies that the first-quantized theory requires $\mathrm{SO}^*\left(D\right)=\mathrm{SO}(D-2,2)$ with two timelike dimensions, confirming the result of the classical analysis recapitulated earlier.

The holographic aspects can be studied in the 2T-field theory. One holographic picture is the $(D-2)$-dimensional massless Klein-Gordon equation, derived originally by Dirac [9]. Another is the nonrelativistic Schrödinger equation, and yet another is the scalar field equation in anti-de Sitter background with a quantized mass, etc. [7]. In each of them, the $\mathrm{SO}(D-2,2)$ automorphism of the 2T-theory arises with different physical interpretation. It is interpreted as conformal symmetry of the Klein-Gordon equation, while, for others, as dynamical symmetry or anti-de Sitter symmetry etc. The existence of this symmetry in some of the 1T-theories is surprising, but it is understood naturally within the 2T-framework. Furthermore, all 1T holographic pictures of the free 2T-physics theory (free massless particle, AdS_d particle, $\mathrm{AdS}_{d-k}\times \mathrm{S}^k$ particle, H-atom, Harmonic oscillator in one less dimension, etc.) occupy the same singleton/doubleton representation described above [5].

Generalizations of the same approach to field theory, including field interactions, and including spinning particles, gauge, and gravitational fields, etc. were accomplished [7].

However, one unsatisfactory aspect is that the equations $\mathbf{q}_{\mu\nu}\psi = \cdots$ are not all treated on an equal footing: the $\mathbf{q}_{22}$ condition, including interactions, is derivable from an interacting 2T-theory action, however, the $\mathbf{q}_{11}$ and $\mathbf{q}_{12}$ conditions do not follow directly from the action and are applied as additional constraints (although one could introduce them by using Lagrange multipliers). One thus anticipates [5] that 2T-field theories ought to be constructed most naturally as noncommutative field theories on the phase-space spanned by $\left(X_1^{\mathrm{M}}, X_2^{\mathrm{M}}\right)$, as this is the space where the $\mathrm{Sp}(2,R)$ transformations are manifest, and all $Q_{\mu\nu}$ appear on an equal footing.

To construct noncommutative field theories that reproduce known results of 2T-physics, we will develop some formalism in the next two sections. We will focus on how to maintain the $\mathrm{Sp}(2,R)\times \mathrm{SO}^*(D)$ covariance manifest and study the theories in cases where $\mathrm{Sp}(2,R)$ symmetry is global or local. The $\mathrm{Sp}(2,R)$ gauge symmetry is the necessary ingredient for 2T-physics and leads to the same results as the classical and the first-quantized 2T-theory. In noncommutative field theories, however, the $\mathrm{Sp}(2,R)$ gauge symmetry renders consistent interactions as well. In the free field limit, field equations in configuration space Eqs.(4,5) follow naturally from the noncommutative field equations. Solutions to these equations and their holographic 1T interpretation coincide with the previous results [7] of the 2T field theory in configuration space. The noncommutative field theories also yield known results when generic background fields are turned on [6][8]. We thus find that the noncommutative $\mathrm{Sp}(2,R)$ gauge theories offer a unified approach to all aspects of 2T-physics, including interactions.

So far, we have considered mainly the phase-space interpretation of the noncommuting coordinates $X_1^{\mathrm{M}} = X^{\mathrm{M}}$, $X_2^{\mathrm{M}} = P^{\mathrm{M}}$. On the other hand, as mentioned in the previous section, we may also consider a noncommutative geometry interpretation of $X_1^{\mathrm{M}}, X_2^{\mathrm{M}}$ as noncommuting *positions* of *two* point-particles, where noncommutativity is induced by a constant magnetic field. This idea naturally occurred in the context of a two-particle system described by $\left(X_1^{\mathrm{M}}, P_1^{\mathrm{M}}\right)$ and $\left(X_2^{\mathrm{M}}, P_2^{\mathrm{M}}\right)$, with constraints $P_1^2 = P_2^2 = P_1\cdot P_2 = 0$ that followed from two-particle gauge symmetries [13]. In the presence of a constant magnetic field with interactions $B\left(\dot{X}_1\cdot X_2 - \dot{X}_2\cdot X_1\right)$ the two position coordinates $X_1^{\mathrm{M}}, X_2^{\mathrm{M}}$ develop mutual non-commutativity. (In the infinite magnetic field limit, the kinetic terms are negligible and then one may reinterpret the system as a single particle, where

$X_1 = X$, $X_2 = P$ are phase-space variables[2]). Such a set-up is analogous in spirit to the interpretation of noncommutative field theories in terms of 'dipole' behavior of an open string theory in background magnetic field.

3 $\star$–Algebra on Relativistic Quantum Phase-Space

In this section we develop an *ab initio* approach. Consider the non-commutative (NC) Moyal products for any two functions $f(x) \star g(x)$ in 2×D dimensions. Instead of the generic NC spacetime x^m which satisfies $x^m \star x^n - x^n \star x^m = i\theta^{mn}$, we are interested in a special form of the non-commutativity parameter θ^{mn} that explicitly exhibits the highest possible symmetry. Recalling that in a real basis for x^m the parameter θ^{mn} may be brought to block diagonal form with skew 2×2 blocks, the highest symmetry is manifest when all the 2×2 diagonal blocks are identical up to signs. Such a θ^{mn} parameter has the symmetry $\mathrm{Sp}^*(2D)$ which contains the subgroup $\mathrm{Sp}(2,R) \times \mathrm{SO}(D-n,n)$ for some n. For notational convenience we will write $\mathrm{SO}^*(D) \equiv \mathrm{SO}(D-n,n)$ and $\mathrm{Sp}(2) \equiv \mathrm{Sp}(2,R)$. To exhibit this subgroup symmetry, it is convenient to use the pair of labels $m = \mu \mathrm{M}$ with $\mu = 1,2$ and $\mathrm{M}=1,2,\cdots,D$, so that spacetime is labelled by X_μ^{M} instead of x^m, and θ^{mn} is replaced by $\hbar\varepsilon_{\mu\nu}\eta^{\mathrm{MN}}$, where $\varepsilon_{\mu\nu}$ and η^{MN} are the invariant metrics for $\mathrm{Sp}(2)$ and $\mathrm{SO}^*(D)$ respectively. In this basis the Moyal $\star$-product takes the form

$$(f \star g)\left(X_\mu^{\mathrm{M}}\right) = \exp\left(\frac{i\hbar}{2}\varepsilon_{\lambda\sigma}\eta^{\mathrm{MN}}\frac{\partial}{\partial X_\lambda^{\mathrm{M}}}\frac{\partial}{\partial \widetilde{X}_\sigma^{\mathrm{N}}}\right) f\left(X_\mu^{\mathrm{M}}\right) g\left(\widetilde{X}_\mu^{\mathrm{M}}\right)\Big|_{X_\mu^{\mathrm{M}}=\widetilde{X}_\mu^{\mathrm{M}}}. \tag{11}$$

We define the $\star$-commutator

$$\Big[f(X), g(X)\Big]_\star := f(X) \star g(X) - g(X) \star f(X). \tag{12}$$

We then have the Heisenberg algebra:

$$\left[X_\mu^{\mathrm{M}}, X_\nu^{\mathrm{N}}\right]_\star = i\varepsilon_{\mu\nu}\eta^{\mathrm{MN}}, \tag{13}$$

which exhibits a global automorphism symmetry $\mathrm{Sp}(2,R) \times \mathrm{SO}^*(D)$. Hereafter we will set $\hbar = 1$.

In the NC limit of 11-dimensional M-theory the highest such symmetry would be $\mathrm{Sp}(2,R) \times \mathrm{SO}(5)$ with Euclidean signature. Our formalism would be useful in this physical setting. More generally, concerning the spacetime signature, for now, we will take the signature arbitrary, say, $(D-n)$ spacelike and n timelike dimensions. Ultimately, we shall be promoting the $\mathrm{Sp}(2,R)$ subgroup to a gauge symmetry, and find that, as a consequence of the gauge invariance, the number of timelike dimensions n is determined uniquely to be $n = 2$.

In this basis, there is no loss of generality if we consider the $\mathrm{Sp}(2,R)$ doublet X_μ^{M} as the doublet of D-dimensional spacetime positions and energy-momenta: $X_\mu^{\mathrm{M}} = \left(X^{\mathrm{M}}, P^{\mathrm{M}}\right)$, spanning $D \times 2$

[2]The derivation of one particle dynamics from a two-particle system with two-times, where non-commutativity is induced by a constant magnetic field, was the historical path that led to the concepts in the first paper in [5].

dimensional relativistic phase-space. The subgroup $\mathrm{SO}^*(D)$ remains as a global subgroup of the relativistic phase-space.

3.1 Symmetry Generators on Quantum Phase-Space

Having identified the $\mathrm{Sp}(2)\times\ \mathrm{SO}^*(D)$ as the global symmetry groups on the relativistic quantum phase-space, we now investigate their Lie algebra, but in terms of the $\star$-product through the Weyl-Moyal map. Denote the Sp(2) generators as $\mathbf{Q}_{\mu\nu}$ and the $\mathrm{SO}^*(D)$ generators as $\mathbf{L}^{\mathrm{MN}}$, respectively. In terms of the $\star$-product, we have found that they are represented by:

$$\begin{aligned}\mathbf{Q}_{\mu\nu} &\equiv \frac{1}{2}\eta_{\mathrm{MN}} X^{\mathrm{M}}_{(\mu}\star X^{\mathrm{N}}_{\nu)} = \frac{1}{2}\eta_{\mathrm{MN}}\left(X^{\mathrm{M}}_{\mu}\star X^{\mathrm{N}}_{\nu} + X^{\mathrm{M}}_{\nu}\star X^{\mathrm{N}}_{\mu}\right) = \eta_{\mathrm{MN}} X^{\mathrm{M}}_{\mu} X^{\mathrm{N}}_{\nu}, \qquad (14)\\ \mathbf{L}^{\mathrm{MN}} &\equiv \frac{1}{2}\varepsilon^{\mu\nu} X^{[\mathrm{M}}_{\mu}\star X^{\mathrm{N}]}_{\nu} = \frac{1}{2}\varepsilon^{\mu\nu}\left(X^{\mathrm{M}}_{\mu}\star X^{\mathrm{N}}_{\nu} - X^{\mathrm{N}}_{\mu}\star X^{\mathrm{M}}_{\nu}\right) = \varepsilon^{\mu\nu} X^{\mathrm{M}}_{\mu} X^{\mathrm{N}}_{\nu}, \qquad (15)\end{aligned}$$

where the symbols enclosed in parentheses or brackets, $(\mu\nu)$, [MN] etc., refer to symmetrization and antisymmetrization, respectively. The last form of $\mathbf{Q}_{\mu\nu}$, after the star products have been evaluated, is identical to the classical form of Eq.(3). The same remark applies to $\mathbf{L}^{\mathrm{MN}}$. These, $\mathbf{Q}_{\mu\nu}$'s and $\mathbf{L}^{\mathrm{MN}}$'s obey the $\mathrm{sp}(2)\oplus\mathrm{so}^*(D)$ Lie algebras under star products

$$\begin{aligned}\left[\mathbf{Q}_{\mu\nu}\,,\,\mathbf{Q}_{\kappa\lambda}\right]_{\star} &= i\,F_{\mu\nu,\kappa\lambda}{}^{\alpha\beta}\ \mathbf{Q}_{\alpha\beta} \qquad (16)\\ \left[\mathbf{L}^{\mathrm{MN}}\,,\,\mathbf{L}^{\mathrm{KL}}\right]_{\star} &= i\,F^{\mathrm{MN,KL}}{}_{\mathrm{RS}}\mathbf{L}^{\mathrm{RS}} \qquad (17)\\ \left[\mathbf{L}^{\mathrm{MN}},\,\mathbf{Q}_{\mu\nu}\right]_{\star} &= 0. \qquad (18)\end{aligned}$$

Here, $F_{\mu\nu,\kappa\lambda}{}^{\alpha\beta}$, $F^{\mathrm{MN,KL}}{}_{\mathrm{RS}}$ denote the structure constants of the $\mathrm{sp}(2)\oplus\ \mathrm{so}^*(D)$ Lie algebras, respectively:

$$F_{\mu\nu,\lambda\sigma}{}^{\alpha\beta} = \frac{1}{2}\delta^{(\alpha}_{(\mu}\ \varepsilon_{\nu)(\kappa}\ \delta^{\beta)}_{\lambda)} \qquad \text{and} \qquad F^{\mathrm{MN,KL}}{}_{\mathrm{RS}} = \frac{1}{2}\delta^{[\mathrm{M}}_{[R}\ \eta^{\mathrm{N}][\mathrm{K}}\ \delta^{\mathrm{L}]}_{\mathrm{S}]}. \qquad (19)$$

From the $\star$-product representation of the generators, we construct the quadratic Casimir operators of Sp(2) and $\mathrm{SO}^*(D)$:

$$\begin{aligned}C_2[\mathrm{SO}^*(\mathrm{D})] &= \frac{1}{2}\mathbf{L}^{\mathrm{MN}}\star\mathbf{L}_{\mathrm{MN}}, \qquad (20)\\ C_2[\mathrm{Sp}\,(2)] &= \frac{1}{2}\mathbf{Q}_{\mu\nu}\star\mathbf{Q}^{\mu\nu}, \qquad (21)\end{aligned}$$

where indices are contracted by using the metrics η^{MN} and $\varepsilon_{\mu\nu}$ respectively. Remarkably, by applying the $\star$-commutator relation Eq.(11), one can show that the two Casimir invariants are related each other just as in (10)

$$C_2[\mathrm{SO}^*(\mathrm{D})] = C_2[\mathrm{Sp}\,(2)] - \frac{1}{4}D\,(D-4)\,. \qquad (22)$$

Note that the relation is *independent* of the signature of the D-dimensional spacetime. In the following discussions, Eq.(22) will play an important role, especially, in relating the resulting noncommutative field theory to two-time physics.

3.2 Differential Calculus on Quantum Phase-Space

On the relativistic quantum phase-space, $\mathcal{M}_\hbar$, differential calculus may be developed from the defining algebra of the $\star$-products. We thus consider left- or right-multiplication of single power of $X_\mu^{\rm M}$'s from the left or the right of a function $\phi(X)$ on phase-space. They are:

$$X_\mu^{\rm M} \star \phi(X) = \left(X_\mu^{\rm M} + \frac{i}{2}\partial_\mu^{\rm M} \right) \phi(X) \equiv \mathcal{D}_\mu^{\rm M} \phi(X) \tag{23}$$

$$\phi(X) \star X_\mu^{\rm M} = \left(X_\mu^{\rm M} - \frac{i}{2}\partial_\mu^{\rm M} \right) \phi(X) \equiv \overline{\mathcal{D}}_\mu^{\rm M} \phi(X). \tag{24}$$

Here, utilizing the invariant metrics $\varepsilon_{\mu\nu}$ and $\eta_{\rm MN}$, we have introduced the notation:

$$\partial_\mu^{\rm M} \equiv \eta^{\rm MN} \varepsilon_{\mu\nu} \frac{\partial}{\partial X_\nu^{\rm N}} \qquad \text{such that} \qquad \partial_\mu^{\rm M} X_\nu^{\rm N} = \varepsilon_{\mu\nu} \eta^{MN}. \tag{25}$$

The multiplications define, as the notations indicate, two inequivalent differential operators – $\mathcal{D}_\mu^{\rm M}\phi(X)$ and $\overline{\mathcal{D}}_\mu^{\rm M}\phi(X)$. However, these differential operators violate the Leibniz rule: $\mathcal{D}_\mu^{\rm M}(\phi_1 \star \phi_2) \neq \left(\mathcal{D}_\mu^{\rm M}\phi_1\right) \star \phi_2 + \phi_1 \star \left(\mathcal{D}_\mu^{\rm M}\phi_2\right)$. On the other hand, a new differential operator obeying the Leibniz rule can be defined by taking *difference* between the above two differential operators:

$$\left(\mathcal{D}_\mu^{\rm M} - \overline{\mathcal{D}}_\mu^{\rm M} \right) \phi(X) = \left[X_\mu^{\rm M}, \phi(X) \right]_\star = i\partial_\mu^{\rm M} \phi(X). \tag{26}$$

The three differential operators, $\mathcal{D}_\mu^{\rm M}, \overline{\mathcal{D}}_\mu^{\rm M}, \partial_\mu^{\rm M}$, form a complete set of first-order differential operators on the quantum phase-space [3].

Next, consider $\star$-multiplication by two powers of $X_\mu^{\rm M}$'s on $\phi(X)$. Of particular interest are the generators, $\mathbf{Q}_{\mu\nu}$ and $\mathbf{L}^{\rm MN}$. Their commutators define derivatives that obey the Leibniz rule

$$D_{\mu\nu}\phi(X) \equiv -i\left[\mathbf{Q}_{\mu\nu}, \phi\right]_\star = \frac{1}{2}\eta_{\rm MN}\left(X_{(\mu}^{\rm M} \star \partial_{\nu)}^{\rm N}\phi + \partial_{(\mu}^{\rm M}\phi \star X_{\nu)}^{\rm N} \right)(X) = \eta_{\rm MN} X_{(\mu}^{\rm M}\partial_{\nu)}^{\rm N}\phi(X) \tag{27}$$

$$D^{\rm MN}\phi(X) \equiv -i\left[\mathbf{L}^{\rm MN}, \phi\right]_\star = \frac{1}{2}\varepsilon^{\mu\nu}\left(X_\mu^{[\rm M} \star \partial_\nu^{\rm N]}\phi + \partial_\mu^{[\rm M}\phi \star X_\nu^{\rm N]} \right)(X) = \varepsilon^{\mu\nu} X_\mu^{[\rm M}\partial_\nu^{\rm N]}\phi(X). \tag{28}$$

Note the $\star$-multiplication ordering in the middle expressions. After applying Eqs.(23,24), however, they are expressible in terms of ordinary products, as shown in the last expressions. Note further that, using Eq.(25), these two derivations can be expressed as total first order derivatives:

$$D_{\mu\nu}\phi(X) = \partial_{(\mu}^{\rm M}\left(\eta_{\rm MN} X_{\nu)}^{\rm N}\phi\right)(X) \qquad \text{and} \qquad D^{\rm MN}\phi(X) = \partial_\mu^{[\rm M}\left(\varepsilon^{\mu\nu} X_\nu^{\rm N]}\phi\right)(X), \tag{29}$$

[3] More generally, one can construct a family of first-order differential operators:

$$\mathcal{D}_1^{\rm M} = \alpha X_1^{\rm M} + \beta\,\eta^{\rm MN}\frac{i\partial}{\partial X_2^{\rm N}} \qquad \text{and} \qquad \mathcal{D}_2^{\rm M} = \gamma X_2^{\rm M} - \frac{1}{\alpha}(1-\beta\gamma)\,\eta^{\rm MN}\frac{i\partial}{\partial X_1^{\rm N}}$$

with arbitrary coefficients α, β, γ. For $\alpha = \gamma = 1$ and $\beta = 1/2$, they reduce to Eqs.(23,24). For $\alpha = 1$ and $\beta = \gamma = 0$, they reduce to the conventional position and momentum operators $\mathcal{D}_1^{\rm M} \equiv X^{\rm M}$ and $\mathcal{D}_2^{\rm M} \equiv P^{\rm M} = -i\partial/\partial X^{\rm M}$. We shall restrict the following discussion to the derivations Eqs.(23,24,26) only.

implying that integrals over the phase-space of these derivations acting on smooth functions vanish identically.

Left-multiplications of the generators $\mathbf{Q}_{\mu\nu}$ and $\mathbf{L}^{\mathrm{MN}}$ on a function $\phi(X)$ define second-order differential operators $\mathcal{D}_{\mu\nu}, \mathcal{D}^{\mathrm{MN}}$:

$$\begin{aligned}
\mathbf{Q}_{\mu\nu} \star \phi(X) &= \frac{1}{2}\eta_{\mathrm{MN}}\left(X^{\mathrm{M}}_{(\mu} + \frac{i}{2}\partial^{\mathrm{M}}_{(\mu}\right)\left(X^{\mathrm{N}}_{\nu)} + \frac{i}{2}\partial^{\mathrm{N}}_{\nu)}\right)\phi(X) && (30)\\
&= \left(X_\mu \cdot X_\nu + \frac{i}{2}D_{\mu\nu} - \frac{1}{4}\partial_\mu \cdot \partial_\nu\right)\phi(X) \quad \equiv \ \mathcal{D}_{\mu\nu}\phi(X) && (31)\\
\mathbf{L}^{MN} \star \phi(X) &= \frac{1}{2}\varepsilon^{\mu\nu}\left(X^{[\mathrm{M}}_{\mu} + \frac{i}{2}\partial^{[\mathrm{M}}_{\mu}\right)\left(X^{\mathrm{N}]}_{\nu} + \frac{i}{2}\partial^{\mathrm{N}]}_{\nu}\right)\phi(X) && (32)\\
&= \left(X^{[\mathrm{M}}_1 X^{\mathrm{N}]}_2 + \frac{i}{2}D^{\mathrm{MN}} - \frac{1}{4}\partial^{[\mathrm{M}}_1\partial^{\mathrm{N}]}_2\right)\phi(X) \ \equiv \mathcal{D}^{\mathrm{MN}}\phi(X). && (33)
\end{aligned}$$

The $\cdot$ refers to contraction of indices with respect to the $\mathrm{SO}^*(D)$ metric η_{MN}. Likewise, from right-multiplications of $\mathbf{Q}_{\mu\nu}$ and $\mathbf{L}^{\mathrm{MN}}$ to the function $\phi(X)$, one obtains another set of second-order differentiations, $\overline{\mathcal{D}}_{\mu\nu}, \overline{\mathcal{D}}^{\mathrm{MN}}$:

$$\begin{aligned}
\phi(X) \star \mathbf{Q}_{\mu\nu} &= \left(X_\mu \cdot X_\nu - \frac{i}{2}D_{\mu\nu} - \frac{1}{4}\partial_\mu \cdot \partial_\nu\right)\phi(X) \ \equiv \overline{\mathcal{D}}_{\mu\nu}\phi(X) && (34)\\
\phi(X) \star \mathbf{L}^{\mathrm{MN}} &= \left(X^{[M}_1 X^{N]}_2 - \frac{i}{2}D^{\mathrm{MN}} - \frac{1}{4}\partial^{[M}_1\partial^{N]}_2\right)\phi(X) \equiv \overline{\mathcal{D}}^{\mathrm{MN}}\phi(X). && (35)
\end{aligned}$$

These various first and second-order left- and right- differential operators violate the Leibniz rule, however, they have interesting properties: from the commutation relations of $\mathbf{Q}_{\mu\nu}$ and $\mathbf{L}^{\mathrm{MN}}$, Eqs.(16-18), it follows immediately that each of the sets of differential operators we have defined $\left(D_{\mu\nu}, D^{\mathrm{MN}}, \partial^{\mathrm{M}}_{\mu}\right)$ or $\left(\mathcal{D}_{\mu\nu}, \mathcal{D}^{\mathrm{MN}}, \mathcal{D}^{\mathrm{M}}_{\mu}\right)$ or $\left(\overline{\mathcal{D}}_{\mu\nu}, \overline{\mathcal{D}}^{MN}, \overline{\mathcal{D}}^{\mathrm{M}}_{\mu}\right)$ provide inequivalent representations for the generators of the $\mathrm{Sp}(2) \times \mathrm{SO}^*(D)$ symmetry group, as they obey the $\mathrm{sp}(2) \oplus \mathrm{so}^*(D)$ Lie algebra.

$$\begin{aligned}
\left[\mathcal{D}_{\mu\nu}\,,\,\mathcal{D}_{\kappa\lambda}\right] &= iF_{\mu\nu,\kappa\lambda}{}^{\alpha\beta}\ \mathcal{D}_{\alpha\beta}, && (36)\\
\left[\mathcal{D}^{\mathrm{MN}}, \mathcal{D}^{\mathrm{KL}}\right] &= iF^{\mathrm{MN,KL}}{}_{\mathrm{PQ}}\mathcal{D}^{\mathrm{PQ}}, && (37)\\
\left[\mathcal{D}^{\mathrm{MN}}, \mathcal{D}_{\mu\nu}\right] &= 0, && (38)
\end{aligned}$$

and rotate the first order derivatives in the appropriate fundamental representation

$$\begin{aligned}
\left[\mathcal{D}_{\mu\nu}\,, \mathcal{D}^{\mathrm{K}}_{\lambda}\right] &= i\varepsilon_{\nu\lambda}\mathcal{D}^{\mathrm{K}}_{\mu} + i\varepsilon_{\mu\lambda}\mathcal{D}^{\mathrm{K}}_{\mu}, && (39)\\
\left[\mathcal{D}^{\mathrm{MN}}, \mathcal{D}^{\mathrm{K}}_{\lambda}\right] &= i\eta^{\mathrm{NK}}\mathcal{D}^{\mathrm{M}}_{\lambda} - i\eta^{\mathrm{MK}}\mathcal{D}^{\mathrm{N}}_{\lambda} && (40)
\end{aligned}$$

Similar commutation relations are obeyed by the other sets of differential operators $\left(D_{\mu\nu}, D^{\mathrm{MN}}, \partial^{\mathrm{M}}_{\mu}\right)$ or $\left(\overline{\mathcal{D}}_{\mu\nu}, \overline{\mathcal{D}}^{MN}, \overline{\mathcal{D}}^{\mathrm{M}}_{\mu}\right)$.

There also exists another class of second-order differential operators of the form $X^{\mathrm{M}}_{\mu} \star \phi(X) \star X^{\mathrm{N}}_{\nu}$'s. One can show, however, that their algebra does not close among themselves and hence is not relevant for the representation of $\mathrm{Sp}(2)\times\mathrm{SO}^*(D)$ symmetry group.

Summarizing the result of this section, we have constructed various first- and second-order differential operators. The Leibniz rule is obeyed by $\partial^{\mathrm{M}}_{\mu}$,$D_{\mu\nu}$,$D^{\mathrm{MN}}$ and violated by $\mathcal{D}^{\mathrm{M}}_{\mu}$,$\overline{\mathcal{D}}^{\mathrm{M}}_{\mu}$, $\mathcal{D}_{\mu\nu}$,$\overline{\mathcal{D}}_{\mu\nu}$, $\mathcal{D}^{\mathrm{MN}}$, $\overline{\mathcal{D}}^{\mathrm{MN}}$. Nevertheless, in the following discussion, all of them will play a role.

3.3 Projective Relations

Extending further the products the field with higher powers of $X_\mu^{\rm M}$'s, consider $\star$-multiplication between fields. Given a set of fields that are well-defined on phase-space, satisfying a suitable fall-off condition at infinity, the $\star$-multiplication between them ought to correspond to another field well-defined on the same phase-space, viz.

$$\star \quad : \quad \mathcal{M}_\theta \otimes \mathcal{M}_\theta \quad \longrightarrow \quad \mathcal{M}_\theta. \tag{41}$$

We will define a complete basis of fields that close under the $\star$-product, and will prove Eq.(41) via explicit calculation. Recall that, in the context of non-relativistic quantum mechanics, the Wigner function defined on the particle's phase-space [2] is the Weyl-Moyal counterpart of the *diagonal* density-matrix operators. We will begin with generalizing this correspondence to a complete set of covariant fields defined on relativistic phase-space by including *off-diagonal* density-matrix operators.

Consider a complete set of covariant fields, $\varphi_m(X_1) \equiv < X_1|\varphi_m >$, $m = 1, 2, 3, \cdots$, defined on the particle's configuration-space, and construct all possible density matrices $\hat{\rho}_{mn} := |\varphi_m\rangle\langle\varphi_n|$ out of them. Then, noncommutative scalar fields $\phi_{mn}(X_1, X_2)$ can be defined by applying the Weyl-Moyal map to the density matrix:

$$\phi_{mn}(X_1, X_2) \quad : \quad = \int d^DY \; \varphi_m(X_1) \star \exp\left(-iX_2 \cdot Y\right) \star \varphi_n^*(X_1), \tag{42}$$

$$= \quad \int d^DY \; \varphi_m\left(X_1 - \frac{Y}{2}\right) \exp\left(-iX_2 \cdot Y\right) \varphi_n^*\left(X_1 + \frac{Y}{2}\right) \tag{43}$$

The phase-space field $\phi_{mn}(X)$ is nothing but the Wigner transformation [2] of the configuration space fields $\varphi_m(X_1), \varphi_n(X_1)$, now extended to a relativistically covariant and off-diagonal form. Assuming completeness, one can construct a *coherent superposition* to represent any noncommutative field in the form

$$\phi(X_1, X_2) := \sum C_{mn}\, \phi_{mn}(X_1, X_2). \tag{44}$$

where C_{mn} are a set of constant coefficients. Therefore it is useful to learn about the properties of the ϕ_{mn}.

We claim that noncommutative fields of the form Eq.(42) form a set that close under the $\star$-multiplication, as in Eq.(41). Explicitly, consider two noncommutative fields, $\phi_{k\ell}(X), \phi_{mn}(X)$, of the form Eq.(42) and take the $\star$-product between them. One calculates that

$$\begin{aligned}
\left(\phi_{k\ell} \star \phi_{mn}\right)(X) \;&=\; \int d^DY d^D\widetilde{Y} \,\left(\varphi_k(X_1) \star e^{-iY\cdot X_2} \star \varphi_\ell^*(X_1)\right) \star \left(\varphi_m(X_1) \star e^{-i\tilde{Y}\cdot X_2} \star \varphi_n^*(X_1)\right) \\
&=\; \int d^DY d^D\widetilde{Y}\, \varphi_k(X_1) \star \left[e^{-iY\cdot X_2} \star \left(\varphi_\ell^*(X_1)\varphi_m(X_1)\right) \star e^{-i\tilde{Y}\cdot X_2}\right] \star \varphi_n^*(X_1) \\
&=\; 2^{-D} \int d^DY_+\, \varphi_k(X_1) \star \left[e^{-iX_2\cdot Y_+} \int d^DY_- \varphi_\ell^*\left(X_1 - \frac{Y_-}{2}\right) \varphi_m\left(X_1 - \frac{Y_-}{2}\right)\right] \star \varphi_n^*(X_1).
\end{aligned}$$

where $Y_\pm^{\rm M} = \left(Y^{\rm M} \pm \tilde{Y}^{\rm M}\right)$. In going from the second to the third line we used the fact that under the $\star$-product, phase-space 'plane-waves', $e^{-ia\cdot X} \equiv \exp(-ia_i^{\rm M} X_j^{\rm N} \varepsilon^{ij} \eta_{\rm MN})$, generate translation on the quantum phase-space: for any function $F(X_i^{\rm M})$ on phase-space,

$$e^{-ib\cdot X} \star F(X_i^{\rm M}) \star e^{-ia\cdot X} = e^{-iX\cdot(a+b)}\, F\left(X_i^{\rm M} - \frac{1}{2}a_i^{\rm M} + \frac{1}{2}b_i^{\rm M}\right). \tag{45}$$

Since the integrals over $Y_\pm^{\rm M}$ are factorized, one finally obtains

$$(\phi_{k\ell} \star \phi_{mn})(X) = \mathcal{N}_{\ell m} \int d^D Y_+\, \varphi_k(X_1) \star e^{-iX_2\cdot Y_+} \star \varphi_n^*(X_1), \tag{46}$$

where $\mathcal{N}_{\ell m}$ is a constant

$$\mathcal{N}_{\ell m} = 2^{-D} \int d^D Y_-\, \varphi_\ell^*(X_1 - Y_-/2)\, \varphi_m(X_1 - Y_-/2) \tag{47}$$

$$= \int d^D X_1\, \varphi_\ell^*(X_1) \varphi_m(X_1) \tag{48}$$

which denotes inner-product between two configuration-space fields, or simply $\mathcal{N}_{\ell m} = \langle \varphi_l | \varphi_m \rangle$. Thus the closure of the algebra satisfied by the ϕ_{mn} under $\star$-products is the same as the one satisfied by density matrices $\hat{\rho}_{mn} := |\varphi_m\rangle\langle\varphi_n|$.

In case the configuration-space fields φ_m's form an orthonormal basis – take, for example, configuration space plane-waves, $e^{iX_1\cdot K}$ –, viz. $\mathcal{N}_{\ell m} = \delta_{\ell,m}$. One then obtains covariant version of the orthogonality relation:

$$\left(\phi_{k\ell} \star \phi_{mn}\right)(X) = \delta_{\ell,m} \phi_{kn}(X) \tag{49}$$

as the fundamental nonlinear relations among the noncommutative fields. A subset closed under the orthogonality relation consists of *diagonal* noncommutative fields $\phi_{mm}(X)$, which have the property of projection operators $|\varphi_m\rangle\langle\varphi_m|$, satisfying:

$$\left(\phi_{mm} \star \cdots \star \phi_{mm}\right)(X) = \phi_{mm}(X). \tag{50}$$

In fact, such a projection operator ϕ_{mm} is a relativistic generalization of the Wigner distribution function:

$$\phi_{mm} = \int d^D Y\, \varphi_m(X_1) \star \exp\left(-iX_2 \cdot Y\right) \star \varphi_m^*(X_1) \tag{51}$$

$$= \int d^D Y\, \varphi_m\left(X_1 - Y/2\right) \exp(-iX_2 \cdot Y)\, \varphi_m\left(X_1 + Y/2\right). \tag{52}$$

In the solution of our NCFT equations we will use the general superposition (44) to relate 2T-physics in noncommutative field theory to 2T-physics in configuration space as discussed in the following subsection.

Incidentally, in recent works on noncommutative solitons [15], both diagonal and off-diagonal Wigner distribution functions have been utilized. Interpreting the $D \times 2$-dimensional phase-space as $D \times 2$-dimensional noncommutative space, diagonal Wigner functions are interpreted as spherically symmetric solitons, while off-diagonal ones are interpreted as asymmetric solitons. Indeed, the two are related each other by $\mathrm{U}(\infty)$ transformations.

3.4 Map Between Phase-Space and Configuration Space

Consider the Fourier transform in the X_2 variable of the general field in NCFT

$$\phi\left(X_1, X_2\right) := \int d^D Y \ \exp\left(-iX_2 \cdot Y\right) \ F\left(X_1 - \frac{Y}{2}, X_1 + \frac{Y}{2}\right) \tag{53}$$

where $F(X_L^{\mathrm{M}}, X_R^{\mathrm{M}}) := f\left(X_1, Y\right)$ is a by-local field in *configuration space*. If one computes the $\star$-products $X_1^{\mathrm{M}} \star \phi\left(X_1, X_2\right)$ and $X_2^{\mathrm{M}} \star \phi\left(X_1, X_2\right)$ acting from the left as in (23), then their effect is reproduced by acting only on the left variable in $F(X_L^{\mathrm{M}}, X_R^{\mathrm{M}})$ like position and derivative in configuration space respectively. A similar result is obtained by acting from the right

$$\begin{aligned}
X_1^{\mathrm{M}} \star \phi\left(X_1, X_2\right) \quad &\rightarrow \quad X_L^{\mathrm{M}} \ F(X_L^{\mathrm{M}}, X_R^{\mathrm{M}}), &(54)\\
X_2^{\mathrm{M}} \star \phi\left(X_1, X_2\right) \quad &\rightarrow \quad -i\frac{\partial}{\partial X_L^{\mathrm{M}}} \ F(X_L^{\mathrm{M}}, X_R^{\mathrm{M}}), &(55)\\
\phi\left(X_1, X_2\right) \star X_1^{\mathrm{M}} \quad &\rightarrow \quad X_R^{\mathrm{M}} \ F(X_L^{M}, X_R^{\mathrm{M}}), &(56)\\
\phi\left(X_1, X_2\right) \star X_2^{\mathrm{M}} \quad &\rightarrow \quad i\frac{\partial}{\partial X_R^{\mathrm{M}}} \ F(X_L^{\mathrm{M}}, X_R^{\mathrm{M}}). &(57)
\end{aligned}$$

The left-hand side of these equations is equal to the Fourier transform of the right hand side as in Eq.(53). Similarly, we may consider the basis $\phi_{mn}\left(X_1, X_2\right)$ of the previous section. From their definition Eq.(42) we see that $F(X_1 - \frac{Y}{2}, X_1 + \frac{Y}{2})$ is replaced by $\varphi_m(X_1 - \frac{Y}{2})\, \varphi_n^*\left(X_1 + \frac{Y}{2}\right)$, and assuming the completeness of the superposition (44), we have

$$F(X_1 - \frac{Y}{2}, X_1 + \frac{Y}{2}) = \sum C_{mn}\, \varphi_m\left(X_1 - \frac{Y}{2}\right)\ \varphi_n^*\left(X_1 + \frac{Y}{2}\right). \tag{58}$$

Using Eq.(23), we readily verify that the $\star$-multiplication of $X_1^{\mathrm{M}}, X_2^{\mathrm{M}}$ on the phase-space field $\phi_{mn}\left(X_1, X_2\right)$ is equivalent to applying X_1^{M} and $-i\partial/\partial X_1^{\mathrm{M}}$, respectively, on configuration-space fields $\varphi_m\left(X_1^{\mathrm{M}}\right)$. Explicitly,

$$\begin{aligned}
X_1^{\mathrm{M}} \star \phi_{mn}\left(X\right) \ &= \ \int d^D Y\, X_1^{\mathrm{M}} \star \left(\varphi_m(X_1) \star e^{-iX_2 \cdot Y} \star \varphi_n^*(X_1)\right) &(59)\\
&= \ \int d^D Y \left(X_1 \varphi_m(X_1)\right) \star e^{-iX_2 \cdot Y} \star \varphi_n^*(X_1) &(60)
\end{aligned}$$

and

$$\begin{aligned}
X_2^{\mathrm{M}} \star \phi\left(X\right) \ &= \ \int d^D Y\, X_2^{\mathrm{M}} \star \left(\varphi_m(X_1) \star e^{-iX_2 \cdot Y} \star \varphi_n^*(X_1)\right) &(61)\\
&= \ \int d^D Y \left(- i\frac{\partial}{\partial X_1^{\mathrm{M}}} \varphi_m(X_1)\right) \star e^{-iX_2 \cdot Y} \star \varphi_n^*(X_1). &(62)
\end{aligned}$$

Therefore, the action X_1^{M} or X_2^{M} on NC fields, from the left or the right, is equivalent to the usual rules for position and momenta acting on a complete set of wavefunctions in configuration space, as illustrated by the expressions in Eqs.(59,61) or in Eqs.(54-57).

Using these results, one can show similarly that for the free $\mathbf{Q}_{\mu\nu}$ or $\mathbf{L}^{\mathrm{MN}}$ we have

$$\begin{aligned} \mathbf{Q}_{\mu\nu} \star \phi_{mn}(X) &= \int d^D Y \left(\mathbf{q}_{\mu\nu}\, \varphi_m(X_1) \right) \star e^{-iX_2 \cdot Y} \star \varphi_n^*(X_1) && (63) \\ \mathbf{L}^{\mathrm{MN}} \star \phi_{mn}(X) &= \int d^D Y \left(\mathbf{l}^{\mathrm{MN}}\, \varphi_m(X_1) \right) \star e^{-iX_2 \cdot Y} \star \varphi_n^*(X_1), && (64) \end{aligned}$$

where $\mathbf{q}_{\mu\nu}\, \varphi_m(X_1)$ and $\mathbf{l}^{\mathrm{MN}}\, \varphi_m(X_1)$ are given in terms of ordinary products or derivatives as in Eqs.(5) and (9) respectively. Thus, acting on the basic fields $\varphi_m(X_1)$ in configuration-space, $\mathbf{q}_{\mu\nu}$, $\mathbf{l}^{\mathrm{MN}}$ are the operators obeying $\mathrm{sp}(2) \oplus \mathrm{so}^*(D)$ Lie algebra:

$$\begin{aligned} [\mathbf{q}_{\mu\nu}, \mathbf{q}_{\alpha\beta}] &= iF_{\mu\nu,\lambda\sigma}{}^{\alpha\beta}\, \mathbf{q}_{\alpha\beta}, \\ \left[\mathbf{l}^{\mathrm{MN}}, \mathbf{l}^{\mathrm{PQ}}\right] &= iF^{\mathrm{MN,PQ}}{}_{\mathrm{RS}}\, \mathbf{l}^{\mathrm{RS}}, \end{aligned} \tag{65}$$

an immediate consequence of Eqs.(16)–(18). We have seen in the previous section that these operators have played a prominent role in the first-quantized approach to 2T-physics.

The above analysis allows us to rewrite the free field equations of 2T-physics in X_1-space given in Eq.(4) as free field equations in NCFT in noncommutative phase space

$$\mathbf{q}_{\mu\nu}\, \varphi_m(X_1) := 0 \quad \Leftrightarrow \quad \mathbf{Q}_{\mu\nu} \star \phi(X_1, X_2) = 0 = \phi(X_1, X_2) \star \mathbf{Q}_{\mu\nu} \tag{66}$$

A complete set of solutions to the free NCFT equations is provided by a complete set of solutions to the configuration space free field equations. These were already solved in [7]. Thus the 2T- to 1T-holographic properties of 2T-physics in NC phase space are closely related to those of the configuration space by the above map. The complete set of states $\varphi_m(X_1)$ form a specific unitary representation of $\mathrm{SO}(D-2,2)$ with quadratic Casimir $D(4-D)/4$, namely the singleton/doubleton, as explained in the paragraphs following Eq.(10). Hence the noncommutative field $\phi(X_1, X_2)$ that satisfies the NC free field equation should be regarded as the direct product of two singletons/doubletons.

For the more general 2T-physics theory in the presence of background fields, the first quantized field equation (2) can also be rewritten simply in the noncommutative field theory approach as

$$\left(\widehat{\mathbf{Q}}_{\mu\nu} \star \phi \right)(X_1, X_2) = 0 = \left(\phi \star \widehat{\mathbf{Q}}_{\mu\nu} \right)(X_1, X_2) \tag{67}$$

where $\widehat{\mathbf{Q}}_{\mu\nu}(X_1, X_2)$ contains all background fields, including scalar, vector (gauge field), tensor (gravitational field) and higher spin fields as analyzed in [8]. The $\widehat{\mathbf{Q}}_{\mu\nu}(X_1, X_2)$ are required to obey the Sp(2) Lie algebra using star products since at the classical level they had to obey the same algebra using Poisson brackets

$$\left[\widehat{\mathbf{Q}}_{\mu\nu}, \widehat{\mathbf{Q}}_{\lambda\sigma} \right]_\star = i \left(\varepsilon_{\nu\lambda} \widehat{\mathbf{Q}}_{\mu\sigma} + \varepsilon_{\mu\lambda} \widehat{\mathbf{Q}}_{\nu\sigma} + \varepsilon_{\nu\sigma} \widehat{\mathbf{Q}}_{\mu\lambda} + \varepsilon_{\mu\sigma} \widehat{\mathbf{Q}}_{\nu\lambda} \right). \tag{68}$$

Having established the desired field equations in NCFT, including background fields (before adding further non-linear interactions among the NC fields ϕ, $\widehat{\mathbf{Q}}_{\mu\nu}$), we will next proceed to developing the methodology for deriving them from first principles directly in the NCFT setting. This requires a study of both global and local $\mathrm{Sp}(2) \times \mathrm{SO}^*(D)$ covariance in NCFT.

4 Field Theory with Global Sp(2)$\times$ SO$^*(D)$ Symmetry

Having identified the symmetry group on relativistic phase-space, we next construct noncommutative field theory, in which the Sp(2) symmetry is global. Since this is a new subject which may have more general applications, we will first develop some general methodology before returning to the 2T-physics problem.

We begin with specifying Sp(2) $\times$ SO$^*(D)$ representations to the noncommutative fields. The generators that act on the relativistic phase-space are $\mathbf{Q}_{\mu\nu}$ and $\mathbf{L}^{\mathrm{MN}}$, Eqs.(14,15). Noncommutative fields of different defining representations are specified, depending on whether the generators act on fields from the left, the right, or as a commutator. Additionally, the fields can carry $\mu, \nu, \cdots; \mathrm{M}, \mathrm{N}, \cdots$ or spinor indices, thus describing states of higher-spin in Sp(2) or SO$^*(D)$.

4.1 'Adjoint' Representations

By a noncommutative scalar field $\phi(X)$ in 'adjoint' representation of Sp(2)$\times$SO$^*(D)$, we refer to the transformation rules:

$$\delta_{\mathrm{sp}}\phi(X) = -\frac{i}{2}\omega^{\alpha\beta}\left[\mathbf{Q}_{\alpha\beta}, \phi\right]_\star(X) = \frac{1}{2}\omega^{\alpha\beta}D_{\alpha\beta}\phi(X), \tag{69}$$

under infinitesimal Sp(2) rotation, where $\omega^{\alpha\beta}$ denote three infinitesimal rotation parameters, and

$$\delta_{\mathrm{so}}\phi(X) = -\frac{i}{2}\omega_{\mathrm{MN}}\left[\mathbf{L}^{\mathrm{MN}}, \phi\right]_\star(X) = \frac{1}{2}\omega_{\mathrm{MN}}D^{\mathrm{MN}}\phi(X), \tag{70}$$

under infinitesimal SO$^*(D)$ rotation, where ω_{MN} denote $D(D-1)$ infinitesimal parameters of SO$^*(D)$ rotations. Note that the latter transformation involves the total "angular momentum" operator $X_1^{[\mathrm{M}}\partial^{1\mathrm{N}]} + X_2^{[\mathrm{M}}\partial^{2\mathrm{N}]}$, rotating both X_1^{M} and X_2^{M} coordinates of the relativistic phase-space.

From Eq.(69), one finds Sp(2) transformation rules for various differential operators acting on the scalar field. Explicitly,

$$\begin{aligned} \delta_{\mathrm{sp}}\left(\partial_\mu^{\mathrm{M}}\phi\right)(X) &= \partial_\mu^{\mathrm{M}}\left(\frac{1}{2}\omega^{\alpha\beta}D_{\alpha\beta}\phi\right)(X) && (71)\\ &= \frac{1}{2}\omega^{\alpha\beta}D_{\alpha\beta}\left(D_\mu^{\mathrm{M}}\phi\right)(X) + \omega_\mu{}^\beta\left(\partial_\beta^{\mathrm{M}}\phi\right)(X). && (72)\end{aligned}$$

The second term arises because $\partial_\mu^{\mathrm{M}}\phi$ transforms as Sp(2) doublet as opposed to $\phi(X)$ itself being Sp(2) singlet. Similarly, $D_{\mu\nu}\phi$, $\mathcal{D}_{\mu\nu}\phi$, $\overline{\mathcal{D}}_{\mu\nu}\phi$ transforms as Sp(2) triplets, while $D^{\mathrm{MN}}\phi$, $\mathcal{D}^{\mathrm{MN}}\phi$, $\overline{\mathcal{D}}^{\mathrm{MN}}\phi$ transforms as Sp(2) singlets. Hence,

$$\begin{aligned} \delta_{\mathrm{sp}}\left(D_{\mu\nu}\phi\right)(X) &= \frac{1}{2}\omega^{\alpha\beta}D_{\alpha\beta}\left(D_{\mu\nu}\phi\right)(X) + \omega_\mu{}^\beta\left(D_{\beta\nu}\phi\right)(X) + \omega_\nu{}^\beta\left(D_{\beta\mu}\phi\right)(X), && (73)\\ \delta_{\mathrm{sp}}\left(D^{\mathrm{MN}}\phi\right)(X) &= \frac{1}{2}\omega^{\alpha\beta}D_{\alpha\beta}\left(D^{\mathrm{MN}}\phi\right)(X), && (74)\end{aligned}$$

and similarly for $\left(\mathcal{D}_{\mu\nu}\phi,\,\mathcal{D}^{\rm MN}\phi\right)$ or $\left(\overline{\mathcal{D}}_{\mu\nu}\phi,\,\overline{\mathcal{D}}^{\rm MN}\phi\right)$. For an infinitesimal SO$^*(D)$ rotation, transformation rules are obtained analogously: $D^{\rm MN}\phi$, $\mathcal{D}^{\rm MN}\phi$, $\overline{\mathcal{D}}^{\rm MN}\phi$ transform as SO$^*(D)$ adjoints, while $D_{\mu\nu}\phi$, $\mathcal{D}_{\mu\nu}\phi$, $\overline{\mathcal{D}}_{\mu\nu}\phi$ transforms as SO$^*(D)$ singlets.

Having identified the adjoint Sp(2) and SO$^*(D)$ transformation rules, we now proceed to the construction of an action functional possessing manifest global Sp(2) $\times$SO$^*(D)$ invariance.

We begin with the potential term. Consider an arbitrary $\star$-product polynomial of ϕ's:

$$V_\star(\phi) = m^2\phi\star\phi + \frac{\lambda_3}{3}\phi\star\phi\star\phi + \frac{\lambda_4}{4}\phi\star\phi\star\phi\star\phi + \cdots. \tag{75}$$

Eqs.(69, 70) and cyclicity of the $\star$-multiplication then imply that its integral is invariant under the Sp(2)$\times$SO$^*(D)$ transformations. Explicitly,

$$\begin{aligned}\delta_{\rm sp}\left(\int d^{2D}X\, V_\star(\phi)\right) &= \int d^{2D}X\, V'_\star(\phi)\star\delta_{\rm sp}\phi = -\frac{i}{2}\omega^{\alpha\beta}\int d^{2D}X\left[\mathbf{Q}_{\alpha\beta}, V_\star(\phi)\right]_\star = 0, &(76)\\ \delta_{\rm so}\left(\int d^{2D}X\, V_\star(\phi)\right) &= \int d^{2D}X\, V'_\star(\phi)\star\delta_{\rm so}\phi = -\frac{i}{2}\omega_{\rm LM}\int d^{2D}X\,\left[\mathbf{L}^{\rm MN}, V_\star(\phi)\right]_\star = 0, &(77)\end{aligned}$$

where cyclicity property of the $\star$-multiplication is used.

Consider next the kinetic term. Possible terms quadratic in differential operators are given by:

$$\begin{aligned}\frac{1}{2}\left(\partial^\mu_{\rm M}\phi\right)\star\left(\partial^{\rm M}_\mu\phi\right) \quad &, \quad \frac{1}{2}\left(\overline{\mathcal{D}}^\mu_{\rm M}\phi\right)\star\left(\mathcal{D}^{\rm M}_\mu\phi\right) &(78)\\ \frac{1}{4}\left(D^{\mu\nu}\phi\right)\star\left(D_{\mu\nu}\phi\right) \quad &, \quad \frac{1}{4}\left(D^{\rm MN}\phi\right)\star\left(D_{\rm MN}\phi\right) &(79)\\ \frac{1}{4}\left(\overline{\mathcal{D}}^{\mu\nu}\phi\right)\star\left(\mathcal{D}_{\mu\nu}\phi\right) &= \frac{1}{4}\phi\star\left(\mathcal{D}^{\mu\nu}\mathcal{D}_{\mu\nu}\phi\right), &(80)\\ \frac{1}{4}\left(\overline{\mathcal{D}}^{\rm MN}\phi\right)\star\left(\mathcal{D}_{\rm MN}\phi\right) &= \frac{1}{4}\phi\star\left(\mathcal{D}^{\rm MN}\mathcal{D}_{\rm MN}\phi\right). &(81)\end{aligned}$$

All indices are raised or lowered by the Sp(2) or SO$^*(D)$ metrics, $\varepsilon_{\mu\nu}$ or $\eta^{\rm MN}$. Because of that, the integrals of the two terms in the first line vanish identically. The rest, which will be denoted collectively as $\mathcal{L}_{\rm KE}$, all behave as scalars under Sp(2) $\times$SO$^*(D)$ transformations. Hence, like the potential term, their integrals are invariant once the cyclicity of the $\star$-multiplication is taken into account:

$$\begin{aligned}\delta_{\rm sp}\int d^{2D}X\,\mathcal{L}_{\rm KE} &= -\frac{i}{2}\,\omega^{\alpha\beta}\int d^{2D}X\,\left[\mathbf{Q}_{\alpha\beta},\mathcal{L}_{\rm KE}\right]_\star = 0, &(82)\\ \delta_{\rm so}\int d^{2D}X\,\mathcal{L}_{\rm KE} &= -\frac{i}{2}\omega_{\rm MN}\int d^{2D}X\left[\mathbf{L}^{\rm MN},\mathcal{L}_{\rm KE}\right]_\star = 0. &(83)\end{aligned}$$

Furthermore, because of the relation Eq.(22), the last two terms are related each other:

$$\frac{1}{2}\phi\star\mathcal{D}^{\rm MN}\mathcal{D}_{\rm MN}\phi = \frac{1}{2}\phi\star\mathcal{D}^{\mu\nu}\mathcal{D}_{\mu\nu}\phi + \frac{1}{4}D(4-D)\phi\star\phi. \tag{84}$$

Overall, the most general Sp(2)$\times$SO$^*(D)$ invariant action functional of the 'adjoint' scalar field is expressible as:

$$I[\phi] = \int d^{2D}X\,\left[\frac{a}{4}\left(\overline{\mathcal{D}}^{\mu\nu}\phi\right)\star\left(\mathcal{D}_{\mu\nu}\phi\right) + \frac{b}{4}\left(D^{\mu\nu}\phi\right)\star\left(D_{\mu\nu}\phi\right) + \frac{c}{4}\left(D^{\rm MN}\phi\right)\star\left(D_{\rm MN}\phi\right) - V_\star(\phi)\right], \tag{85}$$

where a, b, c denote arbitrary constants.

Inclusion of fermions is straightforward. Denote $\mathrm{SO}^*(D)$ spinors as $\psi_\alpha(X)$. The Sp(2) invariant differential operators are extendible to the spinors. By contracting them with $\mathrm{SO}^*(D)$ Dirac matrices, one obtains possible kinetic terms as:

$$iD_{\mathrm{MN}}\left(\Gamma^{\mathrm{MN}}\psi\right)_\alpha, \qquad i\mathcal{D}_{\mathrm{MN}}\left(\Gamma^{\mathrm{MN}}\psi\right)_\alpha, \qquad i\overline{\mathcal{D}}_{\mathrm{MN}}\left(\Gamma^{\mathrm{MN}}\psi\right)_\alpha. \tag{86}$$

As an example, consider a fermion ψ interacting with a scalar field ϕ, all transforming in 'adjoint' representation under Sp(2). The $\mathrm{Sp}(2)\times\mathrm{SO}^*(D)$ invariant action is then given by:

$$I[\overline{\psi}, \psi, \phi] = \int d^{2D}X \left[i\overline{\psi} \star \gamma^{\mathrm{MN}} \left(a'\mathcal{D}_{\mathrm{MN}} + b'\overline{\mathcal{D}}_{\mathrm{MN}} + c' D_{\mathrm{MN}}\right)\psi + g\overline{\psi} \star \phi \star \psi + \cdots\right], \tag{87}$$

where a', b', c' are arbitrary constants and g denotes the Yukawa coupling parameter.

4.2 'Fundamental' Representations

In the previous section, we have shown that left or right multiplication of $\mathbf{Q}_{\alpha\beta}$'s and $\mathbf{L}^{\mathrm{MN}}$'s yield, in addition to commutator multiplication, another representations of the $\mathrm{sp}(2) \oplus \mathrm{so}^*(D)$ Lie algebra. Based on this, we define left- or right-'fundamental' representation of a noncommutative scalar field $\Phi(X)$ by the following transformation rules:

$$\delta^L_{\mathrm{sp}}\Phi(X) = +\frac{i}{2}\omega_L^{\alpha\beta}\mathbf{Q}_{\alpha\beta} \star \Phi(X) := +\frac{i}{2}\omega_L^{\alpha\beta}\left(\mathcal{D}_{\alpha\beta}\Phi\right)(X), \tag{88}$$

$$\delta^R_{\mathrm{sp}}\Phi(X) = -\frac{i}{2}\Phi(X) \star \mathbf{Q}_{\alpha\beta}\,\omega_R^{\alpha\beta} := -\frac{i}{2}\left(\overline{\mathcal{D}}_{\alpha\beta}\Phi\right)(X)\,\omega_R^{\alpha\beta}, \tag{89}$$

where $\omega_L^{\alpha\beta}, \omega_R^{\alpha\beta}$ denotes infinitesimal $\mathrm{Sp}(2)_L$ and $\mathrm{Sp}(2)_R$ transformation parameters. Note that the field Φ ought to be complex-valued, in contrast to the 'adjoint' representation scalar ϕ, which could be real- or complex-valued. Then, the hermitian conjugate field transforms as

$$\delta^L_{\mathrm{sp}}\Phi^\dagger(X) = -\frac{i}{2}\left(\Phi^\dagger(X) \star \mathbf{Q}_{\alpha\beta}\right)\omega_L^{\alpha\beta} = -\frac{i}{2}\left(\overline{\mathcal{D}}_{\alpha\beta}\Phi^\dagger\right)(X)\,\omega_L^{\alpha\beta}, \tag{90}$$

$$\delta^R_{\mathrm{sp}}\Phi^\dagger(X) = +\frac{i}{2}\omega_R^{\alpha\beta}\left(\mathbf{Q}_{\alpha\beta} \star \Phi^\dagger(X)\right) = +\frac{i}{2}\omega_R^{\alpha\beta}\left(\mathcal{D}_{\alpha\beta}\Phi^\dagger\right)(X). \tag{91}$$

Likewise, for $\mathrm{SO}^*(D)$ transformation, left- or right-'fundamental' representations can be defined analogously.

From Eqs.(89, 91), it also follows that $\Phi \star \Phi^\dagger$ and $\Phi^\dagger \star \Phi$ transform as:

$$\delta^L_{\mathrm{sp}}\left(\Phi \star \Phi^\dagger\right)(X) = \frac{1}{2}\omega_L^{\alpha\beta}D_{\alpha\beta}\left(\Phi \star \Phi^\dagger\right)(X) \quad \text{and} \quad \delta^R_{\mathrm{sp}}\left(\Phi \star \Phi^\dagger\right)(X) = 0, \tag{92}$$

$$\delta^L_{\mathrm{sp}}\left(\Phi^\dagger \star \Phi\right)(X) = 0 \quad \text{and} \quad \delta^R_{\mathrm{sp}}\left(\Phi^\dagger \star \Phi\right)(X) = \frac{1}{2}\omega_R^{\alpha\beta}D_{\alpha\beta}\left(\Phi^\dagger \star \Phi\right)(X). \tag{93}$$

Note that the infinitesimal transformations of $\Phi \star \Phi^\dagger$ are all given entirely in terms of the $D_{\alpha\beta} := -i\left[\mathbf{Q}_{\alpha\beta}, \circ\right]$, the derivation satisfying the Leibniz rule, although the transformation of

Φ involves $\overline{\mathcal{D}}_{\alpha\beta}$, the differential operator which does not satisfy the Leibniz rule. It then follows that any function of $\Phi\star\Phi^\dagger$, $V_\star\left(\Phi\star\Phi^\dagger\right)$, is invariant under $\mathrm{Sp}(2)_R$ and transforms as an 'adjoint' representation under $\mathrm{Sp}(2)_L$. Thus,

$$\delta^R_{\mathrm{sp}} V_\star\left(\Phi\star\Phi^\dagger\right)=0 \quad \text{and} \quad \delta^L_{\mathrm{sp}} V_\star\left(\Phi\star\Phi^\dagger\right)=\frac{1}{2}\omega_L^{\alpha\beta}D_{\alpha\beta}V_\star\left(\Phi\star\Phi^\dagger\right) \tag{94}$$

and vice versa for any function of $\Phi^\dagger\star\Phi$, $V_\star(\Phi^\dagger\star\Phi)$. Therefore, taking $V_\star(\Phi^\dagger\star\Phi)$ or $V_\star(\Phi\star\Phi^\dagger)$ as the potential term, its integral is invariant manifestly under both $\mathrm{Sp}(2)_L\times$ $\mathrm{SO}^*(D)_L$ and $\mathrm{Sp}(2)_R\times$ $\mathrm{SO}^*(D)_R$ transformations.

Next, to construct a kinetic term in the action integral, consider various differential operators acting on the fields $\Phi,\Phi^\dagger$. Begin with $D_{\alpha\beta}\Phi$ and $D^{\mathrm{MN}}\Phi$. Under $\mathrm{Sp}(2)_L$ transformation

$$\delta^L_{\mathrm{sp}}\left(D_{\alpha\beta}\Phi\right)(X)=\omega_L\cdot\mathcal{D}\left(D_{\alpha\beta}\Phi\right)(X)-\left(\omega_L\cdot\mathcal{D}\right)_{(\alpha\beta)}\Phi(X) \tag{95}$$

generates the $\mathcal{D}$ differential operator in the second term, hence $D_{\mu\nu}\Phi$ is not covariant under $\mathrm{Sp}(2)_L$. Analogous results apply for $\mathrm{Sp}(2)_R$, $\mathrm{SO}^*(D)_L$, and $\mathrm{SO}^*(D)_R$ transformations. Hence $D_{\mu\nu}\Phi$ and $D^{\mathrm{MN}}\Phi$ do not define covariant differential operators, contrary to the situation for the fields in 'adjoint' representation. It turns out that $\mathrm{Sp}(2)_L$ and $\mathrm{Sp}(2)_R$ covariant differential operators are given by $\mathcal{D}_{\mu\nu}\Phi$ and $\overline{\mathcal{D}}_{\mu\nu}\Phi$, respectively. Explicitly,

$$\begin{aligned}
\delta^L_{\mathrm{sp}}\left(\mathcal{D}_{\alpha\beta}\Phi\right)(X) &= +\frac{i}{2}\omega_L\cdot\mathcal{D}\left(\mathcal{D}_{\alpha\beta}\Phi\right)(X)-\left(\omega_L\cdot\mathcal{D}\right)_{(\alpha\beta)}\Phi(X) && (96)\\
\delta^R_{\mathrm{sp}}\left(\overline{\mathcal{D}}_{\alpha\beta}\Phi\right)(X) &= -\frac{i}{2}\omega_R\cdot\mathcal{D}\left(\overline{\mathcal{D}}_{\alpha\beta}\Phi\right)(X)-\left(\omega_R\cdot\overline{\mathcal{D}}\right)_{(\alpha\beta)}\Phi(X) && (97)
\end{aligned}$$

and analogous expressions for $\mathrm{SO}^*(D)_{L,R}$ transformations for $\mathcal{D}^{\mathrm{MN}}\Phi$ and $\overline{\mathcal{D}}^{\mathrm{MN}}\Phi$ differential operators. Thus, $\mathcal{D}_{\alpha\beta}\Phi$, $\overline{\mathcal{D}}_{\alpha\beta}\Phi$, $\mathcal{D}^{\mathrm{MN}}\Phi$, and $\overline{\mathcal{D}}^{\mathrm{MN}}\Phi$ transform as adjoint representations under $\mathrm{Sp}(2)_L$, $\mathrm{Sp}(2)_R$, $\mathrm{SO}^*(D)_L$, and $\mathrm{SO}^*(D)_R$, respectively, and as singlets otherwise. Similarly, $\mathcal{D}_\mu^M\Phi$ transforms in the fundamental representation of $\mathrm{Sp}(2)_L\times\mathrm{SO}^*(D)_L$ and in the singlet of $\mathrm{Sp}(2)_R\times\mathrm{SO}^*(D)_R$, while $\overline{\mathcal{D}}_\mu^{\mathrm{M}}\Phi$ transforms in the singlet of $\mathrm{Sp}(2)_L\times\mathrm{SO}^*(D)_L$ and in the fundamental representation of $\mathrm{Sp}(2)_R\times\mathrm{SO}^*(D)_R$. The differential operators acting on the hermitian conjugate field $\Phi^\dagger$ exhibit similar transformation rules, related to those of Φ by interchanging the left- and the right-symmetry groups.

Putting the above results together, for the 'fundamental' scalar field Φ, the most general action integral with manifest $\mathrm{Sp}(2)_L\times$ $\mathrm{Sp}(2)_R\times$ $\mathrm{SO}^*(D)_L\times$ $\mathrm{SO}^*(D)_R$ invariance is given by

$$I[\Phi,\Phi^\dagger]=\int d^{2D}X\left[a\,\overline{\mathcal{D}}^{\mu\nu}\Phi^\dagger\star\mathcal{D}_{\mu\nu}\Phi+b\,\mathcal{D}_{\mu\nu}\Phi^\dagger\star\overline{\mathcal{D}}^{\mu\nu}\Phi-V_\star\left(\Phi\star\Phi^\dagger\right)\right]. \tag{98}$$

One could have also added terms of the form $\overline{\mathcal{D}}^{\mathrm{MN}}\Phi^\dagger\star\mathcal{D}_{\mathrm{MN}}\Phi$ and $\mathcal{D}^{\mathrm{MN}}\Phi^\dagger\star\overline{\mathcal{D}}_{\mathrm{MN}}\Phi$. As pointed out in Eq.(84), they are re-expressible in terms of those already included. In the action, a,b are arbitrary coefficients.

The field equation of motion is given by:

$$\frac{1}{2}\left(a\mathcal{D}^{\mu\nu}\mathcal{D}_{\mu\nu}+b\overline{\mathcal{D}}^{\mu\nu}\overline{\mathcal{D}}_{\mu\nu}\right)\Phi=V_\star'\left(\Phi\star\Phi^\dagger\right)\star\Phi. \tag{99}$$

Note that the left-hand side is expressed entirely in terms of the $\mathrm{Sp}(2)_L$ and $\mathrm{Sp}(2)_R$ Casimir operators, viz. $\frac{1}{2}\mathcal{D}^{\mu\nu}\mathcal{D}_{\mu\nu}=\frac{1}{2}\mathbf{Q}^{\mu\nu}\star\mathbf{Q}_{\mu\nu}=\frac{1}{2}\overline{\mathcal{D}}^{\mu\nu}\overline{\mathcal{D}}_{\mu\nu}$ acting on Φ either from the left or from the right.

Extension to fermion or higher-rank tensor field is straightforward. The fermion Ψ_α can be taken as the spinor representation of either $\mathrm{SO}^*(D)_L$ or $\mathrm{SO}^*(D)_R$ and as the 'fundamental' representation of either $\mathrm{Sp}(2)_L$ or $\mathrm{Sp}(2)_R$. Taking, as an example, that Ψ_α is in the left-representations for both, the action integral is expressible as:

$$I[\overline{\Psi},\Psi]=\int d^{2D}X\,\left[\overline{\Psi}\star\left(\Gamma^{\mathrm{MN}}\cdot i\mathcal{D}_{\mathrm{MN}}\Psi\right)+\cdots\right]. \tag{100}$$

The ellipses denote interaction part, whose form is constrained severely by the requirement of both the $\mathrm{Sp}(2)_L\times$ $\mathrm{SO}^*(D)_L$ and the $\mathrm{Sp}(2)_R\times$ $\mathrm{SO}^*(D)_R$ symmetry groups.

4.3 Spacetime Signature and Automorphism Group

We have constructed noncommutative field theories on the relativistic phase-space, in which the phase-space $\mathrm{Sp}(2)\times$ $\mathrm{SO}^*(D)$ is manifest. Because $\mathrm{Sp}(2)$ is part of the manifest symmetry group, X_1, X_2 appear explicitly in the kinetic terms, and break translation symmetry. These field theories are $\mathrm{SO}^*(D)$ Lorentz invariant but not Poincaré invariant in D dimensions. It implies, in particular, energy and momentum cannot be used as quantum numbers labelling states in Hilbert space. The good quantum numbers are associated with the representations of $\mathrm{Sp}(2)\times$ $\mathrm{SO}^*(D)$.

Is this an indication that something is wrong with the theory? Not at all. The lack of translation invariance is a common feature of 2T-physics in all its formulations, and surprisingly it is correct from the lower dimensional 1T physical point of view. When we identify the 1T-dynamics in the lower $(D-2)$ dimensions, the system does have Poincaré symmetry from the $D-2$ dimensional point of view. An example in one of the holographic pictures is that $\mathrm{SO}^*(D)=\mathrm{SO}(D-2,2)$ is the conformal group in $D-2$ dimensions, and it does contain the Poincaré group, including translations. This example shows that embedding the symmetries of the physical space in $\mathrm{SO}^*(D)$ is possible, and that the embedding space may have some unusual signature.

In more general cases beyond 2T-physics, the spacetime signature, which has been left unspecified so far, ought to be determined by consistency and physical properties of the theory. There are several ways of doing so. One is by treating the spacetime coordinates X_1^{M}'s as embedding space coordinates of a true physical spacetime as in the 2T-physics example. For instance, one may formulate Euclidean quantum field theories on a $(D-1)$-dimensional hypersphere in terms of those on D-dimensional Euclidean space [16]. Likewise, quantum field theories on $(D-2)$-dimensional de Sitter space can be recasted in terms of those on a D-dimensional Lorentzian spacetime with *one* timelike dimension, and quantum field theories on $(D-2)$-dimensional anti-de Sitter space in terms of those on a D-dimensional Lorentzian spacetime with *two* timelike dimensions. In all cases, the physical spacetime is defined as a hypersurface defined by an appropriate quadratic equations for coordinates of the D-dimensional embedding space. Moreover, the symmetry group of the physical spacetime is $\mathrm{SO}^*(D)$ and acts linearly on coordinates of the embedding space. Any of these embeddings will require some local symmetry to thin out degrees of freedom, eliminate ghosts, and reduce the theory to the lower dimensional theory.

The above discussion suggests that the noncommutative field theory with global $\mathrm{Sp}(2)\times\mathrm{SO}^*(D)$ automorphism group may be viewed as a sort of theory defined on an embedding phase-space of the physical phase-space. In particular, the signature of the higher dimensional spacetime will be determined depending on the way the physical phase-space is embedded into the higher dimensional space.

5 Field Theory with Local Sp(2) Symmetry

In this section, we will discuss noncommutative Sp(2) gauge theory on relativistic noncommutative phase-space. Of particular interest would be the construction of a theory, whose field equations coincide with Eqs.(67,68) for 2T-physics.

5.1 Action and equations of motion

Consider promoting the global $\mathrm{Sp}(2)_L$ transformation Eq.(88) of the complex scalar field Φ, to a local transformation parametrized by $\omega_L^{\alpha\beta}(X_1, X_2)$:

$$\delta^L\Phi(X) = \frac{i}{2}\omega^{\alpha\beta}(X)\star(\mathcal{D}_{\alpha\beta}\Phi)(X) = \frac{i}{2}\Big(\omega^{\alpha\beta}(X)\star\mathbf{Q}_{\alpha\beta}\Big)\star\Phi(X) \equiv i\omega_L(X)\star\Phi(X). \tag{101}$$

Ordering of the factors in $\frac{1}{2}\omega_L^{\alpha\beta}(X)\star\mathbf{Q}_{\alpha\beta} := \omega_L(X)$ could be more general. With any ordering, the resulting $i\omega_L(X_1, X_2)$ can be regarded as the general noncommutative infinitesimal local phase transformation acting on the left of Φ. So, we will in fact interpret local $\mathrm{Sp}(2)_L$ applied on a scalar to mean the general gauge transformation for any $\omega_L(X)$ applied from the left as in the last expression in Eq.(101). Proceeding as usual, we introduce a gauge potential $\mathbf{A}_{\mu\nu}(X_1, X_2)$ and promote the global $\mathrm{Sp}(2)_L$ differential operator $\mathcal{D}_{\mu\nu}$ to a local covariant differential operator $\widehat{\mathcal{D}}_{\mu\nu}$

$$\widehat{\mathcal{D}}_{\mu\nu}\Phi(X) := \mathcal{D}_{\mu\nu}\Phi(X) + \mathbf{A}_{\mu\nu}\star\Phi(X) = \Big(\mathbf{Q}_{\mu\nu} + \mathbf{A}_{\mu\nu}(X)\Big)\star\Phi(X). \tag{102}$$

The noncommutative local transformations are defined by Eq.(101) along with

$$\delta^L\mathbf{A}_{\mu\nu}(X) = \widehat{D}_{\mu\nu}\,\omega_L = D_{\mu\nu}\,\omega_L(X) - i\left[\mathbf{A}_{\mu\nu}(X), \omega_L(X)\right]_\star = -i\left[\left(\mathbf{Q}_{\mu\nu} + \mathbf{A}_{\mu\nu}\right), \omega_L\right]_\star, \tag{103}$$

where $D_{\mu\nu}$ is the derivation of Eq.(27) that satisfies the Leibniz rule. This ensures the covariance of the differential operator $\hat{\mathcal{D}}_{\mu\nu}\Phi$:

$$\delta^L\left(\widehat{\mathcal{D}}_{\mu\nu}\Phi\right) = i\omega_L \star \widehat{\mathcal{D}}_{\mu\nu}\Phi. \tag{104}$$

Denote the covariantized $\mathbf{Q}_{\mu\nu}$ as $\widehat{\mathbf{Q}}_{\mu\nu}\left(X_1, X_2\right)$

$$\widehat{\mathbf{Q}}_{\mu\nu} := \mathbf{Q}_{\mu\nu} + \mathbf{A}_{\mu\nu} = \frac{1}{2}X^{\mathrm{M}}_{(\mu} \star X^{\mathrm{N}}_{\nu)}\eta_{\mathrm{MN}} + \mathbf{A}_{\mu\nu}\left(X\right). \tag{105}$$

Note that $\widehat{\mathbf{Q}}_{\mu\nu}\left(X_1, X_2\right)$ is the counterpart of the classical $\hat{Q}_{\mu\nu}\left(X, P\right)$ that appeared in the worldline formalism in Eq.(1). The infinitesimal local gauge transformation of Eq.(103) is re-expressed as

$$\delta^L\widehat{\mathbf{Q}}_{\mu\nu} = -i\left[\widehat{\mathbf{Q}}_{\mu\nu}, \omega_L\right]_\star. \tag{106}$$

This is the counterpart of the canonical transformations in the "space of all theories" discussed in the worldline approach [8].

The covariant field strength $\mathbf{G}^L_{\mu\nu,\lambda\sigma}(X)$ is obtained from the $\star$-commutator of the covariant derivatives:

$$\begin{aligned}
\left[\widehat{\mathcal{D}}_{\mu\nu}, \widehat{\mathcal{D}}_{\lambda\sigma}\right]_\star \star \Phi(X) &= \left[\left(\mathcal{D}_{\mu\nu} + \mathbf{A}_{\mu\nu}\right), \left(\mathcal{D}_{\lambda\sigma} + \mathbf{A}_{\lambda\sigma}\right)\right]_\star \star \Phi(X) && (107)\\
&= i\, F_{\mu\nu,\lambda\sigma}{}^{\alpha\beta}\left(\mathcal{D}_{\alpha\beta}\Phi\right) + i\left(D_{\mu\nu}\mathbf{A}_{\lambda\sigma} - D_{\lambda\sigma}\mathbf{A}_{\mu\nu} - i\left[\mathbf{A}_{\mu\nu}, \mathbf{A}_{\lambda\sigma}\right]_\star\right) \star \Phi && (108)\\
&= i\, F_{\mu\nu,\lambda\sigma}{}^{\alpha\beta}\,\widehat{\mathcal{D}}_{\alpha\beta}\Phi + i\,\mathbf{G}_{\mu\nu,\lambda\sigma} \star \Phi, && (109)
\end{aligned}$$

where $F_{\mu\nu,\lambda\sigma}{}^{\alpha\beta}$ refers to the Sp(2) structure constants, Eq.(19), and the covariant field strength is given by

$$\mathbf{G}_{\mu\nu,\lambda\sigma}(X) = D_{\mu\nu}\mathbf{A}_{\lambda\sigma} - D_{\lambda\sigma}\mathbf{A}_{\mu\nu} - i\left[\mathbf{A}_{\mu\nu}, \mathbf{A}_{\lambda\sigma}\right]_\star - i\, F_{\mu\nu,\lambda\sigma}{}^{\alpha\beta}\mathbf{A}_{\alpha\beta}. \tag{110}$$

Note again that $D_{\mu\nu}$ is the derivation of Eq.(27) that satisfies the Leibniz rule. The last term in the field strength originates from the covariantization of the non-Abelian differential operators involved. In terms of the covariant generators $\widehat{\mathbf{Q}}_{\mu\nu}(X)$, the field strength becomes

$$i\,\mathbf{G}_{\mu\nu,\lambda\sigma} = \left[\widehat{\mathbf{Q}}_{\mu\nu}, \widehat{\mathbf{Q}}_{\lambda\sigma}\right]_\star - i\, F_{\mu\nu,\lambda\sigma}{}^{\alpha\beta}\,\widehat{\mathbf{Q}}_{\alpha\beta}. \tag{111}$$

$\mathbf{G}_{\mu\nu,\lambda\sigma}$ has only three independent components which may be rewritten in the form of a symmetric 2×2 tensor $\mathbf{G}^{\mu\nu}$, the latter being obtained from contraction of $\mathbf{G}_{\lambda\sigma,\rho\sigma}$ with the structure constant raised indices $F^{\mu\nu,\lambda\sigma,\rho\sigma}$. Explicitly, three independent components of the $\mathbf{G}^{\mu\nu}$ takes the form

$$\begin{aligned}
\mathbf{G}^{11} &= i\left[\widehat{\mathbf{Q}}_{12}, \widehat{\mathbf{Q}}_{22}\right]_\star + 2\widehat{\mathbf{Q}}_{22}, && (112)\\
\mathbf{G}^{12} &= \frac{i}{2}\left[\widehat{\mathbf{Q}}_{22}, \widehat{\mathbf{Q}}_{11}\right]_\star - 2\widehat{\mathbf{Q}}_{12}, && (113)\\
\mathbf{G}^{22} &= i\left[\widehat{\mathbf{Q}}_{11}, \widehat{\mathbf{Q}}_{12}\right]_\star + 2\widehat{\mathbf{Q}}_{11}. && (114)
\end{aligned}$$

The vanishing of the field strengths $\mathbf{G}^{\mu\nu}$ or $\mathbf{G}_{\mu\nu,\lambda\sigma}$ is equivalent to $\widehat{\mathbf{Q}}_{\mu\nu}$ satisfying the first quantized sp(2) algebra as in Eq.(68). This algebra had emerged as a condition in the first quantized worldline theory Eq.(1), which followed from the identical algebra in the form of Poisson brackets in the classical theory. Thus, we now aim at deriving the equations $\mathbf{G}^{\mu\nu}=0$ as equations of motion (before possible field interactions) from an action principle in the noncommutative field theory. We can easily obtain this result from the following noncommutative field theory, whose structure is analogous to the Chern-Simons gauge theory:

$$\begin{aligned} S_Q &= \int d^{2D}X \left[\left\langle \widehat{\mathbf{Q}},\widehat{\mathbf{Q}}\star\widehat{\mathbf{Q}}\right\rangle - \left\langle \widehat{\mathbf{Q}},\widehat{\mathbf{Q}}\right\rangle\right] \\ &:= \int d^{2D}X \left(\begin{array}{c} i\widehat{\mathbf{Q}}_{11}\star\widehat{\mathbf{Q}}_{12}\star\widehat{\mathbf{Q}}_{22} - i\widehat{\mathbf{Q}}_{22}\star\widehat{\mathbf{Q}}_{12}\star\widehat{\mathbf{Q}}_{11} \\ +\widehat{\mathbf{Q}}_{11}\star\widehat{\mathbf{Q}}_{22} + \widehat{\mathbf{Q}}_{22}\star\widehat{\mathbf{Q}}_{11} - 2\widehat{\mathbf{Q}}_{12}\star\widehat{\mathbf{Q}}_{12} \end{array}\right), \end{aligned} \tag{115}$$

whose variation yields $\delta S_Q = \int d^{2D}X \left(\delta\widehat{\mathbf{Q}}_{\mu\nu}\star\mathbf{G}^{\mu\nu}\right)$.

To obtain also the equation Eq.(67) for the matter field $\Phi\left(X_1,X_2\right)$, consider the covariant derivative Eq.(102), $\widehat{\mathcal{D}}_{\mu\nu}\Phi\equiv\widehat{\mathbf{Q}}_{\mu\nu}\star\Phi$, and add it to the action Eq.(115) after multiplying it with a Lagrange multiplier field $Z^{\mu\nu}\left(X_1,X_2\right)$:

$$S_{\Phi,Q,Z} = -i\int d^{2D}X\left(\overline{Z}^{\mu\nu}\star\widehat{\mathbf{Q}}_{\mu\nu}\star\Phi - \overline{\Phi}\star\widehat{\mathbf{Q}}_{\mu\nu}\star Z^{\mu\nu}\right). \tag{116}$$

The $\overline{Z}^{\mu\nu}$ field equation yields the free part of the the desired matter equation

$$\widehat{\mathbf{Q}}_{\mu\nu}\star\Phi = 0, \tag{117}$$

while $\overline{\Phi}$ field equation yields an equation for $Z^{\mu\nu}$of the form

$$\widehat{\mathbf{Q}}_{\mu\nu}\star Z^{\mu\nu} = 0. \tag{118}$$

The action $S_{\Phi,Q,Z}$ is invariant under the local $\mathrm{Sp}(2)_L$ transformations Eqs.(101,106) provided $Z^{\mu\nu}$ field transfoms as $\delta^L Z^{\mu\nu} = i\omega_L\star Z^{\mu\nu}$, and under the local $\mathrm{Sp}(2)_R$ defined by

$$\delta^R\Phi = -i\Phi\star\omega_R \qquad \text{and} \qquad \delta^R Z^{\mu\nu} = -iZ^{\mu\nu}\star\omega_R. \tag{119}$$

One may accordingly define a Hermitian field $\phi\left(X_1,X_2\right)=\Phi\star\overline{\Phi}$ satisfying $\widehat{\mathbf{Q}}_{\mu\nu}\star\phi=0=\phi\star\widehat{\mathbf{Q}}_{\mu\nu}$, corresponding to the first-quantized matter wavefunction of the worldline theory, Eq.(67).

The addition of matter fields would give rise to a back-reaction to the gauge fields themselves. The field equations derived from the combined action

$$S_{\text{total}} = S_Q + S_{\Phi,Q,Z} \tag{120}$$

are

$$\mathbf{G}^{\mu\nu} = \Phi\star\overline{Z}^{\mu\nu} - Z^{\mu\nu}\star\overline{\Phi}, \quad \widehat{\mathbf{Q}}_{\mu\nu}\star\Phi = 0, \qquad \widehat{\mathbf{Q}}_{\mu\nu}\star Z^{\mu\nu} = 0, \tag{121}$$

plus Hermitian conjugates of the last two equations. From them, one derives the following field equations involving gauge fields only

$$\widehat{\mathbf{Q}}_{\mu\nu} \star \mathbf{G}^{\mu\nu} = 0 = \mathbf{G}^{\mu\nu} \star \widehat{\mathbf{Q}}_{\mu\nu}. \tag{122}$$

Evidently, the structure of these equations is consistent with the first quantization of the worldline theory as given in Eqs.(68,67), in particular, when the matter self interactions are ignored, as then $\mathbf{G}^{\mu\nu} = 0$ and $\widehat{\mathbf{Q}}_{\mu\nu} \star \Phi = 0$. One may setup an expansion around this solution and analyze the classical solution of these equations.

By virtue of the relation to the worldline 2T-physics theory, we are assured that the spectrum of these 2T-field equations is unitary (ghost-free) and causal. Indeed, as in the classical theory, the physical spectrum is empty unless there are two timelike dimensions. Furthermore, the physics described by them has a direct relation to the 1T-physics in $(D-2)$ dimensions by virtue of the holographic property of 2T-physics. As demonstrated below (see also [8]), the $\mathbf{G}^{\mu\nu}(X,P) = 0$ equations describe, when expanded in powers of P_{M}'s, background gauge fields of various higher–spins in the $(D-2)$-dimensional spacetime. The matter field equation, the second in Eq.(121), implies that these higher-spin fields are coupled also to a scalar field (the $\varphi(X)$ in Eq.(42)) in $(D-2)$-dimensional spacetime.

Having noted that we have made the desired connection with 2T-physics, one can generalize the noncommutative Sp(2) gauge theory by including nonlinear (self)-interactions consistently with gauge and spacetime symmetries. The inclusion of such interaction, such as Eq.(122) and those below, would generate kinetic terms describing propagation of the gauge fields, but this has not been studied yet in our setting. This is an interesting issue for furthre study, as it is related to construction of an *interacting* higher-spin gauge field theory, whose satisfactory solution has remained elusive despite considerable progress [17]. Specifically, consider adding terms up to two derivatives of $\mathcal{D}_{\mu\nu}$ or $D_{\mu\nu}$. Of particular interest would be the Yang-Mills action for the Sp(2) gauge field $\mathbf{A}_{\mu\nu}(X)$, which can be taken instead of or in addition to the above Chern-Simons type action:

$$\begin{aligned} S_{G^2} &= -\frac{1}{4g^2}\int d^{2D}X \ \left(\mathbf{G}_{\mu\nu,\lambda\sigma}\right)^2 \\ &= \frac{1}{4g^2}\int d^{2D}X \ \left(\left[\widehat{\mathbf{Q}}_{\mu\nu}, \widehat{\mathbf{Q}}_{\lambda\sigma}\right]_\star - i\, F_{\mu\nu,\lambda\sigma}{}^{\alpha\beta}\widehat{\mathbf{Q}}_{\alpha\beta}\right)_\star^2 . \end{aligned} \tag{123}$$

Similarly, one may add self-interactions of the scalar field $\varphi(X_1, X_2)$ (including the scalar field Φ discussed above):

$$\begin{aligned} S_\varphi &= \int d^{2D}X \left[-\frac{1}{2}\left(\widehat{\mathcal{D}}^{\mu\nu}\varphi\right)^\dagger \star \widehat{\mathcal{D}}_{\mu\nu}\varphi - V_\star\left(\varphi\star\varphi^\dagger\right)\right] \\ &= \int d^{2D}X \left[-\frac{1}{2}\varphi^\dagger \star \left(\widehat{\mathbf{Q}}_{\mu\nu}\right)^2 \star \varphi - V_\star\left(\varphi\star\varphi^\dagger\right)\right], \end{aligned}$$

where $\left(\widehat{\mathbf{Q}}_{\mu\nu}\right)^2$ is the quadratic Casimir operator of Sp(2).

5.2 Classical Solutions

Let us now analyze physical contents of the equations

$$\mathbf{G}^{\mu\nu}=0 \qquad \text{and} \qquad \widehat{\mathbf{Q}}_{\mu\nu}\star\Phi=0. \tag{124}$$

From Eqs.(112-114) it is evident that $\mathbf{G}^{\mu\nu}=\mathbf{0}$ is equivalent to imposing the sp(2) algebra on $\widehat{\mathbf{Q}}_{\mu\nu}(X,P)$. A solution to this situation was found in [8] as follows: using the $\mathrm{Sp}(2)_L$ gauge transformations, one can choose gauges such that the $\widehat{\mathbf{Q}}_{\mu\nu}(X,P)$ takes the following form [4]

$$\begin{aligned}\widehat{\mathbf{Q}}_{11}(X,P) &= X^{\mathrm{M}}X^{\mathrm{N}}\eta_{\mathrm{MN}}, \quad \widehat{\mathbf{Q}}_{12}(X,P)=X^{\mathrm{M}}\left(P_{\mathrm{M}}+A_{\mathrm{M}}(X)\right)\\ \widehat{\mathbf{Q}}_{22}(X,P) &= G_0(X)+G_2^{\mathrm{MN}}(P+A)_{\mathrm{M}}(P+A)_{\mathrm{N}}\\ &\quad+\sum_{s=3}^{\infty}G_s^{\mathrm{M_1\cdots M_s}}(X)\left[(P+A)_{\mathrm{M_1}}\cdots(P+A)_{\mathrm{M_s}}\right]\end{aligned}$$

where η_{MN} is the $\mathrm{SO}(D-2,2)$ metric, $A_{\mathrm{M}}(X)$ is the Maxwell gauge field in D dimensions, $G_0(X)$ is the dilaton, $G_2^{\mathrm{MN}}(X)=\eta^{\mathrm{MN}}+h_2^{\mathrm{MN}}(X)$ is the spacetime metric in D dimensions, and $G_s^{\mathrm{M_1\cdots M_s}}(X)$ for all $s\geq 3$ are the higher-spin gauge fields. To obey the sp(2) algebra, these fields ought to be homogeneous polynomials of degree $(s-2)$ and be orthogonal to X^{M} (using the flat $\mathrm{SO}(D-2,2)$ metric η_{MN}) as follows

$$X\cdot\partial G_s=(s-2)\,G_s, \qquad X_{\mathrm{M_1}}G_s^{\mathrm{M_1\cdots M_s}}=X_{\mathrm{M_1}}h_2^{\mathrm{M_1M_2}}=0, \qquad X^{\mathrm{M}}F_{\mathrm{MN}}=0, \tag{125}$$

where $F_{\mathrm{MN}}=(\partial_{\mathrm{M}}A_{\mathrm{N}}-\partial_{\mathrm{N}}A_{\mathrm{M}})$ is the Maxwell field strength. The Maxwell gauge symmetry can also be partially fixed by taking $X\cdot A=0$. Then, $X^{\mathrm{M}}F_{\mathrm{MN}}=0$ becomes a homogeneity condition $X\cdot\partial A_M=-A_M$. After the gauge-fixing, there still remains local $\mathrm{Sp}(2)_L$ symmetry that does not change the gauge fixed form of the $\widehat{\mathbf{Q}}_{11}(X,P)$ and $\widehat{\mathbf{Q}}_{22}(X,P)$ given above (i.e. $\delta^L\widehat{\mathbf{Q}}_{11}=\delta^L\widehat{\mathbf{Q}}_{22}=0$). From Eq.(106), one finds that the corresponding gauge function $\omega_L(X,P)$ ought to take the form

$$\omega_L(X,P)=\omega_0(X)+\omega_1^{\mathrm{M}}(X)\,(P+A)_{\mathrm{M}}+\sum_{s=2}^{\infty}\omega_s^{\mathrm{M_1\cdots M_s}}(X)\left[(P+A)_{\mathrm{M_1}}\cdots(P+A)_{\mathrm{M_s}}\right], \tag{126}$$

where each coefficient is a homogeneous function of degree s and is transverse to X^{M} :

$$X\cdot\partial\omega_s=s\,\omega_s \qquad \text{and} \qquad X_{\mathrm{M_1}}\omega_s^{\mathrm{M_1\cdots M_s}}=0 \qquad \text{for} \qquad s\geq 0. \tag{127}$$

These residual gauge symmetries are interpreted as follows: $\omega_0(X)$ is the gauge parameter that transforms the Maxwell field, $\omega_1^{\mathrm{M}}(X)$ is the parameter for general coordinate transformations, and

[4] We emphasize that after choosing a gauge for $\widehat{\mathbf{Q}}_{11}$, the remaining gauge symmetry is insufficient to simplify the structure of $\widehat{\mathbf{Q}}_{12}(X,P)$ further. However, if $\widehat{\mathbf{Q}}_{12}$ is restricted to obey the sp(2) algebra, the remaining gauge symmetry can be used to set it to the form shown in the text.

the $\omega_{s-1}^{\mathrm{M_1 \cdots M_{s-1}}}$ are gauge parameters for the high spin fields $G_s^{\mathrm{M_1 \cdots M_s}}$. The gauge transformations mix various gauge fields one another (see [8]), but typically an inhomogeneous term

$$\delta G_s^{\mathrm{M_1 \cdots M_s}} = \partial^{(M_1} \omega_{s-1}^{\mathrm{M_2 \cdots M_s)}} + \cdots \tag{128}$$

remains in the gauge transformations, where the index on ∂^{M} raised as $\partial^{\mathrm{M}} = G_2^{\mathrm{MN}} \partial_{\mathrm{N}}$.

The equations Eqs.(125), taken together with $X^2 = 0$ (which equals to the $\widehat{\mathbf{Q}}_{11} = 0$ condition), describe fields whose independent degrees of freedom reside in $(D-2)$ dimensions, both from the viewpoint of their components and their dependence on spacetime coordinates. Specifically, Eqs.(125), together with $X^2 = 0$, impose the holographic property of 2T-physics. An explicit holographic projection from D dimensional spacetime X^M to $(D-2)$-dimensional spacetime x^μ is presented in [8]. One then sees that the independent degrees of freedom are given by the fields $g_0(x), A_\mu(x), g_{\mu\nu}(x), g_s^{\mu_1 \cdots \mu_s}(x)$ for $s \geq 3$, which are fields in $(D-2)$ dimensions, where the Lorentz components $\mu, \nu, \cdots$ transform according to $\mathrm{SO}(d-1,1)$. All of these $(D-2)$-dimensional fields are consistent with the $(D-2)$-dimensional conformal symmetry $\mathrm{SO}(D-2,2)$, as this is made evident by the D-dimensional formalism of 2T-physics.

The remaining gauge symmetries of Eq.(127) are also holographically projected to $(D-2)$ dimensions, and their independent components are $\varepsilon_0(x)$, $\varepsilon_1^\mu(x)$, and $\varepsilon_s^{\mu_1 \cdots \mu_s}(x)$ for $s \geq 2$. It turns out that these remaining gauge symmetries are strong enough to reduce the fields to pure gauge degrees of freedom, *unless* lower- and higher-spin fields do not coexist in the solution. The exceptional cases therefore lead to two distinct sets of non-trivial solutions: a *lower-spin branch* and a *higher-spin branch*. The lower-spin branch consists only of $g_0(x), A_\mu(x), g_{\mu\nu}(x)$, while all higher-spin fields $(s \geq 3)$ vanish. In the higher-spin branch, $g_0(x), A_\mu(x)$ vanish, while $g_{\mu\nu}(x)$, together with $g_s^{\mu_1 \cdots \mu_s}(x)$ for $s \geq 3$ form a non-trivial basis for the gauge transformations, whose explicit forms are calculated in [8].

Intriguingly, the two disconnected branches of solutions appear to bear a correspondence to massless states of string theories in two extreme limits (or phases). The lower-spin branch with spins $s \leq 2$ coincides with the limiting string spectrum in the zero Regge slope limit (infinite tension), while the higher-spin branch $s \geq 2$ coincides with the limiting string spectrum of the leading graviton trajectory in the infinite Regge slope limit (zero tension). [5]

We have thus found a set of interesting solutions to Eqs.(124) and have succeeded in their physical interpretations . They have important implications; the equations $\mathbf{G}^{\mu\nu} = 0$ encode *all* possible $(D-2)$-dimensional gauge field backgrounds that a spinless point-particle would interact with. Moreover, the interaction with the spinless field is governed by the physical state condition, $\widehat{\mathbf{Q}}_{\mu\nu} \star \Phi(X) = 0$, – a condition which solves the noncommutative field equations when nonlinear

[5] The aforementioned solutions for $\mathbf{G}^{\mu\nu} = 0$ describe gauge fields, but the propagation of these fields is not determined by this equation. Thus the kinetic term must come from terms such as Eqs.(122,123) which have not been included in our consideration so far.

interactions are turned off. Via the covariant Wigner transform, Eq.(42) and the technology developed in Eqs.(59-66), one then obtains the corresponding field equations for the complete set of fields $\varphi_m(X_1)$ defined on the D-dimensional configuration space, but now in the presence of these background fields. The physical state condition then reduces them to $(D-2)$-dimensional field equations, again in the presence of these background fields.

Finally, let us describe how the 2T- to 1T- holography and duality properties emerge in this formalism. The reduction from D-dimensional spacetime to $(D-2)$-dimensional one has followed from solving the D-dimensional field equations. The solution can be presented in a variety of ways of embedding $the(D-2)$ dimensions inside the D dimensions [7]. Different embeddings give rise to different $(D-2)$-dimensional 'holographic' viewpoints of the original D-dimensional field equations. In doing so, which one of the two times becomes *the* timelike dimension in the projected $(D-2)$ dimensions? In principle, an infinite number of choices are available, corresponding to the embedding of a timelike curve in the extra dimensions. The choice made by the embedding determines the dynamical evolution of the holographic projection. Each of the $(D-2)$-dimensional dynamics may look different, even though any one of them represents a gauge invariant physical sector of one and the same D-dimensional theory. This implies that, by a different choice of the Sp(2) gauge, different $(D-2)$-dimensional theories in different background fields are obtained and all these theories are transformed one another by local $\text{Sp}(2,R)$ gauge transformations. What we have succeeded in this work is that this property can now be obtained from *the first principles* by formulating the 2T-physics in terms of noncommutative Sp(2) gauge field theories.

6 Outlook

In this paper we have constructed noncommutative field theories with global or local Sp(2) symmetry defined on relativistic phase-space. We believe these theories deserve further investigation, either as a description of 2T-physics from first principles, or with global Sp(2) symmetry in other applications.

We mention some of the immediate questions that come up by the results in this work. First, in noncommutative Sp(2) gauge theories, there is an important issue concerning gauge-invariant operators. It is known that, in the context of noncommutative field theories formulated as deformation quantization over a noncommutative space, part of noncommutative gauge transformation orbit is identifiable with translation along the noncommutative space [18, 19, 20]. It implies that gauge-invariant observables are necessarily nonlocal. A complete set of such observables are identified with open Wilson lines [18, 19, 20]. By a similar argument, in noncommutative Sp(2) gauge theories formulated in this work, part of noncommutative gauge transformation orbit ought to be identifiable with rotation on the relativistic phase-space so that gauge-invariant observables are nonlocal. We expect that open Wilson lines stretched over the relativistic phase-space constitute an important class of such observables. As they are gauge-invariant, from the viewpoint of the two-time physics, expectation value of the open Wilson lines ought to be universally the same for all classes of $(D-2)$-dimensional theories related to one another via the 'holography property'. In view of conceptual importance of the latter, the role of these observables in understanding the '2T- to 1T- holography' could be extremely rewarding.

Second, a complete classification of noncommutative $\mathrm{Sp}(2,R)$ gauge theories underlying the 2T-physics is desirable. We have already shown that a Chern-Simons type action or its variant is a viable route. For this goal, a BRST approach would offer an economic procedure for construction of the actions. For example, analogous to Witten's open string field theory, one can construct a BRST operator

$$\mathcal{Q}_{\mathrm{BRST}} = \left\langle \mathbf{c}, \widehat{\mathbf{Q}} \right\rangle - \langle \mathbf{c}, \mathbf{c} \star \mathbf{b} \rangle \tag{129}$$

$$\equiv \mathbf{c}^{\mu\nu} \widehat{\mathbf{Q}}_{\mu\nu}(X,P) - \frac{i}{2} F_{\mu\nu,\sigma\lambda}{}^{\alpha\beta}\, \mathbf{c}^{\mu\nu} \mathbf{c}^{\sigma\lambda} \mathbf{b}_{\alpha\beta}, \tag{130}$$

where $\mathbf{c}^{\mu\nu}$, $\mathbf{b}_{\mu\nu}$ are the BRST ghosts and anti-ghosts, with ghost charge $Q_{\mathrm{gh}} = -1, +1$, respectively. The ghosts $\mathbf{c}^{\mu\nu}$ and $\mathbf{b}_{\mu\nu}$ represent three independent fermionic degrees of freedom (one may think of them as 3 creation and 3 annihilation operators acting on fermionic Fock space, equivalent to six (8×8) matrices with the same anticommutation properties). There is no need for a definition of star products for the ghosts (although this is possible via the Weyl correspondence applied to fermions). Instead of star products they can be treated as fermionic quantum operators, or Grassmann numbers, keeping track of their orders as usual. We take an action of the purely cubic Chern-Simons type

$$S_{\mathrm{BRST}} = \int d\mu[X, \mathbf{b}, \mathbf{c}]\ \left(\mathcal{Q}_{\mathrm{BRST}} \star \mathcal{Q}_{\mathrm{BRST}} \star \mathcal{Q}_{\mathrm{BRST}} \right), \tag{131}$$

where the star product refers to Moyal product in phase space (X,P) we have used in the rest of the paper. The integration measure $d\mu[X,\mathbf{b},\mathbf{c}] = \left(d^{2D}X\right)(d^3\mathbf{c})\,(d^3\mathbf{b})\,(\mathbf{b}_{11}\mathbf{b}_{12}\mathbf{b}_{22})$ is invariant under Sp(2) and has ghost number +3, cancelling the ghost number -3 of the Lagrangian density (instead of fermionic integrals one may also use a vacuum expectation value in Fock space, or a trace in 8×8 matrix space). Thus the only term in the Lagrangian that survives the integration is the term that contains the Sp(2) invariant ghost factor $\mathbf{c}^{11}\mathbf{c}^{12}\mathbf{c}^{22}$.. Generalizing Eq.(129), one can take the BRST operator $\mathcal{Q}_{\mathrm{BRST}}(X,P,\mathbf{c},\mathbf{b})$ to be the most general ghost number -1 field, containing phase space fields as coefficients in all the allowed terms (which have the form $\mathbf{c}, \mathbf{ccb}, \mathbf{cccbb}$). One may then define a gauge symmetry on these fields that is given by

$$\delta \mathcal{Q}_{\mathrm{BRST}} = i \left[\mathcal{Q}_{\mathrm{BRST}}, \Lambda \right]_\star, \tag{132}$$

where $\Lambda(X,P,\mathbf{c},\mathbf{b})$ is a general gauge function of ghost number zero. Note that, when expanded in powers of ghosts, the ghost independent term in Λ is precisely the local gauge parameter $\omega_L(X,P)$ discussed earlier. The action, Eq.(131), is the direct counterpart of the background-independent, purely cubic action in Witten's open string field theory. Some comparison points include the fact that $\widehat{\mathbf{Q}}_{\mu\nu} = \mathbf{Q}_{\mu\nu} + A_{\mu\nu}$ where $\mathbf{Q}_{\mu\nu} = \frac{1}{2}X_{(\mu} \cdot X_{\nu)}$ is a particular background, while the general $\mathcal{Q}_{\rm BRST}$, as well as the star product, are background independent. The equation of motion is $\mathcal{Q}_{\rm BRST} \star \mathcal{Q}_{\rm BRST} = 0$ and, when an appropriate Λ gauge is chosen, it leads to the fundamental equation $\mathbf{G}^{\mu\nu} = 0$. This was in the absence of matter. One may add matter fields $\Psi, \overline{\Psi}$ containing a linear combination of ghost charges 0,-1,-2, $\Psi = \Psi_0 + \Psi_{-1} + \Psi_{-2}$, with an action that takes the form

$$S_{\rm matter} = \int d\mu[X, \mathbf{b}, \mathbf{c}] \left(\overline{\Psi} \star \mathcal{Q}_{\rm BRST} \star \Psi \right). \tag{133}$$

The terms that survive integration are those that add up to ghost number -3. The field equations that follow from the total action $S_{\rm BRST} + S_{\rm matter}$ are

$$\mathcal{Q}_{\rm BRST} \star \Psi = 0, \qquad \overline{\Psi} \star \mathcal{Q}_{\rm BRST} = 0, \qquad \mathcal{Q}_{\rm BRST} \star \mathcal{Q}_{\rm BRST} = \left(\Psi \star \overline{\Psi} \right)_{-2}, \tag{134}$$

where the subscript -2 implies the sum of terms in the product with total ghost number -2. Thus each matter field $\Psi_0, \Psi_{-1}, \Psi_{-2}$, is annihilated by $\mathcal{Q}_{\rm BRST}$ separately. These equations lead to

$$\mathcal{Q}_{\rm BRST} \star \mathcal{Q}_{\rm BRST} \star \mathcal{Q}_{\rm BRST} = 0, \tag{135}$$

which is similar to the nonlinear relation following from the action in Eq.(120). This now looks like an equation of motion for the gauge fields since it has the form of $\mathcal{Q}_{\rm BRST}$ (i.e. Klein-Gordon type operator) applied on $\mathcal{Q}_{\rm BRST} \star \mathcal{Q}_{\rm BRST}$ (which is like a field strength for the gauge fields).

Given that there are several viable candidate theories, which one would become eventually 'the' proper 2T-physics field theory? A criterion would be that the kinetic term for the gauge fields ought to be produced correctly by the proper theory. As computations involving the Moyal star product are notoriously difficult in the present setting, primarily because they involve derivatives of all orders, identification of the proper theory would take considerable effort. We will report progress on this project elsewhere in a separate paper.

Third, for any given action, further study and a complete classification of the classical solutions in noncommutative Sp(2) gauge theories are needed. As uncovered in the present work, classical solutions correspond to variety of background fields in the holographically projected configuration space. As such, a complete classification of the classical solutions would lead to better understanding of many important issues in 2T-physics as well as 1T-physics, in particular, a consistent formulation of interacting higher-spin field theories [17]. We anticipate that classical solutions with nonzero field strength, $\mathbf{G}^{\mu\nu}(X) \neq 0$, and nonvanishing scalar self-interactions, $V_\star(\Phi\star\Phi^\dagger) \neq 0$, open up new surprises.

Finally, we also expect diverse applications of our formalism and results to the Euclidean noncommutative field theories arising in string theories and M-theory [10, 11, 12] and even to other physics problems than string theories and M-theory.

We will report progress on these issues elsewhere.

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