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Abstract: This article discusses the motion of particles in multiple time dimensions and in multiple space dimensions. Transformations are presented for the transfer from one inertial frame of reference to another inertial frame of reference for the case of multidimensional time. The implications are indicated of the existence of a large number of time dimensions on physical laws like the Lorentz covariance, *CPT* symmetry, the principle of invariance of the speed of light, the law of addition of velocities, the energy-momentum conservation law, etc. The Doppler effect is obtained for the case of multidimensional time. Relations are derived between energy, mass, and momentum of a particle and the number of time dimensions in which the particle is moving. The energy-momentum conservation law is formulated for the case of multidimensional time. It is proven that if certain conditions are met, then particles moving in multidimensional time are as stable as particles moving in one-dimensional time. This result differs from the view generally accepted until now [J. Dorling, Am. J. Phys. **38**, 539 (1970)]. It is proven that luxons may have nonzero rest mass, but only provided that they move in multidimensional time. The causal structure of space-time is examined. It is shown that in multidimensional time, under certain circumstances, a particle can move in the causal region faster than the speed of light in vacuum. In the case of multidimensional time, the application of the proper orthochronous transformations at certain conditions leads to movement backwards in the time dimensions. It is concluded that the number of different antiparticles in the k-dimensional time is equal to $3^k - 2^k$. Differences between tachyons and particles moving in multidimensional time are indicated. It is shown that particles moving faster than the speed of light in vacuum can have a real rest mass (unlike tachyons), provided that they move in multidimensional time. © *2012 Physics Essays Publication*. [DOI: 10.4006/0836-1398-25.3.403]

Résumé: L'article traite du mouvement des particules dans un temps et espace multi-dimensionnels. Les transformations de référentiels inertiels d'un système à l'autre sont déduites dans un temps multidimensionnel. Les conséquences de l'existence d'un plus grand nombre de dimensions temporelles sur les lois physiques sont démontrées: sur l'invariance de Lorentz, sur la symétrie *CPT*, sur le principe de l'invariance de la vitesse de la lumière, sur la loi d'accumulation des vitesses, sur la loi de conservation de l'énergie-impulsion etc. L'effet Doppler est obtenu dans un temps multidimensionnel. Les corrélations entre l'énergie, la masse, l'impulsion d'une particule donnée sont déduites, ainsi que le nombre des dimensions temporelles dans lesquelles cette particule se meut. Formulée à été la loi de conservation de l'énergie-impulsion en cas de temps multidimensionnel. Il est démontré que si ont été satisfaites certaines conditions, les particules qui se déplacent dans un temps multidimensionnel sont tout aussi stables que les particules se déplaçant dans un temps unidimensionnel. Se résultat tranche avec le point de vue adopté jusqu'à présent [J. Dorling, Am. J. Phys. **38**, 539–540 (1970)]. Il est démontré que les luxons peuvent avoir en repos une masse non égale à zéro, mais à la condition qu'ils se meuvent dans un temps multidimensionnel. La structure causale de l'espace-temps est étudiée. Il est démontré que dans un temps multidimensionnel, dans certaines conditions, une particule peut se mouvoir dans le champ causal plus rapidement que la vitesse de la lumière dans du vide. En cas de temps multidimensionnel, l'application des propres transformations orthochrones mène, dans certaines conditions, à une marche en arrière dans la mesure du temps. Nous atteignons la conclusion que le nombre des différentes antiparticules dans un temps k-dimensionnel est égal à $(3^k - 2^k)$. Les différences entre les tachyons et les particules se mouvant dans un temps multidimensionnel sont montrées. Il est démontré que les particules qui se meuvent plus rapidement que la vitesse de la lumière dans du vide peuvent avoir une masse réelle en repos (à la différence des tachyons), mais à la condition qu'elles se meuvent dans un temps multidimensionnel.

Key words: Multidimensional Time; Special Relativity; Mass-Energy Equivalence; Energy-Momentum Conservation Law; Antiparticles; Tachyons; Lorentz Transformations; Invariance of the Speed of Light.

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I. INTRODUCTION

The concept of multidimensional time has been introduced and has become more important in contemporary physical theories.[1–5] According to the inflation theory of the big bang, the visible universe is only a small part of the multiverse, and it is possible that many other universes have emerged in which conditions are entirely different from the conditions of our universe.[6] Up to now, no physical principle or law has been found that determines the possible number of spatial dimensions and of temporal dimensions (or limits the number of spatial and temporal dimensions to a value which differs from the observed number in our universe). Due to this fact, the number of spatial and temporal dimensions in our universe is more probably a result of chance than a result of unknown processes acting during the initial development phases of the universe. Through the anthropic principle it is explained that we live in a universe with more than three dimensions of space (or 10 dimensions, as predicted by M-theory) but only one dimension of time. Therefore, in the other universes that are part of the multiverse it is quite possible that space and time have entirely different numbers of dimensions than the dimensions in our universe. We can assume the existence of universes having two, three, four, or more temporal dimensions. The relation between the anthropic principle and the number of spatial and temporal dimensions is considered by Tegmark.[5] Here we do not discuss this matter.

As shown in some studies,[1,2] it is possible to formulated physically meaningful theories with two time dimensions. Bars noted that "two-time physics could be viewed as a device for gaining a better understanding of one-time physics, but beyond this, two- time physics offers new vistas in the search of the unified theory while raising deep questions about the meaning of spacetime."[2] For systems that are not yet understood or even constructed, such as M-theory, two-time physics points to a possible approach for a more symmetric and more revealing formulation in $11 + 2$ dimensions[a] that can lead to deeper insights, including a better understanding of space and time. The two-time physics approach could be one of the possible avenues to construct the most symmetric version of the fundamental theory.[1,2]

As noted by Tegmark, "Even when $m > 1$, there is no obvious reason why an observer could not, none the less, perceive time as being one-dimensional, thereby maintaining the pattern of having 'thoughts' in a one-dimensional succession that characterizes our own reality perception. If the observer is a localized object, it will travel along an essentially one-dimensional (timelike) world line through the $(n + m)$-dimensional space-time manifold."[5,b] Thus it is fully reasonable to ask the question "What relations, effects, and features would exist if we examined an object moving in multidimensional time?"

In order to find experimental evidence for the existence of particles moving in multidimensional time, it is necessary to know their physical properties. As noted by Recami in another context—the experimental search for the hypothetical particles named tachyons—"it is not possible to make a meaningful experiment without a good theory."[7]

The main objective of this article is to generalize the special theory of relativity (STR) for the cases of multidimensional time and multidimensional space. There is a need to clarify not only the mathematical but also the physical meaning of multidimensional time.

In this respect, the study raises several basic tasks:

- deriving transformations for the transition between inertial frames of reference for the case where the number of time dimensions is greater than one;
- establishing the implications arising from the existence of a large number of dimensions of time on physical laws—the Lorentz covariance, *CPT* symmetry, the constancy of the speed of light, the law of addition of velocities, the energy-momentum conservation law, etc.;
- deriving the Doppler effect for the case of multidimensional time;
- examining the causal structure of space-time;
- deriving formulas for momentum and energy for the case of more than one time dimension;
- establishing the exact relationship between the energy of a particle and the number of time dimensions in which the particle is moving;
- formulating the energy-momentum conservation law;
- considering antiparticles in multidimensional time; and
- distinguishing between tachyons and particles moving in multidimensional time.

The problem with the generalization of STR for the case of multidimensional time is still not sufficiently studied and is only briefly mentioned in different studies concerning the topic. The consequences on physical laws of the existence of multidimensional time have also not been well studied. Up to now there has been no distinction between tachyons and particles moving in multidimensional time.

II. GENERAL CONSIDERATIONS

Important for this study is following question: Are there physical arguments and grounds allowing general conclusions concerning the dimension of time? Related to this question is another: Is Minkowski space-time real and should we accept time as the fourth dimension, given the fact that STR can be equally formulated in a three-dimensional or a four-dimensional language? As noted by Petkov, of course, we have to solve this issue before seriously discussing a theory involving a large number of

[a] Two-time physics introduces one additional space dimension and one additional time dimension.

[b] Here m is the number of time dimensions and n is the number of space dimensions.

dimensions (of space or of time).[8] This question is important because later in this article, additional time dimensions are introduced. The arguments applying to one time dimension will be valid for two, three, or more time dimensions—i.e., they will be applicable also to these cases. It has been shown that the block universe view, regarding the universe as a timelessly existing four-dimensional world, is the only one that is consistent with special relativity.[9] Some arguments have been made in favor of the statement that special relativity alone can resolve the debate on whether the world is three-dimensional or four-dimensional. If the world were three-dimensional, the kinematic consequences of special relativity—and, more importantly, the experiments confirming them—would be impossible.[9,10] Therefore, time is indeed an extra dimension and it is fully justified and reasonable to set and examine the issue of the dimensionality of time.

The interval in Minkowski space-time is an invariant and is given by the expression $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. But instead of the four-dimensional space-time of Minkowski, we can consider a general five-dimensional space-time (dx, dy, dz, cdt, $d\xi$). Here the fifth dimension $d\xi$ reflects the proper time cdt_0 or the proper length dl_0 or is equal to 0. (In Ref. 11 a similar model is considered, but the fifth dimension corresponds to the proper time cdt_0.) By definition, $cdt_0 = \sqrt{ds^2}$ if $ds^2 \geq 0$ and $dl_0 = \sqrt{-ds^2}$ if $ds^2 \leq 0$. Therefore, if $ds^2 > 0$, the fifth dimension $d\xi = cdt_0$ is spacelike; and if $ds^2 < 0$, then $d\xi = dl_0$ is timelike. If $ds^2 = 0$, then $d = 0$ is lightlike. If $ds^2 < 0$, then $d\xi$ (or $d\xi/c$) can be regarded as a second, additional time dimension. It should be noted that the fifth dimension is invariant. If the quantity $d\xi$ is not invariant, then the time is not one-dimensional in the usual sense (see Section IV). (A five-dimensional model of space-time is used in Sections X and XI.)

The number of time dimensions we will denote with k, and that of space dimensions with n. The time dimensions themselves we will denote with $x_1 = ct_1$, $x_2 = ct_2, \ldots, x_k = ct_k$, and the space dimensions with x_{k+1}, $x_{k+2}, \ldots, x_{k+n}$.

The metric signature in the case of k-dimensional time and n-dimensional space will be $(\underbrace{+,+,\ldots,+}_{k},\underbrace{-,-,\ldots,-}_{n})$.[c] Therefore, the $(n + k)$-dimensional interval $ds_{n,k}$ is given by the expression $ds^2_{n,k} = c^2dt_1^2 + \cdots + c^2dt_k^2 - \cdots - dx^2_{k+n}$. It is clear that the $(n+k)$-dimensional interval $ds_{n,k}$ is invariant.

In the case of multidimensional time, the velocities of a particle according to different time dimensions cannot be defined as a set of partial derivatives of the independent variables $t_1, t_2, \ldots, t_k$. Indeed, the movement of a pointlike particle in the general case can be presented as a one-dimensional (timelike) world line in $(k + n)$-dimensional space-time. Let us set $\mathbf{r} = \mathbf{r}(t_1, t_2, \ldots, t_k)$; accordingly, $d\mathbf{r} = (\partial\mathbf{r}/\partial t_1)dt_1 + (\partial\mathbf{r}/\partial t_2)dt_2 + \cdots + (\partial\mathbf{r}/\partial t_k)dt_k$, where $\mathbf{r} = (x_{k+1}, x_{k+2}, \ldots, x_{k+n})$ is the radius vector, defining the position of the particle. Then we have $x_\eta = x_\eta(t_1, t_2, \ldots, t_k)$ and $dx_\eta = (\partial x_\eta/\partial t_1)dt_1 + (\partial x_\eta/\partial t_2)dt_2 + \cdots + (\partial x_\eta/\partial t_k)$, where $\eta = k+1, k+2, \ldots, k+n$. In the general case, each one of the functions $x_\eta = x_\eta(t_1, t_2, \ldots, t_k)$ must be represented by a k-dimensional hypersurface in the $(k + 1)$-dimensional space-time $t_1, t_2, \ldots, t_k, x_\eta$—that is, it will not be presented as a one-dimensional world line. Therefore, the functions $x_\eta = x_\eta(t_1, t_2, \ldots, t_k)$ could not describe the movement of a pointlike particle in the space-time. Each one-dimensional world line in the $(k + 1)$-dimensional space-time $t_1, t_2, \ldots, t_k, x_\eta$ is defined through a system of k equations: $F_{1\eta}(t_1, t_2, \ldots, t_k, x_\eta) = 0$, $F_{2\eta}(t_1, t_2, \ldots, t_k, x_\eta) = 0, \ldots, F_{k\eta}(t_1, t_2, \ldots, t_k, x_\eta) = 0$. (During the uniform and rectilinear movement of the particle, which can be presented through a straight world line, the functions $F_{1\eta}, F_{2\eta}, \ldots, F_{k\eta}$ are linear.) From here we can derive the equalities $x_\eta = f_{1\eta}(t_1) = f_{2\eta}(t_2) = \cdots = f_{k\eta}(t_k)$, where $f_{1\eta}, f_{2\eta}, \ldots, f_{k\eta}$ are different (linear) functions of the variables $t_1, t_2, \ldots, t_k$, respectively. Thus, we have $dx_\eta = f'_{\theta\eta}(t_\theta)dt_\theta$, where $\theta = 1, 2, \ldots, k$. For the case where the particle moves along a straight world line, the derivatives $f'_{1\eta}(t_1), f'_{2\eta}(t_2), \cdots, f'_{k\eta}(t_k)$ are constants which define the velocities of the particle in relation to $t_1, t_2, \ldots, t_k$, respectively. (We will find that $f'_{\theta\eta}(t_\theta) = dx_\eta/dt_\theta = V_{\theta\eta}$; see the considerations at the end of Section II.)

In order to determine the velocities of a given particle for the case of multidimensional time, we are going to use following considerations. Let us consider a particle moving uniformly and rectilinearly with velocity $\mathbf{U}$ in relation to the frame of reference K. Let us assume that at the moment $\mathbf{T} = (t_1, t_2, \ldots, t_k)$, the location of this particle is defined by the radius vector $\mathbf{r} = (x_{k+1}, x_{k+2}, \ldots, x_{k+n})$, and at the moment $\mathbf{T} + d\mathbf{T} = (t_1 + dt_1, t_2 + dt_2, \ldots, t_k + dt_k)$, the location of the particle is defined by the radius vector $\mathbf{r} + d\mathbf{r} = (x_{k+1} + dx_{k+1}, x_{k+2} + dx_{k+2}, \ldots, x_{k+n} + dx_{k+n})$. In the case of multidimensional time we will have $d\mathbf{T} \times \mathbf{U} = d\mathbf{r}$, where $d\mathbf{T} = (dt_1, dt_2, \ldots, dt_k)$ and $d\mathbf{r} = (dx_{k+1}, dx_{k+2}, \ldots, dx_{k+n})$. As can be easily seen, the velocity $\mathbf{U}$ is a $k \times n$ matrix having elements $u_{\theta\eta}$ ($\theta = 1, 2, \ldots, k$; $\eta = k+1, k+2, \ldots, k+n$), i.e., $\mathbf{U} = [u_{\theta\eta}]_{k\times n}$. Let us denote

$$\mathbf{u}_\eta = \begin{pmatrix} u_{1\eta} \\ u_{2\eta} \\ \vdots \\ u_{k\eta} \end{pmatrix},$$

where $\eta = k+1, k+2, \ldots, k+n$. Then we have $d\mathbf{T} \times \mathbf{u}_\eta = dx_\eta$, that is:

$$\frac{dt_1}{dx_\eta}u_{1\eta} + \frac{dt_2}{dx_\eta}u_{2\eta} + \cdots + \frac{dt_k}{dx_\eta}u_{k\eta} = 1. \quad (1)$$

It is clear that if for a given δ ($1 \leq \delta \leq k$) we have $dt_\delta = 0$, then the components of the velocity $u_{\delta(k+1)}$, $u_{\delta(k+2)}, \ldots, u_{\delta(k+n)}$ will be undefined quantities. If for a given ρ ($k + 1 \leq \rho \leq k + n$) we have $dx_\rho = 0$, then it follows that $u_{1\rho} = u_{2\rho} = \cdots = u_{k\rho} = 0$.

[c] For the purpose of this article, we will use the timelike convention for the metric signature (i.e., we choose positive signs for the squares of timelike dimensions and negative signs for the squares of spacelike dimensions).

Let us set $u_{\theta\eta} = \lambda_{\theta\eta}(dx_\eta/dt_\theta)$, where $\sum_{\theta=1}^{k} \lambda_{\theta\eta} = 1$ and $\eta = k+1, k+2, \ldots, k+n$. Let us further denote

$$dT = \sqrt{\sum_{\theta=1}^{k} dt_\theta^2} > 0,$$

$$dX = \sqrt{\sum_{\eta=k+1}^{k+n} dx_\eta^2} > 0,$$

$$dt_\theta = \alpha_\theta dT,$$

$$dx_\eta = \chi_\eta dX.$$

Then we have $\sum_{\theta=1}^{k} \alpha_\theta^2 = 1$—that is, $|\alpha_\theta| \leq 1$—and $\sum_{\eta=k+1}^{k+n} \chi_\eta^2 = 1$—that is, $|\chi_\eta| \leq 1$.

Let us set

$$U_\eta = \sqrt{\sum_{\theta=1}^{k} u_{\theta\eta}^2} = \frac{|dx_\eta|}{dT}\sqrt{\sum_{\theta=1}^{k}\left(\frac{\lambda_{\theta\eta}}{\alpha_\theta}\right)^2},$$

$$U_\theta = \sqrt{\sum_{\eta=k+1}^{k+n} u_{\theta\eta}^2} = \frac{dX}{|dt_\theta|}\sqrt{\sum_{\eta=k+1}^{k+n}(\lambda_{\theta\eta}\chi_\eta)^2},$$

$$U = \sqrt{\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k} u_{\theta\eta}^2}.$$

For the velocity U we have

$$U = \sqrt{\sum_{\eta=k+1}^{k+n} U_\eta^2} = \sqrt{\sum_{\theta=1}^{k} U_\theta^2} = \frac{dX}{dT}\sqrt{\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}\left(\frac{\lambda_{\theta\eta}\chi_\eta}{\alpha_\theta}\right)^2}. \tag{2}$$

We will show the conditions for the velocities U_η, U_θ, and U. These conditions are imposed due to physical considerations (see also Subsection IX.A). When the time-axes basis of the frame K is changed (these are the so-called passive linear transformations) and the value of dT^2 is still the same, then the quantity $dX^2 = c^2dT^2 - ds_{n,k}^2$ remains the same as well. Likewise, when the basis of the space axes of the frame K is changed, then the values dX^2 and $c^2dT^2 = ds_{n,k}^2 + dX^2$ remain the same. It is clear that the choice of time axes of the frame K can be made independently from the choice of space axes of the frame K, and vice versa (see also Section VII and Subsection IX.A). Thus, when these transformations are applied, some physical quantities (in our case the velocities) must remain invariant.

Since in the expression for the velocity U_η all the time axes of the frame K are represented in the denominator ($dT = \sqrt{\sum_{\theta=1}^{k} dt_\theta^2}$), we accept that the velocity U_η is invariant when a change of the time-axes basis of the frame K is made. Thus, the velocity U_η can be presented as $U_\eta = \gamma_\eta(|dx_\eta|/dT)$. Here $\gamma_\eta > 0$ is a parameter not depending on the quantities dt_θ, respectively on the numbers α_θ ($\theta = 1, 2, \ldots, k$).

Since in the expression for the velocity U_θ all the space axes of the frame K are represented in the numerator ($dX = \sqrt{\sum_{\eta=k+1}^{k+n} dx_\eta^2}$), we accept that the velocity U_θ remains invariant when the space-axes basis of the frame K is changed. Thus, the velocity U_θ can be presented as $U_\theta = \gamma_\theta(dX/|dt_\theta|)$. Here $\gamma_\theta > 0$ is a parameter not depending on the quantities dx_η and consequently on the numbers χ_η ($\eta = k+1, k+2, \ldots, k+n$).

From these considerations we can conclude that the velocity U is invariant when the space-axes basis and the time-axes basis of the frame K are changed—i.e., the velocity U can be presented as $U = \gamma(dX/dT)$. Here $\gamma > 0$ is a parameter not depending on the values dt_θ and dx_η and consequently on the numbers α_θ and χ_η [$\theta = 1, 2, \ldots, k$; $\eta = k+1, k+2, \ldots, k+n$; see Eq. (2)]. Indeed,

$$U = \sqrt{\sum_{\eta=k+1}^{k+n} U_\eta^2} = \frac{dX}{dT}\sqrt{\sum_{\eta=k+1}^{k+n}\chi_\eta^2\gamma_\eta^2} = \sqrt{\sum_{\theta=1}^{k} U_\theta^2} = \frac{dX}{dT}\sqrt{\sum_{\theta=1}^{k}\frac{\gamma_\theta^2}{\alpha_\theta^2}}.$$

Let us set $\gamma = \sqrt{\sum_{\eta=k+1}^{k+n}\chi_\eta^2\gamma_\eta^2} = \sqrt{\sum_{\theta=1}^{k}(\gamma_\theta^2/\alpha_\theta^2)}$. Since the values χ_η and γ_η do not depend on the numbers α_θ, neither does the parameter $\gamma = \sqrt{\sum_{\eta=k+1}^{k+n}\chi_\eta^2\gamma_\eta^2}$. Since the values α_θ and γ_θ do not depend on the numbers χ_η, neither does the parameter $\gamma = \sqrt{\sum_{\theta=1}^{k}(\gamma_\theta^2/\alpha_\theta^2)}$. Thus, the parameter γ does not depend on α_θ or χ_η, and consequently does not depend on the values dt_θ or dx_η.

Let us assume that $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1/\sqrt{k}$ (that is, $dt_1 = dt_2 = \cdots = dt_k = dT/\sqrt{k} > 0$) and $\chi_{k+1} = \chi_{k+2} = \cdots = \chi_{k+n} = 1/\sqrt{n}$ (that is, $dx_{k+1} = dx_{k+2} = \cdots = dx_{k+n} = dX/\sqrt{n} > 0$). Let us apply a proper or improper rotation of the time axes of the frame K in the hyperplane of time. The transformation under consideration can be presented through the orthogonal matrix $\mathbf{A} = [a_{\theta\varsigma}]_{k\times k}$, belonging to the orthogonal group $\mathrm{O}(k, \mathbb{R})$, where $\mathbb{R}$ denotes the real numbers field (see also Section VII). (We have $\mathbf{A}^{\mathrm{tr}} = \mathbf{A}^{-1}$, $\det(\mathbf{A}) = \pm 1$. Here $\mathbf{A}^{\mathrm{tr}}$ denotes the transpose of the matrix $\mathbf{A}$.) The new time axes, obtained after applying this transformation, we will denote with t'_θ, ($\theta = 1, 2, \ldots, k$). Since $\sum_{\theta=1}^{k}(\alpha'_\theta)^2 = \sum_{\theta=1}^{k}\alpha_\theta^2 = 1$, we have

$$\begin{pmatrix}\alpha'_1\\ \alpha'_2\\ \vdots\\ \alpha'_k\end{pmatrix} = \mathbf{A}\times\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_k\end{pmatrix},$$

that is,

$$\alpha'_\theta = \sum_{\varsigma=1}^{k} a_{\theta\varsigma}\alpha_\varsigma = \frac{1}{\sqrt{k}}\sum_{\varsigma=1}^{k} a_{\theta\varsigma}. \tag{3}$$

Since the velocity U_η remains invariant when the time-axes basis is changed for the frame K, the expression $\sum_{\theta=1}^{k}(\lambda_{\theta\eta}/\alpha_\theta)^2$ is constant when there is a change in the values of α_θ—that is,

$$\sum_{\theta=1}^{k}\left(\frac{\lambda'_{\theta\eta}}{\alpha'_\theta}\right)^2 = \sum_{\theta=1}^{k}\left(\frac{\lambda_{\theta\eta}}{\alpha_\theta}\right)^2 = k\sum_{\theta=1}^{k}\lambda^2_{\theta\eta} = \gamma^2_\eta, \tag{4}$$

where γ_η is a parameter not depending on the numbers α_θ ($\theta = 1, 2, \ldots, k$).

Let us consider the orthogonal matrix $\mathbf{B} = [b_{\theta v}]_{k\times k}$, belonging to the orthogonal group $O(k, \mathbb{R})$. Taking into account Eq. (4), we likewise have

$$\frac{\lambda'_{\theta\eta}}{\alpha'_\theta} = \sum_{v=1}^{k}\frac{b_{\theta v}\lambda_{v\eta}}{\alpha_v} = \sqrt{k}\sum_{v=1}^{k} b_{\theta v}\lambda_{v\eta}. \tag{5}$$

From Eqs. (3) and (5) we can define the values $\lambda'_{\theta\eta}$:

$$\lambda'_{\theta\eta} = \sum_{v=1}^{k}\sum_{\varsigma=1}^{k}\frac{a_{\theta\varsigma}\alpha_\varsigma b_{\theta v}\lambda_{v\eta}}{\alpha_v} = \sum_{v=1}^{k}\sum_{\varsigma=1}^{k} a_{\theta\varsigma}b_{\theta v}\lambda_{v\eta}. \tag{6}$$

Let us apply a proper or improper rotation of the space axes of the frame K in the hyperplane of space. The transformation under consideration can be presented through the orthogonal matrix $\mathbf{H} = [h_{\pi\eta}]_{n\times n}$, belonging to the orthogonal group $O(n, \mathbb{R})$. Here $\eta = k+1, k+2, \ldots, k+n$; $\pi = k+1, k+2, \ldots, k+n$. The new space axes, obtained after applying this transformation, we will denote with x''_η ($\eta = k+1, k+2, \ldots, k+n$). Since $\sum_{\eta=k+1}^{k+n}(\chi''_\eta)^2 = \sum_{\eta=k+1}^{k+n}\chi^2_\eta = 1$, we have $(\chi''_{k+1}, \chi''_{k+2}, \cdots \chi''_{k+n} = \chi_{k+1}, \chi_{k+2}, \cdots, \chi_{k+n}) \times \mathbf{H}$—that is,

$$\chi''_\eta = \sum_{\pi=k+1}^{k+n}\chi_\pi h_{\pi\eta} = \frac{1}{\sqrt{n}}\sum_{\pi=k+1}^{k+n} h_{\pi\eta}. \tag{7}$$

Since the velocity U_θ is invariant when the space-axes basis of K is changed, the expression $\sum_{\eta=k+1}^{k+n}(\lambda'_{\theta\eta}\chi_\eta)^2$ remains constant when the values of χ_η are changed—that is,

$$\sum_{\eta=k+1}^{k+h}(\lambda''_{\theta\eta}\chi''_\eta)^2 = \sum_{\eta=k+1}^{k+n}(\lambda'_{\theta\eta}\chi_\eta)^2 = \frac{1}{n}\sum_{\eta=k+1}^{k+n}\lambda'_{\theta\eta} = \gamma^2_\theta. \tag{8}$$

Here γ_θ is a parameter not depending on χ_η ($\eta = k+1, k+2, \ldots, k+n$).

Let us consider the orthogonal matrix $\mathbf{Q} = [q_{\rho\eta}]_{n\times n}$, belonging to the orthogonal group $O(n, \mathbb{R})$. Taking into account Eq. (8), we likewise have

$$\lambda''_{\theta\eta}\chi''_\eta = \sum_{\rho=k+1}^{k+n}\lambda'_{\theta\rho}\chi_\rho q_{\rho\eta} = \frac{1}{\sqrt{n}}\sum_{\rho=k+1}^{k+n}\lambda'_{\theta\rho}q_{\rho\eta}. \tag{9}$$

From Eqs. (6), (7), and (9) we can define the values $\lambda''_{\theta\eta}$:

$$\begin{aligned}\lambda''_{\theta\eta} &= \frac{1}{\sum_{\pi=k+1}^{k+n}\chi_\pi h_{\pi\eta}}\left(\sum_{\rho=k+1}^{k+n}\sum_{v=1}^{k}\sum_{\varsigma=1}^{k}\frac{a_{\theta\varsigma}\alpha_\varsigma b_{\theta v}\lambda_{v\rho}\chi_\rho q_{\rho\eta}}{\alpha_v}\right)\\ &= \frac{\sum_{\rho=k+1}^{k+n}\sum_{v=1}^{k}\sum_{\varsigma=1}^{k} a_{\theta\varsigma}b_{\theta v}\lambda_{v\rho}q_{\rho\eta}}{\sum_{\pi=k+1}^{k+n} h_{\pi\eta}}.\end{aligned} \tag{10}$$

Here the values $\lambda_{v\rho}$ are defined provided that $\alpha_\theta = 1/\sqrt{k}$, $\chi_\eta = 1/\sqrt{n}$ are fulfilled. Since the parameter γ does not depend of the numbers α_θ or χ_η, we can set $\alpha_\theta = 1/\sqrt{k}$, $\chi_\eta = 1/\sqrt{n}$. In this case, we have

$$\gamma = \sqrt{\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}\left(\frac{\lambda_{\theta\eta}\chi_\eta}{\alpha_\theta}\right)^2} = \sqrt{\frac{k}{n}}\sqrt{\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}\lambda^2_{\theta\eta}}. \tag{11}$$

[See Eq. (2).]

Let us set $\Lambda''_\eta = \sum_{\theta=1}^{k}\lambda''_{\theta\eta}$, $\eta = k+1$, $k+2$, $k+n$ [see Eq. (10)]. In the general case (i.e., at arbitrary values of α'_θ and χ''_η), we will have

$$d\mathbf{T} \times \mathbf{U}^{\Lambda''} = d\mathbf{r},$$

where

$$\mathbf{U}^{\Lambda''} = \left[u^{\Lambda''}_{\theta\eta}\right]_{k\times n}, u^{\Lambda''}_{\theta\eta} = \frac{\lambda''_{\theta\eta}\chi''_\eta dX}{\Lambda''_\eta\alpha'_\theta dT}.$$

[See Eqs. (1), (3), (7), and (10).] If $\sum_{\theta=1}^{k}\lambda''_{\theta\eta} \neq \sum_{\theta=1}^{k}\lambda_{\theta\eta}$ (that is, $\Lambda''_\eta \neq 1$, $\eta = k+1, k+2, \cdots, k+n$), then

$$u^{\Lambda''}_{\theta\eta} = \frac{\lambda''_{\theta\eta}\chi''_\eta dX}{\Lambda''_\eta\alpha'_\theta dT} \neq u''_{\theta\eta} = \frac{\lambda''_{\theta\eta}\chi''_\eta dX}{\alpha'_\theta dT}.$$

Let us denote $dx_\eta/dt_\theta = u_{\theta\eta}/\lambda_{\theta\eta} = V_{\theta\eta}$. It is clear that $dx_\eta = V_{1\eta}\, dt_1 = V_{2\eta}\, dt_2 = \cdots = V_{k\eta}\, dt_k$. Let us denote $\mathbf{V}_\theta = (d\mathbf{r}/dt_\theta)$, where $d\mathbf{r} = (dx_{k+1}, dx_{k+2}, \ldots, dx_{k+n})$. Then we have $\mathbf{V}_\theta = [V_{\theta(k+1)}, V_{\theta(k+2)}, \ldots, V_{\theta(k+n)}]$. We will say that $\mathbf{V}_\theta$ is the velocity of the particle under consideration in relation to the frame K, defined in relation to the time dimension t_θ. Let us set $V_\theta = ||\mathbf{V}_\theta|| = dX/|dt_\theta| = U_\theta/\gamma_\theta > 0$. Then the equation $|dt_\theta|/|dt_\varsigma| = V_\varsigma/V_\theta$ is fulfilled, where θ, $\varsigma = 1, 2, \ldots, k$ and $dX \neq 0$.

Let us set $u = dX/dT = U/\gamma$. We will say that u is the total coordinate velocity of the considered particle in relation to K. It is easy to see that

$$u = \frac{1}{\sqrt{\sum_{\theta=1}^{k}\left(\frac{1}{\sum_{\eta=1}^{n}V^2_{\theta\eta}}\right)}} = \frac{1}{\sqrt{\sum_{\theta=1}^{k}\frac{1}{V^2_\theta}}}.$$

We will say that a particle is at rest relative to the frame of reference if $\sum_{\eta=k+1}^{k+n}dx^2_\eta = 0$ and $V_\theta = 0$ ($\theta = 1, 2, \ldots, k$).

We assume that all time dimensions we are going to consider—t_1, t_2, …, t_k (or t, τ, . . .)—are homogeneous (i.e., all moments of a given time dimension are equal), and that space is homogeneous and isotropic (i.e., all points and all directions of the space are equal).

III. TRANSFORMATIONS FROM ONE INERTIAL FRAME OF REFERENCE TO ANOTHER FOR THE CASE OF (*n*, *k*) = (3, 2)

A. Derivation of the transformations

First, we will consider case with $(n, k) = (3, 2)$, i.e., three space dimensions and two time dimensions. The three space dimensions we will denote with x, y, z. The first time dimension we will denote with t, and the second with τ.

The interval in the five-dimensional space-time under consideration is given by the expression $ds_{3,2}^2 = c^2dt^2 + c^2d\tau^2 - dx^2 - dy^2 - dz^2$. The interval $ds_{3,2}$ is invariant.

Let us consider the two inertial frames K and K' (moving uniformly and rectilinearly to each other). We assume that the velocity of the frame K' against K, defined in relation to the first time dimension t, is equal to the vector $\mathbf{v}$, and that defined in relation to the second time dimension τ it is equal to the vector $\mathbf{w}$ (see Section II).

Let us denote with x, y, and z the axes of the frame K, and with x', y', and z' the axes of the frame K'. The two time dimensions defined in the frame K we will denote with t and τ; in the frame K', with t' and τ'. Let us denote with point Q the origin of the spatial frame of reference K (that is, $x = 0$, $y = 0$, $z = 0$), and with point Q' the origin of the spatial frame of reference K' (that is, $x' = 0$, $y' = 0$, $z' = 0$). We choose the frames K and K' in such a way that point Q' is moving along the axis x in the direction of increasing values of x. Further, we can choose the axes of the frames K and K' in such a way that for an observer connected to K, the axis x coincides with the axis x'; the axes y and z are parallel to the axes y' and z', respectively; and the homonymous axes have the same direction. As the initial moment we accept (t, t'), (τ, τ'), where point Q' coincides with point Q (i.e., at the moments $t = t' = 0$ and $\tau = \tau' = 0$, point $Q' \equiv$ point Q). Having all these conditions fulfilled, we can say for K and K' that they are in a standard configuration. Let us set $\mathbf{v} = (v, 0, 0)$ and $\mathbf{w} = (w, 0, 0)$, where $(v, 0, 0)$ are the respective projections of the velocity vector $\mathbf{v}$ on the axes x, y, z of the frame K and $(w, 0, 0)$ are the respective projections of the velocity vector $\mathbf{w}$ on the axes x, y, z of K. Let us assume that a particle has coordinates (t, τ, x, y, z) in K and (t', τ', x', y', z') in K'.

Let us denote $x_1 = ict$, $x_2 = ic\tau$, $x_3 = x$, $x_4 = y$, $x_5 = z$ and $x_1' = ict'$, $x_2' = ic\tau'$, $x_3' = x'$, $x_4' = y'$, $x_5' = z'$. In order to derive the transformations between K and K', we will use the same approach as for the Lorentz transformations in a more general case, the so-called Lorentz boost in an arbitrary direction (transformations between two inertial frames of reference whose x, y, z axes are parallel and whose space-time origins coincide, i.e., Lorentz transformations with no rotation)—see, for example, Ref. 12. First we will consider a proper rotation in the plane x_1-x_2 through angle α, where the other three dimensions (x_3, x_4, x_5) remain invariant. This transformation is described by the matrix

$$\mathbf{R} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us denote with x_{1R} and x_{2R} the new axes which arise from the rotation $\mathbf{R}$ of the axes x_1 and x_2, respectively. Then we will consider a proper rotation in the plane x_{1R}-x_3 through angle φ, where the other three dimensions (x_{2R}, x_4, x_5) remain invariant. This transformation is described by the matrix

$$\mathbf{L} = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order to derive the transformations between K and K', we will consecutively apply the operations $\mathbf{R}$, $\mathbf{L}$, and $\mathbf{R}^{-1}$.

First, we will apply the $\mathbf{R}$ transformation. As can be easily seen, $\tan\alpha = x_2/x_1$. If $x_3' = 0$, then $x_3 = -i(v/c)x_1 = -i(w/c)x_2 > 0$ and thus $\tan\alpha = v/w$. Here the angle α is a real number. Let us set $\beta = 1/\sqrt{(c^2/v^2) + (c^2/w^2)}$, $\zeta = 1/\sqrt{1 - \beta^2}$. (We will have $0 \leq \beta \leq 1$ and $\zeta \geq 1$; see also Section IV.) We have $\cos\alpha = c\beta/v$, $\sin\alpha = c\beta/w$.

It is clear that if $v = 0$, then $w = 0$, and vice versa. We will consider more specific cases in relation to the value of α. Let us assume that $\alpha = b\pi$, where $b = 0, \pm 1, \pm 2, \ldots$ (i.e., the motion occurs only along the axis $x_1 = ict$). It is clear that if $v \neq 0$, then $w = \pm\infty$. If we assume that $\alpha = \pi/2 + b\pi$ (i.e., the motion occurs only along the axis $x_2 = ic\tau$), it is clear that if $w \neq 0$, then $v = \pm\infty$.

We have accepted that point Q' is moving in the direction of increasing values of x, which means that if $x' = 0$, then $x = vt = w\tau > 0$. We have the following: If $\alpha \in [0; \pi/2]$, then $t \geq 0$ and $\tau \geq 0$, and therefore $v > 0$ and $w > 0$; if $\alpha \in (\pi/2; \pi]$, then $t < 0$ and $\tau \geq 0$, and therefore $v < 0$ and $w > 0$; if $\alpha \in (\pi; 3\pi/2)$, then $t < 0$ and $\tau < 0$, and therefore $v < 0$ and $w < 0$; if $\alpha \in [3\pi/2; 2\pi)$, then $t \geq 0$ and $\tau < 0$, and therefore $v > 0$ and $w < 0$.

Let us now apply the transformation $\mathbf{L}$. It can be easily seen that $\tan\varphi = x_3/x_{1R} = x_3/\sqrt{x_1^2 + x_2^2}$. Further, if $x_3' = 0$, then $x_3 = -i(v/c)x_1 = -i(w/c)x_2 > 0$. Therefore, we have $\tan\varphi = -i\beta$. Here the angle φ is an imaginary number. We have $\sin\varphi = -i\beta\zeta$, $\cos\varphi = \zeta$. The signs in these expressions are chosen so that when $v \to 0$ and $w \to 0$, we have $t' \to t$, $\tau' \to \tau$,$x' \to x$, $y' \to y$, $z' \to z$.

Finally, let us apply the transformation

$$\mathbf{R}^{-1} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which is the inverse of the first transformation $\mathbf{R}$. The matrix for the transfer from the coordinates x_1, x_2, x_3, x_4, x_5 to x'_1,x'_2, x'_3, x'_4, x'_5 is equal to the product

$$\mathbf{R}^{-1}\times\mathbf{L}\times\mathbf{R} = \begin{pmatrix} 1+(\cos\varphi-1)\cos^2\alpha & (\cos\varphi-1)\sin\alpha\cos\alpha & \cos\alpha\sin\varphi & 0 & 0 \\ (\cos\varphi-1)\sin\alpha\cos\alpha & 1+(\cos\varphi-1)\sin^2\alpha & \sin\alpha\sin\varphi & 0 & 0 \\ -\cos\alpha\sin\varphi & -\sin\alpha\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us express $\sin\alpha$, $\cos\alpha$, $\sin\varphi$, $\cos\varphi$ through v/c and w/c and go back to the old coordinates t, τ, x, y, z, t', τ', x', y', z'. Then finally we obtain the following expression:

$$\begin{pmatrix} t' \\ \tau' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1+(\zeta-1)\dfrac{c^2}{v^2}\beta^2 & (\zeta-1)\dfrac{c^2}{vw}\beta^2 & -\dfrac{1}{v}\beta^2\zeta & 0 & 0 \\ (\zeta-1)\dfrac{c^2}{vw}\beta^2 & 1+(\zeta-1)\dfrac{c^2}{w^2}\beta^2 & -\dfrac{1}{w}\beta^2\zeta & 0 & 0 \\ -\dfrac{c^2}{v}\beta^2\zeta & -\dfrac{c^2}{w}\beta^2\zeta & \zeta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ \tau \\ x \\ y \\ z \end{pmatrix}. \tag{12}$$

We can easily prove that if $v=0$ and $w=0$, then $t'=t$, $\tau'=\tau$, $x'=x$, $y'=y$, $z'=z$. The transformations in Eq. (12) belong to the group of proper orthochronous transformations, which we will denote with $\Lambda_+^{\uparrow\rightarrow}$ (see Section III.C). The transformations in Eq. (12) are equivalent to the Lorentz transformations as $\tau\to 0$ and accordingly as $w\to\pm\infty$.

Let us set $\varphi=i\Phi$. Taking into account the fact that $\cos\varphi=\cos(i\Phi)=\cosh\Phi$ and $\sin\varphi=\sin(i\Phi)=i\sinh\Phi$, we obtain $\cosh\Phi=\zeta$ and $\sinh\Phi=-\beta\zeta$.

In the plane of time t'-τ', one can apply a proper or an improper rotation. These transformations present change of the time-axes basis of the frame K (the so-called passive linear transformation). The orthogonal matrix $\mathbf{M}$ for the transfer from the old coordinates t', τ', x', y', z' to the new coordinates t''_1, t''_2, x'', y'', z'', is

$$\mathbf{M} = \begin{pmatrix} \cos\sigma & \sin\sigma & 0 & 0 & 0 \\ -\varepsilon\sin\sigma & \varepsilon\cos\sigma & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\varepsilon=+1$ (proper rotation) or $\varepsilon=-1$ (improper rotation, rotary reflection). The following equations are fulfilled: $(t''_1)^2+(t''_2)^2=(t')^2+(\tau')^2$, $x''=x'$, $y''=y'$, $z''=z'$. It is clear that for these transformations, the five-dimensional interval $ds^2_{3,2}$ is invariant. If a proper or improper rotation is applied in the plane of time t'-τ', the transformation $\mathbf{D}$ between $K(t, \tau, x, y, z)$ and $K''(t''_1, t''_2, x'', y'', z'')$ is obtained by multiplying the matrix $\mathbf{\Lambda}$ from Eq. (12) for the transfer from K to K' by the matrix $\mathbf{M}$; that is, $\mathbf{D}=\mathbf{M}\times\mathbf{\Lambda}$.

Let us apply a (proper or improper) rotation in the plane t'-τ'. We obtain the equation $(dt''_1)^2+(dt''_2)^2=(dt')^2+(d\tau')^2$. Here $dt''_1=dt'\cos\sigma+d\tau'\sin\sigma$, $dt''_2=-\varepsilon dt'\sin\sigma+\varepsilon d\tau'\cos\sigma$, where $\varepsilon=\pm1$. It is easy to prove that if $dt''_1=dt''_2>0$, then $\sigma=\alpha'-\varepsilon(\pi/4)+2b\pi$, where $\tan\alpha'=d\tau'/dt'$, $b=0, \pm1, \pm2, \ldots$ Since the vector $d\mathbf{T}'=(dt', d\tau')$ makes an angle α' with the axis t', the axis t''_1 makes an angle σ with the axis t', and the axis t''_2 makes an angle $\sigma+\varepsilon(\pi/2)$ with the axis t', the vector $d\mathbf{T}'$ makes an angle $\alpha'-\sigma=\varepsilon(\pi/4)-2b\pi$ with the axis t''_1 and an angle $\alpha'-[\sigma+\varepsilon(\pi/2)]=-\varepsilon(\pi/4)-2b\pi$ with the axis t''_2. The size of the angle between the vector $d\mathbf{T}'$ and each of the axes t''_1 and t''_2 is equal to $\pi/4$ (see Section VII).

Let us consider two events which are causally connected, i.e., for which $\Delta s^2_{3,2}=c^2\Delta t^2+c^2\Delta\tau^2-\Delta x^2-\Delta y^2-\Delta z^2\geq 0$ (see Section IV). From Eq. (12) it is seen that if $\Delta t=0$,$\Delta x=0$,$\Delta y=0$,$\Delta z=0$, then $\Delta t'=(\zeta-1)(c^2/vw)\beta^2\Delta\tau$ and $\Delta x'=-(c^2/w)\beta^2\zeta\Delta\tau$. Thus, if two events are causally connected and if in an inertial reference frame K the coordinates of these events, defined according to x, y, z, t, coincide, then in another inertial frame K' the coordinates of the events according to x', y', z', t' cannot coincide. In this case it is possible that $\Delta t'\neq 0$ and thus $\Delta x'\neq 0$ (provided that $\Delta\tau\neq 0$ and $w\neq\pm\infty$).

B. Backwards motion in the two-dimensional time

In STR, the condition $s^2 = c^2t^2 - x^2 - y^2 - z^2 > 0$, $t > 0$ cannot be changed through proper orthochronous Lorentz transformations $L_+^{\uparrow}$ into the condition $s^2 > 0$, $t < 0$. If it could be done, then on account of continuity one could also have $t = 0$—which is impossible if $s^2 > 0$. For this reason, the region $s^2 > 0$, $t > 0$ (inside of the positive light cone) is called the absolute future. Similarly, the region $s^2 > 0$, $t < 0$ (inside of the negative light cone) is called the absolute past relative to $t = 0$. These considerations are not valid for the case of multidimensional time. For example, for the case $k = 2$, the condition $s_{3,2}^2 = c^2t^2 + c^2\tau^2 - x^2 - y^2 - z^2 > 0$, $t > 0$, $\tau > 0$ can be changed through proper orthochronous transformations $\Lambda_+^{\uparrow\rightarrow}$ (see Subsection III.C) into the condition $s_{3,2}^2 > 0$, $t \leq 0$, $\tau \leq 0$, the condition $s_{3,2}^2 > 0$, $t \leq 0$, $\tau \geq 0$, or the condition $s_{3,2}^2 > 0$, $t \geq 0$, $\tau \leq 0$. Indeed, since t and τ are independent variables, it is possible that following conditions are simultaneously fulfilled: $t = 0$, $\tau \neq 0$, $c^2\tau^2 - x^2 - y^2 - z^2 > 0$. Thus the equality $t = 0$ does not contradict the inequality $s_{3,2}^2 > 0$. Due to the same considerations, the equality $\tau = 0$ also does not contradict the inequality $s_{3,2}^2 > 0$.

The application of the transformations in Eq. (12), which belong to the group of the proper orthochronous transformations $\Lambda_+^{\uparrow\rightarrow}$, at certain conditions leads to movement backward in the time dimensions t and τ. Let us assume $s_{3,2}^2 = c^2\Delta t^2 + c^2\Delta\tau^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \geq 0$. We accept that $\Delta x > 0$, $\Delta y = 0, \Delta z = 0$, $\Delta \mathrm{t} > 0, \Delta\tau \geq 0$. According to Eq. (12), following equality will be fulfilled:

$$\Delta t' = \left[1 + (\zeta - 1)\frac{c^2}{v^2}\beta^2\right]\Delta t + (\zeta - 1)\frac{c^2}{vw}\beta^2\Delta\tau - \frac{1}{v}\beta^2\zeta\Delta x. \tag{13}$$

We assume that $v > 0$. Further, we will examine for which values of the velocities v and w the condition $\Delta t' < 0$ is fulfilled.

First, we will assume, that $\Delta\tau = 0$. Since $\Delta s_{3,2}^2 \geq 0$, we will have $\Delta x \leq c\Delta t$. Let us set $\Delta x = c\Delta t$. In this case the inequality $\Delta t' < 0$ is equivalent to the inequality

$$\frac{\Delta t'}{\Delta t} = 1 + (\zeta - 1)\frac{c^2}{v^2}\beta^2 - \frac{c}{v}\beta^2\zeta < 0. \tag{14}$$

If we set $r = c/v$, then we obtain a quadratic inequality in relation to the parameter r. Since in the expressions for β and ζ there are two independent variables c/v and c/w, for which the only restriction imposed is $(c/v)^2 + (c/w)^2 \geq 1$ (see Section IV), we can set $(c/v)^2 + (c/w)^2 = \text{const} \geq 1$ and therefore $\beta = \text{const} \leq 1$ and $\zeta = \text{const} \geq 1$. Further, we can find the first and second derivatives of the function $f(r) = (\zeta - 1)\beta^2r^2 - \beta^2\zeta r + 1$. Then we discover that the function $f(r)$ has a minimum at $r = \zeta/2(\zeta - 1)$ and $\zeta > 1$. This means that at a fixed value of $1 \geq \beta > 0$ (and accordingly of $\zeta > 1$), the function $f(r)$ reaches its minimum at $r = \zeta/2(\zeta - 1)$. Let us set $r = c/v = \zeta/2(\zeta - 1)$ and $\zeta > 1$. According to the inequality in Eq. (14), we have

$$-\beta^2\zeta^2 + 4\zeta - 4 < 0. \tag{15}$$

Taking into consideration that $\zeta = 1/\sqrt{1 - \beta^2}$, we find that at $\beta \to 1$, the expression in Eq. (15) tends toward $-\infty$. Therefore, if $r = c/v = \zeta/2(\zeta - 1)$ and $\beta \to 1$ (that is, $\zeta \to \infty$ and $r \to 1/2$), then $\Delta t'/\Delta t \to -\infty$. Further, if $\beta = (2\sqrt{2})/3$ (that is, $\zeta = 3$ and $r = 3/4$), then the expression in Eq. (15) becomes equal to 0. It is easy to find that if $r \in [1/2; 3/4)$, then Eq. (14) is fulfilled and therefore $\Delta t'/\Delta t < 0$, which we wanted to prove. If $r = 3/4$, then $\Delta t'/\Delta t = 0$—that is, $\Delta t' = 0$—and if $r \to 1/2$, then $\Delta t'/\Delta t \to -\infty$—that is, $\Delta t' \to -\infty$.

According to Eq. (12), the following equality will be valid:

$$\Delta\tau' = (\zeta - 1)\frac{c^2}{vw}\beta^2\Delta t + \left[1 + (\zeta - 1)\frac{c^2}{w^2}\beta^2\right]\Delta\tau - \frac{1}{w}\beta^2\zeta\Delta x. \tag{16}$$

Since $\Delta\tau = 0$ and $\Delta x = c\Delta t$, we have

$$\Delta\tau' = \left[(\zeta - 1)\frac{c^2}{vw}\beta^2 - \frac{c}{w}\beta^2\zeta\right]\Delta t.$$

Let us assume that $w > 0$. It is easy to prove that if $c/v = \zeta/2(\zeta - 1)$, then $(\zeta - 1)(c^2/vw)\beta^2 - (c/w)\beta^2\zeta < 0$. Therefore, in this case we have $\Delta\tau' < 0$. If $\beta \to 1$ and accordingly $\zeta \to \infty$, $r \to 1/2$, then $\Delta\tau'/\Delta t \to -\infty$—that is, $\Delta\tau' \to -\infty$.

Let us now assume that $\Delta\tau > 0$. In this case $\Delta t'$ is given by Eq. (13) and $\Delta\tau'$ is given by Eq. (16). Further, from the condition $\Delta s_{3,2}^2 \geq 0$ (and $\Delta y = 0$, $\Delta z = 0$) it follows that $\Delta x \leq c\sqrt{\Delta t^2 + \Delta\tau^2}$. Let us set $\Delta x = \chi c\sqrt{\Delta t^2 + \Delta\tau^2}$, where $0 \leq \chi \leq 1$. We are going to examine for which values of the velocities v and w the conditions $\Delta t' < 0$ and $\Delta\tau' < 0$ are fulfilled. We accept that $r = c/v = \zeta/2(\zeta - 1)$ and $r \in [1/2; 3/4)$. In this case the following inequalities are fulfilled: $1 + (\zeta - 1)(c^2/v^2)\beta^2 - (c/v)\beta^2\zeta < 0$ and $(\zeta - 1)(c^2/vw)\beta^2 - (c/w)\beta^2\zeta < 0$. These two expressions tend toward $-\infty$ at $\beta \to 1$, and accordingly $\zeta \to \infty$, $r \to 1/2$. Since Δt, $\Delta\tau$, and Δx are independent variables, we can choose Δt large enough, $\Delta\tau$ small enough, and χ close enough to 1 that the following expressions take arbitrarily small values:

$$(\zeta - 1)\frac{c^2}{vw}\beta^2\frac{\Delta\tau}{\Delta t},$$

$$\frac{c}{v}\beta^2\zeta\left(1 - \chi\sqrt{1 + \frac{\Delta\tau^2}{\Delta t^2}}\right),$$

$$\left[1 + (\zeta - 1)\frac{c^2}{w^2}\beta^2\right]\frac{\Delta\tau}{\Delta t},$$

$$\frac{c}{w}\beta^2\zeta\left(1 - \chi\sqrt{1 + \frac{\Delta\tau^2}{\Delta t^2}}\right).$$

Therefore, if $r \in [1/2; 3/4)$, then for appropriate values of Δt, $\Delta\tau$, and Δx the following inequalities will be fulfilled:

TABLE I. Some values of $\Delta t'$ and $\Delta\tau'$, provided that $\Delta t = 1$, $\Delta\tau = 0.3$, $\chi = 0.999$.

β	ζ	$r = \frac{c}{v}$	$\frac{c}{w}$	$\Delta t'$	$\Delta\tau'$
0.799	1.663	1.254	Imaginary number	Complex number	Complex number
0.800	1.667	1.250	0.000	0.276	0.300
0.840	1.843	1.093	0.472	0.320	0.007
0.841649	**1.852**	**1.087**	**0.480**	**0.320**	**0.000**
0.950	3.203	0.727	0.761	0.189	−0.549
0.960	3.571	**0.694**	0.776	**0.142**	−0.659
0.970	4.113	0.661	0.791	0.071	−0.813
0.976556	**4.645**	**0.637**	**0.802**	**0.000**	**-0.958**
0.980	5.025	0.624	0.807	−0.051	−1.060
0.990	7.089	0.582	0.825	−0.336	−1.594
0.999	22.366	0.523	0.853	−2.487	−5.384
0.999999	707.107	0.501	0.866	−99.434	−173.329

$$\left\{\left[1+(\zeta-1)\frac{c^2}{v^2}\beta^2\right]-\frac{c}{v}\beta^2\zeta\right\}+(\zeta-1)\frac{c^2}{vw}\beta^2\frac{\Delta\tau}{\Delta t}+\frac{c}{v}\beta^2\zeta\left(1-\chi\sqrt{1+\frac{\Delta\tau^2}{\Delta t^2}}\right)<0,$$

$$\left[(\zeta-1)\frac{c^2}{vw}\beta^2-\frac{c}{w}\beta^2\zeta\right]+\left[1+(\zeta-1)\frac{c^2}{w^2}\beta^2\right]\frac{\Delta\tau}{\Delta t}+\frac{c}{w}\beta^2\zeta\left(1-\chi\sqrt{1+\frac{\Delta\tau^2}{\Delta t^2}}\right)<0;$$

that is,

$$\Delta t' = \left[1+(\zeta-1)\frac{c^2}{v^2}\beta^2\right]\Delta t+(\zeta-1)\frac{c^2}{vw}\beta^2\Delta\tau-\frac{c}{v}\beta^2\zeta\chi\sqrt{\Delta t^2+\Delta\tau^2}<0,$$

$$\Delta\tau' = (\zeta-1)\frac{c^2}{vw}\boldsymbol{\beta^2}\Delta t+\left[1+(\zeta-1)\frac{c^2}{w^2}\beta^2\right]\Delta\tau-\frac{c}{w}\boldsymbol{\beta^2}\zeta\chi\sqrt{\Delta t^2+\Delta\tau^2}<0.$$

An example of this is shown in Table I. From Table I one can see that if $\beta < 0.800$, then the value c/w is an imaginary number and $\Delta t'$ and $\Delta\tau'$ are complex numbers; if $\beta = 0.800$, then $\Delta t' > 0, \Delta\tau' > 0$; if $\beta \approx 0.841649$, then $\Delta t' > 0, \Delta\tau' = 0$; if $\beta = 0.950$, then $\Delta t' > 0, \Delta\tau' < 0$; if $\beta \approx 0.976556$, then $\Delta t' = 0, \Delta\tau' < 0$; if $\beta = 0.999$, then $\Delta t' < 0, \Delta\tau' < 0$. According to Table I, for the case of two-dimensional time, it is possible that the conditions $v > c$ and $\Delta t' > 0$ can be simultaneously fulfilled (unlike the case of one-dimensional time in STR). For example, if $r = c/v = 0.694$, then $\Delta t' = 0.142 > 0$.

C. General properties of the transformations

We are going to examine some of the properties of the general transformations between K and K'. Let us denote with x^1, x^2 the time dimensions and with x^3, x^4, x^5 the space dimensions; for example $x^1 = ct$, $x^2 = c\tau$, $x^3 = x$, $x^4 = y$, $x^5 = z$. The following equality is fulfilled:

$$(x^1)^2+(x^2)^2-\sum_{\eta=3}^{5}(x^\eta)^2 = (x^{1\prime})^2+(x^{2\prime})^2-\sum_{\eta=3}^{5}(x^{\eta\prime})^2. \tag{17}$$

The general relations which fulfill this condition must have the form

$$x^\mu/ = a^\mu_\rho x^\rho + b^\mu, \tag{18}$$

where $\mu, \rho = 1, 2, 3, 4, 5$.[d] Here b^μ are five constant values, which are equal to the values of $x^{\mu\prime}$ for the case when $x^\mu = 0$ ($\mu = 1, 2, 3, 4, 5$)—that is, b^μ is a five-dimensional vector of translation in the space-time. If the origins of both reference systems are the same, then $b^\mu = 0$ ($\mu = 1, 2, 3, 4, 5$). We will further examine the transformations that do not include translations in space and time, i.e.,

$$x^{\mu\prime} = a^\mu_\rho x^\rho. \tag{19}$$

Let us introduce the notation

$$g_{\mu\rho} = g_{\rho\mu} = \begin{cases} 1 & \mu=\rho=1,2 \\ -1 & \mu=\rho=3,4,5 \\ 0 & \mu\neq\rho. \end{cases}$$

In this way, Eq. (17) can be presented in the form

$$g_{\mu\rho}x^\mu x^\rho = g_{\mu\rho}x^{\mu\prime}x^{\rho\prime}. \tag{20}$$

If we substitute Eq. (19) into the right-hand side of Eq. (20) and compare the coefficients in front of x, we have

$$g_{\mu\rho} = g_{\lambda\sigma}a^\lambda_\mu a^\sigma_\rho. \tag{21}$$

(Here $\lambda, \sigma = 1, 2, 3, 4, 5$). Let us define a 5×5 matrix $(\mathbf{A})_{\mu\rho}$, with elements $a^\mu{}_\rho$: $(\mathbf{A})_{\mu\rho} \equiv a^\mu{}_\rho$. Similarly, let us define the elements of the matrix $(\mathbf{G})_{\mu\rho}$: $(\mathbf{G})_{\mu\rho} \equiv g_{\mu\rho}$. In matrix presentation, Eq. (21) can be written as $(\mathbf{G})_{\mu\rho} = (\mathbf{A}^{\mathrm{tr}}\mathbf{G}\mathbf{A})_{\mu\rho}$; therefore $\det(\mathbf{G}) = \det(\mathbf{A}^{\mathrm{tr}}\mathbf{G}\mathbf{A}) = \det(\mathbf{A}^{\mathrm{tr}})\det(\mathbf{G})\det(\mathbf{A})$. Taking into consideration the fact that $\det(\mathbf{A}^{\mathrm{tr}}) = \det(\mathbf{A})$ and $\det(\mathbf{G}) = -1$, we obtain

[d] In this and the following formulas, the Einstein summation convention for repeating indices is used.

TABLE II. Decomposition of the group of nonzero transformations.

Transformations between K and K′	det(**A**)	sign of $a^1{}_1$	sign of $a^2{}_2$	Operations applied on $\Lambda_+^{\uparrow\rightarrow}$:
$\Lambda_+^{\uparrow\rightarrow}$	+1	+1	+1	**1** (unit matrix) = diag(+1, +1, +1, +1, +1)
$\Lambda_+^{\downarrow\rightarrow}$	+1	+1	−1	$\mathbf{\Pi\Xi}$ = diag(+1, −1, −1, −1, −1)
$\Lambda_+^{\uparrow\leftarrow}$	+1	−1	+1	$\mathbf{\Pi\Omega}$ = diag(−1, +1, −1, −1, −1)
$\Lambda_+^{\downarrow\leftarrow}$	+1	−1	−1	$\mathbf{\Gamma}$ = diag(−1, −1, +1, +1, +1)
$\Lambda_-^{\uparrow\rightarrow}$	−1	+1	+1	$\mathbf{\Pi}$ = diag(+1, +1, −1, −1, −1)
$\Lambda_-^{\downarrow\rightarrow}$	−1	+1	−1	$\mathbf{\Xi}$ = diag(+1, −1, +1, +1, +1)
$\Lambda_-^{\uparrow\leftarrow}$	−1	−1	+1	$\mathbf{\Omega}$ = diag(−1, +1, +1, +1, +1)
$\Lambda_-^{\downarrow\leftarrow}$	−1	−1	−1	$\mathbf{\Pi\Gamma}$ (total inversion) = diag(−1, −1, −1, −1, −1)

$$\det(\mathbf{A}) = \pm 1. \tag{22}$$

In Eq. (21), if we set $\mu = \rho = 1$ and $\mu = \rho = 2$, we accordingly obtain

$$1 = (a_1^1)^2 + (a_1^2)^2 - \sum_{\eta=3}^{5}(a_1^\eta)^2,$$

$$1 = (a_2^1)^2 + (a_2^2)^2 - \sum_{\eta=3}^{5}(a_2^\eta)^2;$$

that is,

$$a_1^1 = \pm\sqrt{1 + \sum_{\eta=3}^{5}(a_1^\eta)^2 - (a_1^2)^2},$$

$$a_2^2 = \pm\sqrt{1 + \sum_{\eta=3}^{5}(a_2^\eta)^2 - (a_2^1)^2}.$$

Therefore, we have nine possible cases: $a^1{}_1 > 0$ and $a^2{}_2 > 0$; $a^1{}_1 > 0$ and $a^2{}_2 < 0$; $a^1{}_1 < 0$ and $a^2{}_2 > 0$; $a^1{}_1 < 0$ and $a^2{}_2 < 0$; $a^1{}_1 = 0$ and $a^2{}_2 > 0$; $a^1{}_1 = 0$ and $a^2{}_2 < 0$; $a^1{}_1 > 0$ and $a^2{}_2 = 0$; $a^1{}_1 < 0$ and $a^2{}_2 = 0$; and $a^1{}_1 = 0$ and $a^2{}_2 = 0$. Adding into consideration Eq. (22), we have altogether 18 cases in general.

In the case of two-dimensional time it is possible that the equalities $a^1{}_1 = 0$ and $a^2{}_2 = 0$ are fulfilled [e.g., if $x^1 = ct, x^2 = c\tau$, $x^3 = x$, $x^4 = y$, $x^5 = z$, then they are fulfilled at the transformation $(x^1)' = c\tau, (x^2)' = -ct, (x^3)' = x, (x^4)' = y, (x^5)' = z$]. The transformations where $a^1{}_1 \neq 0$ and $a^2{}_2 \neq 0$ we will call nonzero transformations.

Let us now consider only the nonzero transformations. There are in general eight nonzero transformations (see Table II). The transformations conserving the orientation—i.e., for which $\det(\mathbf{A}) = 1$, $a^1{}_1 > 0$, $a^2{}_2 > 0$—we will call proper orthochronous transformations and will denote with $\Lambda_+^{\uparrow\rightarrow}$. The transformations which do not change the signs in front of t and τ (that is, $a^1{}_1 > 0$ and $a^2{}_2 > 0$) we will call orthochronous in relation to t and τ; we will denote them with $\Lambda^{\uparrow\rightarrow} = \{\Lambda_+^{\uparrow\rightarrow} \cup \Lambda_-^{\uparrow\rightarrow}\}$. The transformations which do not change the sign in front of t (that is, $a^1{}_1 > 0$) we will call orthochronous in relation to t; we will denote them with $\Lambda^{\uparrow\rightarrow} = \{\Lambda_+^{\uparrow\rightarrow} \cup \Lambda_+^{\downarrow\rightarrow} \cup \Lambda_-^{\uparrow\rightarrow} \cup \Lambda_-^{\downarrow\rightarrow}\}$. The transformations which do not change the sign in front of τ (that is, $a^2{}_2 > 0$) we will call orthochronous in relation to τ; we will denote them with $\Lambda^{\uparrow} = \{\Lambda_+^{\uparrow\rightarrow} \cup \Lambda_+^{\uparrow\leftarrow} \cup \Lambda_-^{\uparrow\rightarrow}\Lambda_-^{\uparrow\leftarrow}\}$. The transformations which change the signs in front of t and τ we will call nonorthochronous in relation to the time dimensions; we will denote them with $\Lambda^{\downarrow\leftarrow} = \{\Lambda_+^{\downarrow\leftarrow} \cup \Lambda_-^{\downarrow\leftarrow}\}$. It is clear that the same transformation can be orthochronous in relation to a given time dimension and nonorthochronous in relation to another time dimension (for example, the transformation $\Lambda_+^{\downarrow\rightarrow}$). The transformations responsible for the condition $\det(\mathbf{A}) = +1$ we will call proper transformations and will denote with $\Lambda_+ = \{\Lambda_+^{\uparrow\rightarrow} \cup \Lambda_+^{\downarrow\rightarrow} \cup \Lambda_+^{\uparrow\leftarrow} \cup \Lambda_+^{\downarrow\leftarrow}\}$. The transformations for which $\det(\mathbf{A}) = -1$ we will call nonproper transformations and will denote with $\Lambda_- = \{\Lambda_-^{\uparrow\rightarrow} \cup \Lambda_-^{\downarrow\rightarrow} \cup \Lambda_-^{\uparrow\leftarrow} \cup \Lambda_-^{\downarrow\leftarrow}\}$.

Let us define the following discrete operations which present spatial or temporal reflection: $x^{\mu\prime} = \Pi^\mu_\rho x^\rho$, $x^{\mu\prime} = \Gamma^\mu{}_\rho x^\rho$, $x^{\mu\prime} = \Omega^\mu{}_\rho x^\rho$, $x^{\mu\prime} = \Xi^\mu{}_\rho x^\rho$, where $\mathbf{\Pi}^\mu{}_\rho$ = diag(+1, +1, −1, −1, −1), $\mathbf{\Gamma}^\mu{}_\rho$ = diag(−1, −1, +1, +1, +1), $\mathbf{\Omega}^\mu{}_\rho$ = diag(−1, +1, +1, +1, +1), $\mathbf{\Xi}^\mu{}_\rho$ = diag(+1, −1, +1, +1, +1). Since after these operations the scalar product remains invariant, they belong also to the full group of transformations. We can connect these operations to the transformations obtained previously (see Table II).

The results obtained here will play a very important role relative to an antiparticle moving in multidimensional time (see Section X).

If we assume that thes transformations are valid, then the Lorentz covariance must be violated. The physical laws will not be Lorentz covariant, i.e., they will not be transformed according to the Lorentz group. The four-dimensional space-time interval in this case will not be invariant, while the five-dimensional interval will be—see Section IV.

In the case of multidimensional time, the *CPT* symmetry will be violated. STR and consequently the Lorentz covariance is placed at the base of *CPT* symmetry. Indeed, the even number of reflections of the coordinates in Minkowski space-time (*PT*-symmetry) is formally reduced to a rotation by an imaginary angle. Due to this fact, the existing physical theories, which are invariant relative to the Lorentz transforms (i.e., rotations in Minkowski space-time) turn out to be automatically *CPT* invariant. We discuss the problem with *CPT* symmetry in multidimensional time more precisely in Section X.

D. Waves in two-dimensional time and three-dimensional space and the Doppler effect

It is clear that in the case of two-dimensional time the d'Alembert operator is not a scalar. Instead, another operator must be used, obtained from the scalar product:

$$\Box_{t\tau} \equiv g^{\mu\rho}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\rho} = \left(\frac{\partial}{c\partial t}\right)^2 + \left(\frac{\partial}{c\partial \tau}\right)^2 - \sum_{\eta=3}^{5}\left(\frac{\partial}{\partial x^\eta}\right)^2,$$

where

$$g^{\mu\rho} = \begin{cases} 1 & \mu=\rho=1,2 \\ -1 & \mu=\rho=3,4,5 \\ 0 & \mu\neq\rho. \end{cases}$$

(Here $x^1 = ct, x^2 = c\tau$.) Since the $\Box_{t\tau}$ operator is a scalar, it follows that

$$\Box_{t\tau} = \Box'_{t\tau} = \left(\frac{\partial}{c\partial t'}\right)^2 + \left(\frac{\partial}{c\partial \tau'}\right)^2 - \sum_{\eta=3}^{5}\left(\frac{\partial}{\partial x^{\eta\prime}}\right)^2.$$

For the case of two-dimensional time and three-dimensional space, the wave equation takes the form

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2} = \left(\frac{\kappa}{\omega_t}\right)^2\frac{\partial^2 F}{\partial t^2} + \left(\frac{\kappa}{\omega_\tau}\right)\frac{\partial^2 F}{\partial \tau^2},$$

where ω_t and ω_τ are the angular frequencies of the wave, defined in relation to the time dimensions t and τ respectively, and κ is the angular wavenumber (ultra-hyperbolic partial differential equations—see Ref. 5). We have set $x^3 = x$, $x^4 = y, x^5 = z$.) Here $u_t = \omega_t/\kappa$ and $u_\tau = \omega_\tau/\kappa$ are phase velocities, defined in relation to t and τ, respectively.

We will examine the Doppler effect in two-dimensional time and three-dimensional space. In this case it is possible that waves exist which have properties depending only on one or both time dimensions. We will consider the general case, when a wave is moving in the two time dimensions t, τ and in the three space dimensions x, y, z.

Let us denote with ω_t and ω_τ the angular frequencies of the wave in the frame K, defined relative to the time dimensions t and τ, and with $\boldsymbol{\kappa} = (\kappa_x, \kappa_y, \kappa_z)$ the wave vector of this wave in K. The phase of the wave in K is given by the expression $\omega_t t + \omega_\tau \tau - \boldsymbol{\kappa}\mathbf{R}$, where $\mathbf{R} = (x, y, z)$. Let us denote with ω'_t and ω'_τ the angular frequencies of the wave in K', defined relative to t' and τ', and with $\boldsymbol{\kappa}' = \kappa'_x, \kappa'_y, \kappa'_z$ the wave vector of the wave in K'. The phase of the wave in K' is given by $\omega'_t t' + \omega'_\tau \tau' - \boldsymbol{\kappa}'\mathbf{R}'$, where $\mathbf{R}' = (x', y', z')$. In order to determine the Doppler effect for the wave under consideration, we have to set to set the wave phase to be invariant—that is, $\omega_t t + \omega_\tau \tau - \boldsymbol{\kappa}\mathbf{R} = \omega'_t t' + \omega'_\tau \tau' - \boldsymbol{\kappa}'\mathbf{R}'$.

We set $\kappa' = ||\boldsymbol{\kappa}'|| > 0$ (angular wavenumber); $u'_t = \omega'_t/\kappa'$, $u'_\tau = \omega'_\tau/\kappa'$ (phase velocities determined according to t' and τ', respectively); $\kappa'_x/\kappa' = \cos\gamma'$. Applying the transformations in Eq. (12) from K to K', we define the relation between ω_t, ω_τ, κ_x, κ_y, κ_z and ω'_t, $\omega'_\tau \kappa'_x$, κ'_y, κ'_z:

$$\omega_t = \omega'_t\left[1 + (\zeta - 1)\frac{c^2}{v^2}\beta^2 + \frac{c^2}{v u'_t}\beta^2\zeta.\cos\gamma'\right] + \omega'_\tau(\zeta - 1)\frac{c^2}{vw}\beta^2,$$

$$\omega_\tau = \omega'_\tau\left[1 + (\zeta - 1)\frac{c^2}{w^2}\beta^2 + \frac{c^2}{w u'_\tau}\beta^2\zeta.\cos\gamma'\right] + \omega'_t(\zeta - 1)\frac{c^2}{vw}\beta^2,$$

$$\kappa_x = \kappa'_x\zeta\left(1 + \frac{u'_t}{v\cos\gamma'}\beta^2\right) + \kappa'_x\beta^2\zeta\frac{u'_\tau}{w\cos\gamma'},$$
$$\kappa_y = \kappa'_y, \qquad \kappa_z = \kappa'_z.$$

The expressions $\omega'_\tau(\zeta - 1)(c^2/vw)\beta^2$ and $\kappa'_x\beta^2\zeta(u'_\tau/w\cos\gamma')$ in the formulas for the angular frequency ω_t and the wave vector κ_x, respectively, can be regarded as corrections in the formulas for the Doppler effect for the case of two-dimensional time. If we set $\omega'_\tau = 0$, we will obtain the formulas for the Doppler effect for the case in which the wave is moving in only one time dimension, t'. Obviously in this case, the expressions for ω_t and κ_x differ from the formulas for the relativistic Doppler effect in STR.

IV. CAUSAL STRUCTURE OF SPACE-TIME (*n*, *k*) = (3, 2)

Let us have a five-dimensional vector $\mathbf{A}^\mu$ in space-time $(n, k) = (3, 2)$. The scalar product of the vector $\mathbf{A}^\mu$ with itself will be[e]

$$(\mathbf{A})^2 = A^\mu A_\mu = g_{\mu\rho}A^\mu A^\rho = (A^1)^2 + (A^2)^2 - \sum_{\eta=3}^{5}(A^\eta)^2,$$

where μ, $\rho = 1, 2, 3, 4, 5$ and

$$g_{\mu\rho} = g_{\rho\mu} = \begin{cases} 1 & \mu=\rho=1,2 \\ -1 & \mu=\rho=3,4,5 \\ 0 & \mu\neq\rho. \end{cases}$$

Let us denote $(\mathbf{A}_{3,1})^2 = (A^1)^2 - \sum_{\eta=3}^{5}(A^\eta)^2$. Then we will have $(\mathbf{A})^2 = (\mathbf{A}_{3,1})^2 + (A^2)^2$. While in Minkowski space-time $[(n, k) = (3, 1)]$ the value $(\mathbf{A}_{3,1})^2$ is invariant, in a space-time $(n, k) = (3, 2)$ it is not invariant—but the value $(\mathbf{A})^2$ is invariant. If, for example, the value $(\mathbf{A}_{3,1})^2$ is spacelike in one frame of reference K $[(\mathbf{A}_{3,1})^2 < 0]$, it can be 0 $[(\mathbf{A}'_{3,1})^2 = 0]$ or timelike $[(\mathbf{A}'_{3,1})^2 > 0]$ in another frame of reference K'.

The causal region of the space-time $(n, k) = (3, 2)$ encompasses the region $(\mathbf{A})^2 \geq 0$, and the causal region of the space-time $(n, k) = (3, 1)$ encompasses the region $(\mathbf{A}_{3,1})^2 \geq 0$.

Let us set: $A^1 = cdt$, $A^2 = cd\tau$, $A^3 = dx$, $A^4 = dy$, $A^5 = dz$, $q_1 = (\sqrt{dx^2 + dy^2 + dz^2})/(c|dt|) = V/c \geq 0$, $q_2 = (\sqrt{dx^2 + dy^2 + dz^2})/(c|d\tau|) = W/c \geq 0$, $ds_{3.2} = ||\mathrm{A}|| = \sqrt{(\mathbf{A})^2}$.

[e] In the formula, the Einstein summation convention is used.

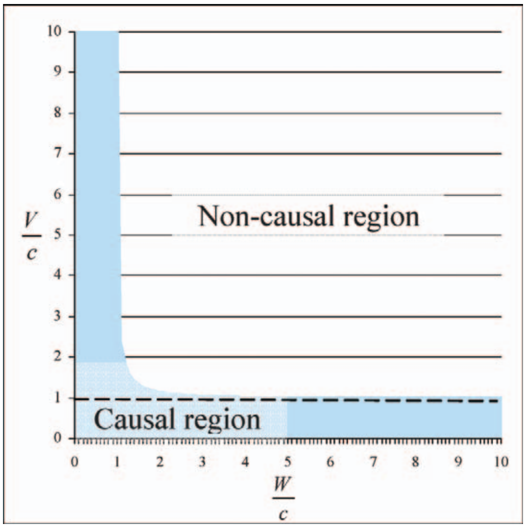

FIG. 1. (Color online) The values of the velocities V and W at which an object is moving in the causal regions of the space-time $(n, k) = (3, 2)$ and $(n, k) = (3, 1)$.

We will consider three cases for the value of $(\mathbf{A})^2$.

First case: $(\mathbf{A})^2 > 0$—that is, $ds_{3,2}^2 > 0$. If $A^2 = cd\tau \neq 0$, then there are three possible cases: $(\mathbf{A}_{3,1})^2 > 0, (\mathbf{A}_{3,1})^2 = 0, (\mathbf{A}_{3,1})^2 < 0$. If $A^2 = cd\tau = 0$, then $(\mathbf{A}_{3,1})^2 > 0$. Let us assume that in the frame of reference K, the following are fulfilled: $A^2 = cd\tau \neq 0$ and $(\mathbf{A}_{3,1})^2 < 0$. It is clear that in the system K' moving uniformly and rectilinearly in relation to K, we have $(\mathbf{A}')^2 = (\mathbf{A}'_{3,1})^2 + (A^{2\prime})^2 = (\mathbf{A})^2 > 0$. If we assume that $A^{2\prime} = cd\tau' = 0$, then we have $(\mathbf{A}'_{3,1})^2 > 0$. So we have obtained that in the frame K the value $\mathbf{A}_{3,1}$ is spacelike $[(\mathbf{A}_{3,1})^2 < 0]$ and in K' the value $\mathbf{A}'_{3,1}$ is timelike $[(\mathbf{A}'_{3,1})^2 > 0]$. It is easy to prove that the condition $(\mathbf{A})^2 > 0$ is equivalent to the following inequality:

$$\frac{1}{q_1^2} + \frac{1}{q_2^2} > 1. \tag{23}$$

If $0 \leq q_1 < 1$ (that is, $0 \leq V < c$), then the inequality in Eq. (23) will be fulfilled for all values of $q_2 \leq \infty$ (i.e., for all values of W, including $W = \infty$). If $q_1 = 1$ (that is, $V = c$), then the inequality in Eq. (23) will be fulfilled provided that $q_2 < \infty$ (that is, $W < \infty$). Similar considerations are valid for q_2 (and accordingly for the velocity W). Therefore, if the velocity of a particle defined in relation to the one time dimension (as absolute value) is less than or equal to the speed of light in vacuum, then the velocity of this particle defined in relation to the other time dimension can have an arbitrary value without violating the causality principle. If simultaneously $q_1 > 1$ and $q_2 > 1$, then the inequality in Eq. (23) will be fulfilled for an appropriate choice of the parameters q_1 and q_2 (e.g., $q_1 = 10/9, q_2 = 2$). It is clear that the condition $(\mathbf{A}_{3,1})^2 > 0$ is equivalent to the inequality $V < c$, the condition $(\mathbf{A}_{3,1})^2 = 0$ is equivalent to the equality $V = c$, and the condition $(\mathbf{A}_{3,1})^2 < 0$ is equivalent to the inequality $V > c$.

Second case: $(\mathbf{A})^2 = 0$—that is, $ds_{3,2}^2 = 0$. In this case, $(\mathbf{A}_{3,1})^2 = -c^2 d\tau^2$, and therefore it is not possible that $(\mathbf{A}_{3,1})^2 > 0$. If $A^2 = cd\tau = 0$, then $(\mathbf{A}_{3,1})^2 = 0$. If $A^2 = cd\tau \neq 0$, then we have $(\mathbf{A}_{3,1})^2 < 0$. The condition $(\mathbf{A})^2 = 0$ is equivalent to the following equality:

$$\frac{1}{q_1^2} + \frac{1}{q_2^2} = 1. \tag{24}$$

If $q_1 < 1$ (that is, $V < c$) or $q_2 < 1$ (that is, $W < c$), then Eq. (24) is impossible. If $q_1 = 1$ (that is, $V = c$), then Eq. (24) is possible under the condition that $q_2 = \infty$ (that is, $W = \infty$). This means that $dx^2 + dy^2 + dz^2 > 0$, $dt^2 > 0$, $d\tau^2 = 0$—i.e., the considered particle moves in space and in the time dimension t but not in the time dimension τ. The same applies for the case $q_2 = 1$ (that is, $W = c$). From Eq. (24) follows the equality $(c^2/V^2) + (c^2/W^2) = 1$. Therefore, if the considered particle is moving with velocity

$$V = \frac{c}{\sqrt{1 - \frac{c^2}{W^2}}},$$

then we have $(\mathbf{A})^2 = ds_{3,2}^2 = 0$. According to the results of Section VIII, if the equality $(c^2/V^2) + (c^2/W^2) = 1$ is fulfilled for the velocities V and W of a particle in the frame of reference K, then for the velocities V' and W' of this particle in the frame K', the similar equality

$$\frac{c^2}{(V')^2} + \frac{c^2}{(W')^2} = 1$$

is fulfilled.

Third case: $(\mathbf{A})^2 < 0$—that is, $ds_{3,2}^2 < 0$. We have $(\mathbf{A}_{3,1})^2 < 0$. In this case it is not possible that $(\mathbf{A}_{3,1})^2 > 0$ or $(\mathbf{A}_{3,1})^2 = 0$. The condition $(\mathbf{A})^2 < 0$ is equivalent to the following inequality:

$$\frac{1}{q_1^2} + \frac{1}{q_2^2} < 1. \tag{25}$$

The inequality in Eq. (25) is fulfilled only if the following inequalities are simultaneously fulfilled: $q_1 > 1$ and $q_2 > 1$ (that is, $V > c$ and $W > c$).

From these considerations we can conclude that the causal region of (3 + 2)-dimensional space-time includes the causal region of (3 + 1)-dimensional space-time and presents a larger part of it.

Figure 1 shows graphs of the functions $(c^2/V^2) + (c^2/W^2) = 1$ and $c/V = 1$ for non-negative values of V and W.

It is clear that for each point in the darker region and on the border of this region [i.e., the region confined by the coordinate axes and the graph of the function $(c^2/V^2) + (c^2/W^2) = 1$] corresponds to a combination of values for the velocities V and W where a given particle is moving in the causal region of the space-time $(n, k) = (3, 2)$, that is, $ds_{3,2}^2 \geq 0$. For each point outside this region there is a combination of values of the velocities V and W where the given point does not move in the causal region of the space-time $(n, k) = (3, 2)$, that is, $ds_{3,2}^2 < 0$. For each point in the region between the abscissa and the straight line $c/V = 1$ there exists a combination of values for the velocities V and W where the particle is moving in the causal region of space-time $(n, k) = (3, 1)$; this is the case of STR $[(\mathbf{A}_{3,1})^2 \geq 0.]$

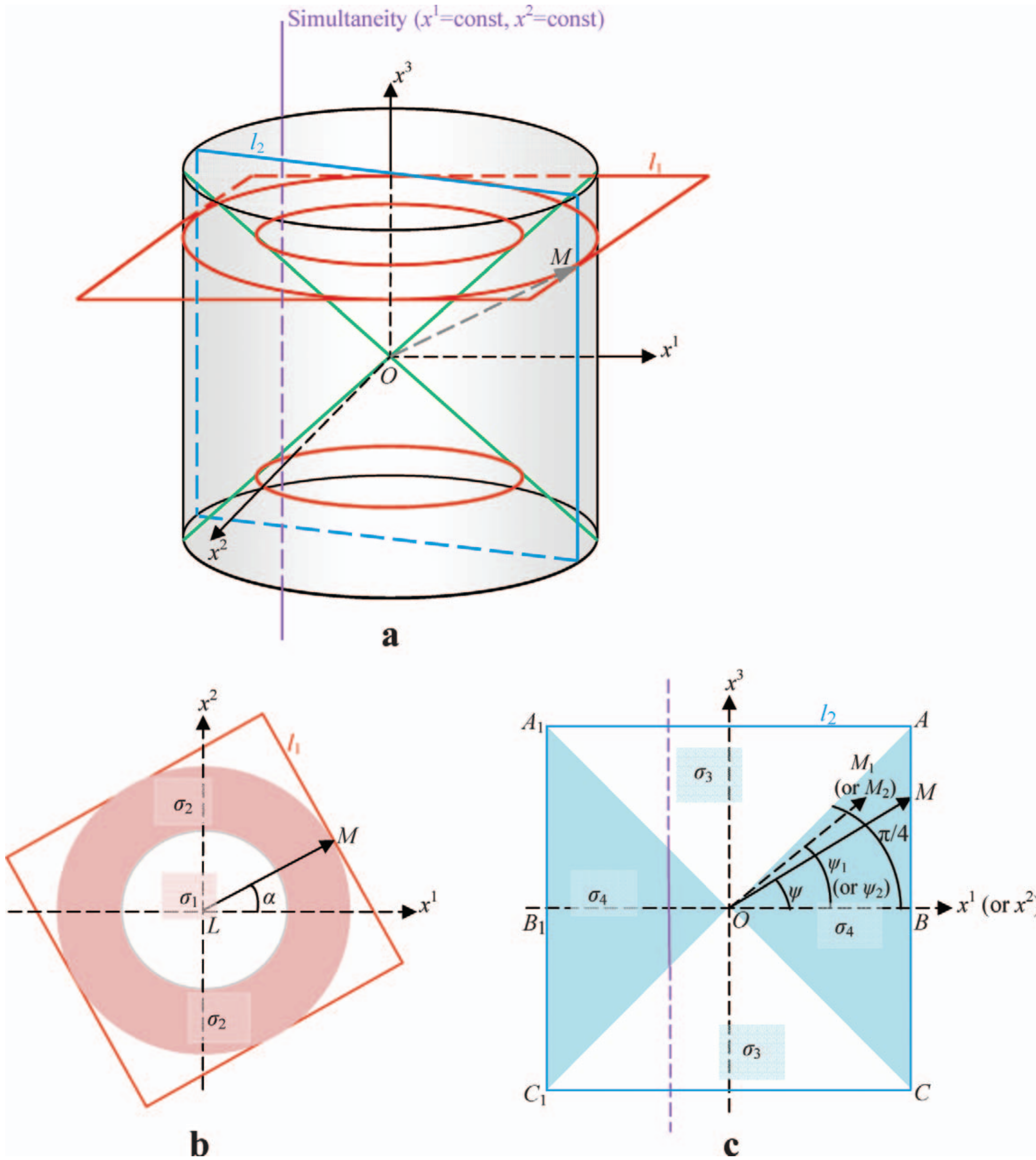

FIG. 2. (Color online) Causal structure of the space-time (a); causal region in the plane l_1 (b) and in the plane l_2 (c).

Let us denote with x^1, x^2 the time dimensions and with x^3, x^4, x^5 the space dimensions. Let us further for simplicity consider only one space dimension x^3 and the two time dimensions x^1 and x^2 (that is, $x^1 = ct$, $x^2 = c\tau$, $x^3 = x$, $x^4 = 0$, $x^5 = 0$). Let us assume that at point O, having coordinates $x^1 = 0$, $x^2 = 0$, $x^3 = 0$ ($x^4 = 0$, $x^5 = 0$) defined in the frame K, there has been an event E. Let us further assume that a given interval of time $\Delta T = \sqrt{\Delta t^2 + \Delta\tau^2} \geq 0$ has passed since the event. Our task is to examine the causal region attached to the event E. Since the time is two-dimensional, all possible combinations of coordinates x^1, x^2 for which the inequality $(x^1)^2 + (x^2)^2 \leq c^2\Delta T^2$ is valid form a circle in the plane x^1-x^2 with center at point O and radius equal to $c\Delta T$. If we add the space dimension x^3 in such a way that for an arbitrary value of x^3 the inequality $(x^1)^2 + (x^2)^2 \leq c^2\Delta T^2$ is fulfilled, then we will obtain a right circular cylinder with the obtained circle as its base. According to the previous considerations, in order for a exist causal relation between two events to exist, the inequality $s_{3,2}^2 \geq 0$ must be fulfilled—that is, $(x^1)^2 + (x^2)^2 - (x^3)^2 \geq 0$. This inequality can be represented graphically by a double right circular cone. The generatrices of the lateral surface of the cone make an angle of $\pi/4$ with the plane x^1-x^2. Therefore, we obtain a double cone inscribed in the cylinder [Fig. 2(a)]. From the inequalities $(x^1)^2 + (x^2)^2 - (x^3)^2 \geq 0$ and $(x^1)^2 + (x^2)^2 \leq c^2\Delta T^2$ we obtain $|x|^3 \leq c\Delta T$—that is, $c\Delta T \geq x^3 \geq -c\Delta T$. The cone and the cylinder have common bases: a circle with its center at the point $x^3 = -c\Delta T$ ($x^1 = 0, x^2 = 0$) and radius $c\Delta T$, and a circle with its center at the point $x^3 = c\Delta T$ ($x^1 = 0$, $x^2 = 0$) and radius $c\Delta T$.

Let us with C_1 denote the border region which includes all points lying on the lateral surface of the cone and the lateral surface of the cylinder. The region C_1 includes also the origin O—i.e., it includes all points (x^1, x^2, x^3) for which the conditions $\{(x^1)^2 + (x^2)^2 - (x^3)^2 = 0$ and $|x|^3 \le c\Delta t\}$ or $\{(x^1)^2 + (x^2)^2 = c^2\Delta T^2$ and $|x|^3 \le c\Delta t\}$ are fulfilled.

Let us denote with C_2 the inner region limited by the lateral surface of the cone and the lateral surface of the cylinder. The region C_2 does not include the origin O or the lateral surfaces of the cone and the cylinder (i.e., it does not include the border region C_1)—it includes all points (x^1, x^2, x^3), for which the inequalities $(x^1)^2 + (x^2)^2 - (x^3)^2 > 0$ and $(x^1)^2 + (x^2)^2 < c^2\Delta T^2$ are fulfilled. From these two inequalities we obtain $|x|^3 \le c\Delta T$. If $x^3 = 0$, then $0 < (x^1)^2 + (x^2)^2 < c^2\Delta T^2$.

Let us denote with C_3 the region which includes all points lying in the inner volume of the cone. The region C_3 does not include the origin O or the lateral surface of the cone. It includes the inner parts of the circles that form the bases of the cones (and the cylinder), but does not include the points lying on the directrix circumference restricting these circles (bases)—i.e., the region C_3 includes all points (x^1, x^2, x^3), for which the inequalities $(x^1)^2 + (x^2)^2 - (x^3)^2 < 0$ and $|x|^3 \le c\Delta T$ are fulfilled. It is clear that if $x^3 = \pm c\Delta T$, then $(x^1)^2 + (x^2)^2 < c^2\Delta T^2$.

Let us denote with C_4 the region including all points outside the cylinder. It does not include the lateral surface of the cylinder or the two bases of the cylinder—i.e., it includes all points (x^1, x^2, x^3) for which the inequality $(x^1)^2 + (x^2)^2 > c^2\Delta T^2$ or the inequality $|x|^3 > c\Delta T$ is fulfilled.

The causal region includes the regions C_1 and C_2; the noncausal region includes the regions C_3 and C_4.

In Fig. 2(a), the vector **OM** shows the motion of a pointlike particle M. The point M_1 is the projection of point M on the plane x^1-x^3, and point M_2 is the projection of point M on the plane x^2-x^3.

Let us consider the plane l_1, which is parallel to the plane x^1-x^2 and includes point M, and the plane l_2, which is perpendicular to the plane x^1-x^2 and includes points O and M. Let us set $\tan\,\alpha = x^2/x^1$ and $\tan\,\psi = x^3/\sqrt{(x^1)^2 + (x^2)^2}$. If we set $x^1 = ct, x^2 = c\tau, x^3 = x > 0$, then $\tan\,\alpha = v/w$ and $\tan\,\psi = 1/\sqrt{(c^2/v^2) + (c^2/w^2)}$. (In this case, v and w are the velocities of the particle M defined relative to the time dimensions t,and τ—see Subsection III.A). Let us denote with ψ_1 the angle between x^1 and $\mathbf{OM}_1$, and with ψ_2 the angle between x^2 and $\mathbf{OM}_2$. It is easy to see that $\tan\,\psi_1 = (\tan\,\psi)/(\cos\,\alpha)$ [if in Fig. 2(c) one can imagine the plane x^1-x^3] and $\tan\,\psi_2 = (\tan\,\psi)/(\sin\,\alpha)$ (if one can imagine the plane x^2-x^3). Let us denote with L the point where the axis x^3 intersects the plane l_1—i.e., point L is the projection of the origin O ($x^1 = 0$, $x^2 = 0$, $x^3 = 0$) on the plane l_1.

The causal region σ_2 in Fig. 2(b) represents the region between the two circumferences having a common center; the outer circumference has a radius $R = c\Delta T$, and the inner one has a radius $r = c\Delta T \tan\,\psi$. The causal region σ_4 in Fig. 2(c) represents two congruent right-angled isosceles triangles, namely OAC and OA_1C_1, which have a common apex at point O. It is clear that $|AC| = |A_1C_1| = 2c\Delta T$, $|OB| = |AB| = |OB_1| = |A_1B_1| = c\Delta T$, $|OA| = |OC| = |OA_1| = |OC_1| = c\Delta T\sqrt{2}$. Further, we have $|OL| = c\Delta T \tan\,\psi$.

We have assumed that since event E an interval of time ΔT has passed. In this case, the causal region in the plane l_1 (that is, σ_2) includes all events which are causally connected with event E and happen at a distance $|OL| = c\Delta T \tan\,\psi$ from O (along the axis x^3). The causal region in the plane l_2 (that is, σ_4) includes all events which are causally connected with event E and happen in a plane making an angle α with the plane x^1-x^3.

If $\alpha = 0$ (i.e., the vector **OM** is in the plane x^1-x^3), then the plane l_2 coincides with the plane x^1-x^3. (We will have $M_1 \equiv M$.) This case is considered in STR. If $\alpha = \pi/2$ (i.e., the vector **OM** is in the plane x^2-x^3), then the plane l_2 coincides with the plane x^2-x^3. (We will have $M_2 \equiv M$.)

We can consider two more specific cases depending on the value of the angle ψ.

First case: $\psi = 0$—that is, the plane l_1 coincides with the plane x^1-x^2. In this case the particle M is at rest; i.e., it does not move in the space dimension x^3, but only in the time dimensions x^1 and x^2. We have $\tan\,\psi = \tan\,0 = 0$. In this case the radius of the inner circle σ_1 in Fig. 2(b) is equal to 0 ($r = c\Delta T \tan\,\psi = 0$), i.e., the causal region σ_2 of event E includes all points of the outer circle. Therefore, the causal region of event E in the plane l_1 (and accordingly in the plane x^1-x^2) in this case coincides with the circle for which $(x^1)^2 + (x^2)^2 \le ||\mathbf{OM}||^2 = c^2\Delta T^2$. In this case we have $\tan\,\psi_1 = 0$, $\psi_1 = 0$ if Fig. 2(c) shows the plane x^1-x^3 and $\tan\,\psi_2 = 0$, $\psi_2 = 0$ if Fig. 2(c) shows the plane x^2-x^3. One can say that the two-dimensional time "flows" from the origin O in all directions in the plane x^1-x^2; as a result of this, the described circle is obtained. If we assume that the moments lying on the circumference of this circle are contemporary moments, then the moments of the inner part of the circle are past moments and the moments outside the circle are future moments (according to the moments defined as contemporary). These considerations are valid also for the case of k-dimensional time. It is evident that k-dimensional time will "flow" in the form of a k-dimensional hypersphere having its center at the origin O. Let us assume that between point O and the event E a period of time $\Delta T > 0$ has passed, which is determined according to the frame of reference K. In this case all moments inside the k-dimensional hypersphere with center O and radius ΔT are in the past and all moments outside the k-dimensional hypersphere are future moments. The surface which defines the k-dimensional hypersphere is a $(k-1)$-dimensional hypersphere. All moments lying on the mentioned $(k-1)$-dimensional hypersphere are present moments.

Second case: $\psi = \pi/4$—that is, the upper bases of the cylinder and the cone lie in the plane l_1. In this case the vector **OM** lies on one of the generatrices of the conical surface. We have $\tan\,\psi = \tan(\pi/4) = x^3/\sqrt{(x^1)^2 + (x^2)^2} = 1$. If we further set $x^1 = ct$, $x^2 = c\tau$, $x^3 = x > 0$, then for the velocities v and w of the particle M defined in relation to the time dimensions t and τ, respectively, the equality $(c^2/v^2) + (c^2/w^2) = 1$ will be fulfilled (see Subsection III.A). Since $\cos\,\alpha = 1/v\sqrt{(1/v^2) + (1/w^2)}$, $\sin\,\alpha = 1/w\sqrt{(1/v^2) + (1/w^2)}$, in this case we can express the velocities v and w in terms of

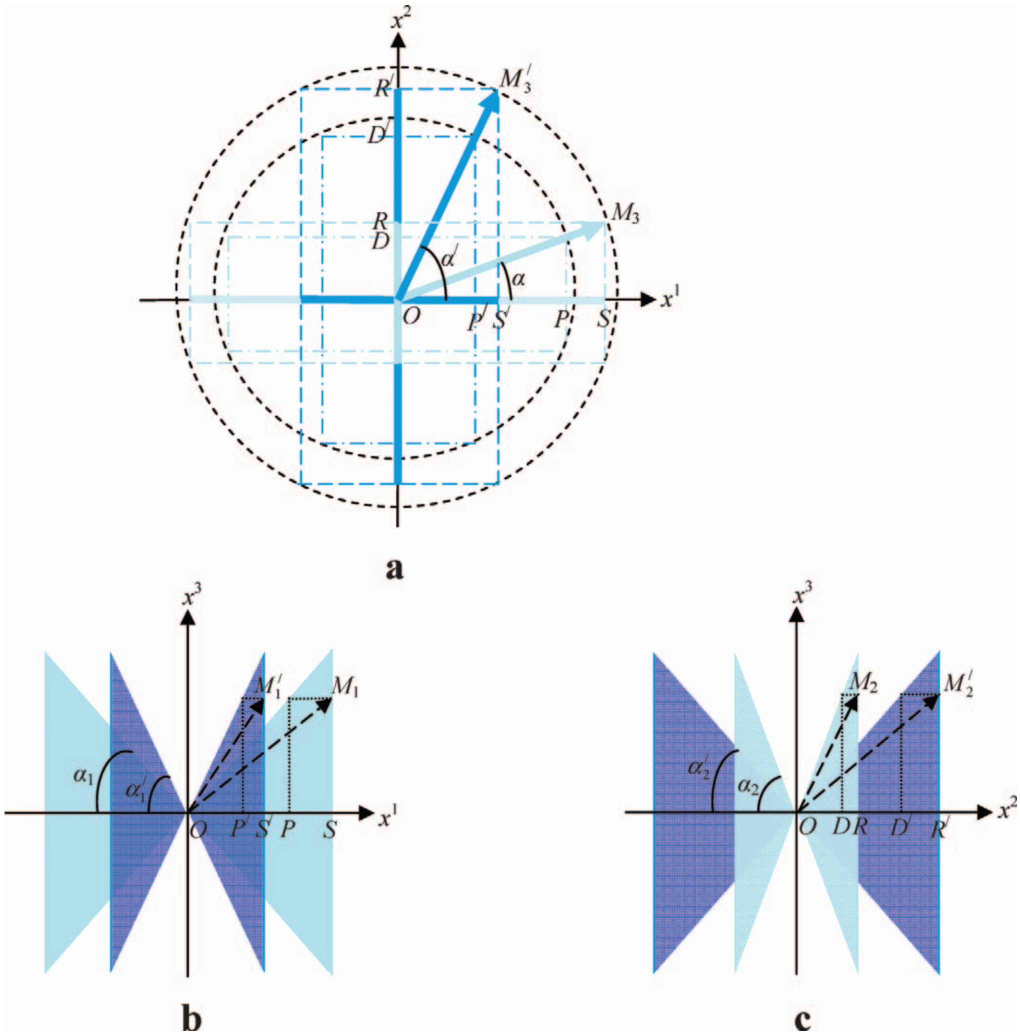

FIG. 3. (Color online) Projection of the causal region (a) on the plane x^1-x^2, (b) on the plane x^1-x^3, and (c) on the plane x^2-x^3, for a change of the angle α (from α to α').

angle α: $v = (c/\cos\,\alpha)$ and $w = (c/\sin\,\alpha)$—see Subsection III.A. Since $\tan\,\psi = 1$, in this case the radius r of the inner circle σ_1 from Fig. 2(b) is equal to the radius R of the outer circle ($r = c\Delta T\,\tan\,\psi = c\Delta T = R$), i.e., the causal region σ_2 includes only the points lying on the outer circumference. In this case we have $\tan\,\psi_1 = (1/\cos\,\alpha)$ if Fig. 2(c) shows the plane x^1-x^3 and $\tan\,\psi_2 = (1/\sin\,\alpha)$ if Fig. 2(c) shows the plane x^2-x^3.

Let us project the intersection of the plane l_1 with the cylinder and with the cone [shown in Fig. 2(a)] onto the plane x^1-x^2 . Let us do the same with the intersection of the plane l_2 with the cylinder and the cone onto x^1-x^3 and x^2-x^3. Then we obtain Fig. 3(a) and the darker areas shown in Figs. 3(b) and 3(c), respectively. If the vector **OM** presents the motion of a given particle M, then the projections of the motion of this particle onto the plane x^1-x^3 (i.e., the vector $\mathbf{OM}_1$) or onto the plane x^2-x^3 (i.e., the vector $\mathbf{OM}_2$) can lie only in the darker regions of Figs. 3(b) 3(c), respectively. In the opposite case—if the projections $\mathbf{OM}_1$ or $\mathbf{OM}_2$ do not lie in the indicated regions—then the motion shown by the vector **OM** is not causally related. In Fig. 3(a), the point M_3 is the projection of point M onto the plane x^1-x^2.

Figures 3(a), 3(b), and 3(c) show what happens when the angle α is changed (i.e., the angle α changes to angle α'). Here $\tan\,\alpha_1 = (1/\cos\,\alpha)$, $\tan\,\alpha_1' = (1/\cos\,\alpha')$ [Fig. 3(b)] and $\tan\,\alpha_2 = (1/\sin\,\alpha)$, $\alpha_2' = (1/\sin\,\alpha')$ [Fig. 3(c)].

At $\alpha = 0$ or $\alpha = \pi/2$ we obtain the well-known case of motion along only one axis of time (x^1 or x^2, respectively). This case ($\alpha = 0$) is considered in STR.

If, for example, $\alpha = 0$ and $x^1 = ct$, $x^3 = x$, then the projection of the causal region onto the plane x^1-x^3 [Fig. 3(b)] will coincide with the causal region according to STR. If $\alpha = 0$, then $\cos\,\alpha = 1$, $\tan\,\alpha_1 = 1$, $\alpha_1 = \pi/4$, $\tan\,\alpha_2 = \infty$, $\alpha_2 = \pi/2$. If $\alpha = 0$, then the projection of the causal region onto the plane x^2-x^3 [Fig. 3(c)] will be a line segment lying on the axis x^3 ($\alpha_2 = \pi/2$). In this case, if $||\mathbf{OM}_2|| = 0$, then the velocity of the particle M determined in relation to the two time dimensions x^1 and x^2 is equal to 0; but if $||\mathbf{OM}_2|| \neq 0$, then the velocity of the particle M defined in relation to x^1 is greater than 0 and less than or equal to the speed of light in a vacuum, and the velocity of the particle defined in relation to x^2 is infinitely large. (See the considerations at the beginning of Section IV.) Likewise, one can apply the same considerations for the case $\alpha = \pi/2$.

For the angle $\alpha' = \pm\alpha + b\pi$ ($b = 0, \pm1, \pm2, \ldots$), one can obtain the same projections of the causal region [Figs. 3(b) and 3(c)] as for angle α.

For an observer in the frame K, each coordinate plane $x^1 = \text{const}$, $x^2 = \text{const}$ represents space in the respective instant of time—i.e., these are a set of events which occur for the observer simultaneously. These regions in STR are called simultaneous spaces of the given observer. By analogy with STR, each timelike straight line [or line segment—see, for example, OM in Fig. 2(a)] is parallel to one of the time axes of a frame of reference K' moving uniformly and rectilinearly against K. Therefore, each timelike straight line "separates" the three-dimensional space into countless simultaneous regions.

Let us assume that the line segment OM lies in the plane x^1-x^3 and not on the axis x^1. In this case, the three-dimensional velocity of K' in relation to K defined in relation to the time dimension x^2 is infinitely large. However, as noted earlier, the three-dimensional velocity of K' against K determined in relation to the time dimension x^1 must be less than the speed of light in a vacuum. Obviously, in this causal region there exist infinitely many inertial frames like K' which posses the previously mentioned properties. Similar considerations are valid if the line segment OM lies in the plane x^2-x^3 and not on the axis x^2.

In space-times with at least two time dimensions, it is always possible to construct closed timelike curves.[f,4] In the space-time $(n, k) = (3, 2)$, we can consider a "motion" in the causal region in the plane t-τ which begins at the origin O ($t = 0, \tau = 0$) and ends at the same point. Indeed, let us set $t = \Omega.\sin\Theta$, $\tau = \Omega(1 - \cos\Theta)$; x, y, z = const. (Here Ω = const, $\Theta \in [0; 2\pi]$.) Then we have $ds^2_{3,2} = c^2dt^2 + c^2d\tau^2 - dx^2 - dy^2 - dz^2 = \Omega^2 d\Theta^2 > 0$—i.e., the world line is everywhere timelike.[4] As pointed out by Foster and Müller,[4]

> The existence of closed time-like curves implies that an observer can revisit the past and, if we accept the tenet of "free will," change it in a manner that is incompatible with the already experienced future. Obviously, anything resembling the common notion of causality cannot be maintained under such circumstances.

It has been argued, however, that such cases arise only due to the misidentification of different space-time points as identical points of the manifold.[13] Other authors have argued that the paradoxes arising from closed timelike curves are real, but can be resolved by an appropriate extension of the space-time manifold[14] (see Subsection III.B).

V. GENERALIZATION OF THE TRANSFORMATION FOR *n*-DIMENSIONAL SPACE AND *k*-DIMENSIONAL TIME

First, we will consider the simplest case, without rotations or translations in space or time (i.e., generalization of the Lorentz boost in a fixed direction in the field of multidimensional space and multidimensional time). Equation (12) can be easily generalized for an arbitrary number of space and time dimensions ($n \geq 1$, $k \geq 1$). The time dimensions we will denote with $t_1, t_2, \ldots, t_k$, and the space dimensions we will denote with $x_{k+1}, x_{k+2}, \ldots, x_{k+n}$. We assume that the frames K and K' are in a standard configuration. Let us denote with

$$\mathbf{v}_1 = (v_1, \underbrace{0, \ldots, 0}_{n-1}),$$

$$\mathbf{v}_2 = (v_2, \underbrace{0, \ldots, 0}_{n-1}),$$

$$\vdots$$

$$\mathbf{v}_k = (v_k, \underbrace{0, \ldots, 0}_{n-1})$$

the vectors of the velocities of K' against K, defined respectively in relation to the time dimensions $t_1, t_2, \ldots, t_k$. Let us set

$$\beta = \frac{1}{\sqrt{\sum_{\vartheta=1}^{k} \frac{c^2}{v_\vartheta^2}}}$$

and $\zeta = 1/\sqrt{1-\beta^2}$. Let us denote with $\mathbf{\Lambda}_{\mu\rho}$ the matrix for transfer between

$$\mathbf{X} = \begin{pmatrix} t_1 \\ \vdots \\ t_k \\ x_{k+1} \\ \vdots \\ x_{k+n} \end{pmatrix}$$

and

$$\mathbf{X}' = \begin{pmatrix} t'_1 \\ \vdots \\ t'_k \\ x'_{k+1} \\ \vdots \\ x'_{k+n} \end{pmatrix},$$

where $\mu, \rho = 1, 2, \ldots, k+n$. Here we have $\mathbf{X}' = \mathbf{\Lambda}_{\mu\rho}\mathbf{X}$. If $\mu \leq k$ and $\rho \leq k$, then $\mathbf{\Lambda}_{\mu\rho} = \delta_{\mu\rho} + (\zeta - 1)(c^2/v_\mu v_\rho)\beta^2$. Here $\delta_{\mu\rho}$ is the Kronecker delta, i.e.,

$$\delta_{\mu r} = \begin{cases} 1 & \mu = \rho \\ 0 & \mu \neq \rho \end{cases}.$$

If $\mu = \rho = k+1$, then $\mathbf{\Lambda}_{\mu\rho} = \zeta$. Furthermore, $\Lambda_{\sigma(k+1)} = -(1/v_\sigma)\beta^2\zeta$, $\Lambda_{(k+1)\theta} = -(c^2/v_\theta)\beta^2\zeta$, where $\sigma, \theta = 1, 2, \ldots, k$. If $\mu \geq k+2$ or $\rho \geq k+2$, then $\mathbf{\Lambda}_{\mu\rho} = \delta_{\mu\rho}$. Therefore, we have

[f] This is different from the case of curved manifolds, where closed timelike curves may arise under certain circumstances.

$$t'_\sigma = \sum_{\theta=1}^{k}\left[\delta_{\sigma\theta}t_\theta + \frac{c^2}{v_\sigma v_\theta}\beta^2(\zeta-1)t_\theta\right] - \frac{1}{v_\sigma}\beta^2\zeta x_{k+1}, \tag{26}$$

$$x'_{k+1} = -c^2\beta^2\zeta\sum_{\theta=1}^{k}\frac{t_\theta}{v_\theta} + \zeta x_{k+1}, \qquad x'_\varphi = x_\varphi, \tag{27}$$

where σ, $\theta = 1, 2, \ldots, k$, and $\varphi = k+2, k+3, \ldots, k+n$. The transformations in Eqs. (26) and (27) are equivalent to the Lorentz transformations at $t_2 \to 0$, $t_3 \to 0, \ldots, t_k \to 0$ and accordingly $v_2 \to \pm\infty$, $v_3 \to \pm\infty, \ldots, v_k \to \pm\infty$.

Let us set $(dx_{k+1}/dt_1) = v_1$, $(dx_{k+1}/dt_2) = v_2, \ldots (dx_{k+1}/dt_k) = v_k$. Applying Eq. (26), we obtain

$$dt'_\sigma = dt_\sigma\sqrt{1-\beta^2},$$

that is,

$$(dt'_1)^2 + \cdots + (dt'_k)^2 = (dt_1^2 + \cdots + dt_k^2)(1-\beta^2).$$

(We used the equalities

$$\frac{dt_\theta}{dt_\sigma} = \frac{dt_\theta}{dx_{k+1}} \times \frac{dx_{k+1}}{dt_\sigma} = \frac{v_\sigma}{v_\theta},$$

where σ, $\theta = 1, 2, \ldots, k$.) Let us apply rotation in the hyperplane of time t'_1-t'_2- $\cdots$-t'_k (see Section VII). We obtain the equality

$$\begin{aligned}(dt''_1)^2 + \cdots + (dt''_k)^2 &= (dt'_1)^2 + \cdots + (dt'_k)^2 \\ &= (dt_1^2 + \cdots + dt_k^2)(1-\beta^2).\end{aligned} \tag{28}$$

If we set $dt''_1 = dt''_2 = \cdots = dt''_k > 0$, then we obtain the following equalities:

$$\begin{aligned}dt''_1 &= dt''_2 = \cdots = dt''_k \\ &= \frac{1}{\sqrt{k}}\sqrt{(dt_1^2 + dt_2^2 + \cdots + dt_k^2)(1-\beta^2)}.\end{aligned}$$

We will use these equations in Section IX.A.

Let us consider more general transformations, which do not include translation in space and time. Let us introduce the indefinite metric

$$\mathbf{I}_{k,n} = \mathrm{diag}(\underbrace{1,\ldots,1}_{k}, \underbrace{-1,\ldots,-1}_{n}).$$

Let us denote with $O(k, n, \mathbb{R})$ the group of all $(k+n)\times(k+n)$ matrices $\mathbf{M}$ for which $\mathbf{M}^{\mathrm{tr}}(\mathbf{I}_{k,n})^{-1}\mathbf{M} = \mathbf{I}_{k,n}$. (Here $\mathbf{M}^{\mathrm{tr}}$ denotes the transpose of the matrix $\mathbf{M}$ and $\mathbb{R}$ denotes the field of real numbers.) The inverse of $\mathbf{M}$ is given by $\mathbf{M}^{-1} = (\mathbf{I}_{k,n})^{-1}\mathbf{M}^{\mathrm{tr}}\mathbf{I}_{k,n}$. The indefinite orthogonal group $O(k, n, \mathbb{R})$ conserves the quadratic form defined through the metrics $\mathbf{I}_{k,n}$ and is isomorphic to the group of all proper and improper rotations in the space-time being considered. The indefinite special orthogonal group $SO(k, n, \mathbb{R})$ is the subgroup of $O(k, n, \mathbb{R})$ consisting of all elements with determinant +1. The group $SO(k, n, \mathbb{R})$ corresponds to the group of all proper rotations in the space-time. The proper Lorentz group of four-dimensional Minkowski space-time is $SO(1, 3, \mathbb{R})$; together with parity and time-reversal it becomes $O(1, 3, \mathbb{R})$. If we introduce an $(n+k)$-dimensional vector of translation in the space-time, then by analogy with $(n, k) = (3, 1)$—that is, the inhomogeneous Lorentz group or Poincaré group in STR—we can obtain the most general transformations [see Eq. (18)].

VI. VELOCITY-ADDITION LAW

From the transformations obtained in Eqs. (26) and (27), it is easy to derive the velocity-addition formulas. Let us denote $(dx_\eta/dt_\sigma) = V_{\sigma\eta}$ and $x'_\eta/dt'_\sigma = V'_{\sigma\eta}$, where $\sigma = 1, 2, \ldots, k$ and $\eta = k+1, k+2, \ldots, k+n$. The velocity-addition formulas are given as follows:

$$V'_{\sigma(k+1)} = \frac{V_{\sigma(k+1)}\zeta\left[1 - \beta^2\sum_{\theta=1}^{k}\frac{c^2}{v_\theta V_{\theta(k+1)}}\right]}{1 + \frac{V_{\sigma(k+1)}}{v_\sigma}\beta^2\left[(\zeta-1)\sum_{\theta=1}^{k}\frac{c^2}{v_\theta V_{\theta(k+1)}} - \zeta\right]}, \tag{29}$$

$$V'_{\sigma\varphi} = \frac{V_{\sigma\varphi}}{1 + \frac{V_{\sigma(k+1}}{v_\sigma}\beta^2\left[(\zeta-1)\sum_{\theta=1}^{k}\frac{c^2}{v_\theta V_{\theta(k+1)}} - \zeta\right]}, \tag{30}$$

where σ, $\theta = 1, 2, \ldots, k$ and $\varphi = k+2, k+3, \ldots, k+n$. We have used the equalities

$$\frac{dt_\theta}{dt_\sigma} = \frac{dx_\eta}{dt_\sigma} \times \frac{dt_\theta}{dx_\eta} = \frac{V_{\sigma\eta}}{V_{\theta\eta}}.$$

We note that $(V_{1\eta}/V_{1\pi}) = (V_{2\eta}/V_{2\pi}) = \cdots = (V_{k\eta}/V_{k\pi})$, where η, $\pi = k+1, k+2, \ldots, k+n$. Indeed, $(V_{\sigma\eta}/V_{\sigma\pi}) = (dx_\eta/dx_\pi)$, where $\sigma = 1, 2, \ldots, k$.

VII. ROTATIONS IN THE HYPERPLANE OF TIME AND IN THE HYPERPLANE OF SPACE

Let us assume that a particle under consideration is moving in k time dimensions ($t_1, t_2, \ldots, t_k$) and in n space dimensions ($x_{k+1}, x_{k+2}, \ldots, x_{k+n}$). According to our considerations, separately and independently from each other we can apply some rotations of the time axes ($t_1, t_2, \ldots, t_k$) in the hyperplane of time and some rotations of the space axes ($x_{k+1}, x_{k+2}, \ldots, x_{k+n}$) in the hyperplane of space (see Section II and Subsection IX.A). These transformations are expressed in a change of the basis of the time axes or of the space axes of the frame of reference under consideration (these are the so-called passive linear transformations). The group of all proper and improper rotations in the hyperplane of time is isomorphic to the orthogonal group $O(k, \mathbb{R})$, and the group of all proper and improper rotations in the hyperplane of space is isomorphic to the orthogonal group $O(n, \mathbb{R})$, where $\mathbb{R}$ denotes the field of real numbers. During the simultaneous rotation of the space axes in the hyperplane of the space and the time axes in the hyperplane of time, the origin will remain invariant. The time interval $dT^2 = dt_1^2 + dt_2^2 + \cdots + dt_k^2$, the space interval $dX^2 = dx_{k+1}^2 + dx_{k+2}^2 + \cdots + dx_{k+n}^2 dx_{k+2}^2$, and the $(n+k)$-dimensional interval $ds_{n,k}^2 = c^2 dT^2 - dX^2$ are invariant during these operations.

Let us apply a proper or improper rotation of the time axes $t_1, t_2, \ldots, t_k$ in the hyperplane of time. The new time axes obtained after this transformation we will denote with $t'_1, t'_2, \ldots, t'_k$. Let us denote $d\mathbf{T} = (dt_1, dt_2, \ldots, dt_k)$. For the vector components of $d\mathbf{T}$ before and after the transformation we have $(dt'_1)^2 + (dt'_2)^2 + \cdots + (dt'_k)^2 = dt_1^2 + dt_2^2 + \cdots + dt_k^2$ .The transformation under consideration can be presented through the orthogonal matrix $\mathbf{A} = [a_{\varsigma\sigma}]_{k\times k}$, which belongs to the orthogonal group $O(k, \mathbb{R})$. Here, $\varsigma, \sigma = 1, 2, \ldots, k$. Then we will have $\mathbf{A}^{\mathrm{tr}} = \mathbf{A}^{-1}$, $\det(\mathbf{A}) = +1$ [proper rotation, $\mathbf{A} \in SO(k, \mathbb{R})$] or $\det(\mathbf{A}) = -1$ (improper rotation). The relation between the components $dt'_1, dt'_2, \cdots, dt'_k$ and the components $dt_1, dt_2, \ldots, dt_k$ is given by $d\mathbf{T}' = d\mathbf{T} \times \mathbf{A}$—that is,

$$dt'_\sigma = \sum_{\varsigma=1}^{k} dt_\varsigma a_{\varsigma\sigma}.$$

Let us set $dt'_1 = dt'_2 = \cdots = dt'_k = dT/\sqrt{k} > 0$. This is possible if the size of the angle made by the vector $d\mathbf{T}$ and each of the time axes $t'_1, t'_2, \ldots, t'_k$ is equal to $\pi/4$.

We can similarly consider a proper or improper rotation of the space axes $x_{k+1}, x_{k+2}, \ldots, x_{k+n}$ in the hyperplane of space.

VIII. GENERALIZATION OF THE PRINCIPLE OF INVARIANCE OF THE SPEED OF LIGHT IN STR

In the case of multidimensional time, the principle of invariance of the speed of light defined in STR is violated. This requires a generalization of this principle for the case $k \geq 1$.

Let us assume that the particle under consideration is moving in k time dimensions, denoted with $t_1, t_2, \ldots, t_k$, and in n space dimensions, denoted with $x_{k+1}, x_{k+2}, \ldots, x_{k+n}$. We consider the motion of the particle in relation to the frame of reference K. The $(n+k)$-dimensional interval in this case is expressed as $ds_{n,k}^2 = c^2 dt_1^2 + \cdots + c^2 dt_k^2 - dx_{k+1}^2 - \cdots - dx_{k+n}^2$. Let us set

$$V_\theta = \frac{\sqrt{\sum_{\eta=k+1}^{k+n} dx_\eta^2}}{|dt_\theta|},$$

where $\theta = 1, 2, \ldots, k$. The total coordinate velocity in this case (see Section II) is equal to

$$u = \sqrt{\frac{\sum_{\eta=k+1}^{k+n} dx_\eta^2}{\sum_{\theta=1}^{k} dt_\theta^2}} > 0.$$

Let us now assume that $ds_{n,k}^2 = 0$. It is easy to prove that in this case we have $u = c$—that is,

$$\sum_{\theta=1}^{k} \frac{c^2}{V_\theta^2} = 1. \tag{31}$$

In Eq. (31), let us set $V_1 = V_2 = \cdots = V_k$. This can be done by appropriate rotation in the hyperplane of time so that the condition $dt_1 = dt_2 = \cdots = dt_k$ is fulfilled (see Sec. VII). Then we will obtain

$$V_1 = V_2 = \cdots = V_k = c\sqrt{k}. \tag{32}$$

Let us consider the motion of the particle in relation to the reference frame K', which is moving uniformly and rectilinearly in relation to K. The $(n+k)$-dimensional interval in this case is given by the expression $(ds'_{n,k})^2 = c^2(dt'_1)^2 + \cdots + c^2(dt'_k)^2 - (dx'_{k+1})^2 - \cdots - (dx'_{k+n})^2$. Let us denote

$$V'_\theta = \frac{\sqrt{\sum_{\eta=k+1}^{k+n} (dx'_\eta)^2}}{|dt'_\theta|},$$

$(\theta = 1, 2, \ldots, k)$, and

$$u' = \sqrt{\frac{\sum_{\eta=k+1}^{k+n} (dx'_\eta)^2}{\sum_{\theta=1}^{k} (dt'_\theta)^2}}.$$

Since the $(n+k)$-dimensional interval $ds_{n,k}^2$ is invariant, the equality $ds_{n,k}^2 = (ds'_{n,k})^2 = 0$ will be fulfilled. From here it follows that $u' = c$—that is,

$$\sum_{\theta=1}^{k} \frac{c^2}{(V'_\theta)^2} = 1. \tag{33}$$

In Eq. (33), let us set $V'_1 = V'_2 = \cdots = V'_k$. This can be obtained through an appropriate rotation in the hyperplane of time so that the condition $dt'_1 = dt'_2 = \cdots = dt'_k$ is fulfilled (see Section VII). Then we will obtain

$$V'_1 = V'_2 = \cdots = V'_k = c\sqrt{k}. \tag{34}$$

Let us summarize: If for the velocities u and V_θ of a particle in K Eq. (31) is fulfilled, then for the velocities u' and V'_θ of the particle in K' the similar Eq. (33) will be fulfilled. If for the velocities V_θ of a particle in K Eq. (32) is fulfilled, then for the velocities V'_θ of this particle in K' the similar Eq. (34) will be fulfilled. These two statements are equivalent. For $k = 1$ we obtain the principle of constancy of the speed of light in a vacuum, defined in STR: If $V = c$, then $V' = c$.

IX. MOTION OF A PARTICLE IN *n*-DIMENSIONAL SPACE AND IN *k*-DIMENSIONAL TIME

A. Proper time and generalized velocity

Let us assume that the particle under consideration is moving in k temporal dimensions, denoted with $t_1, t_2, \ldots, t_k$, and in n spatial dimensions, denoted with

x_{k+1}, x_{k+2}, …, x_{k+n}. By analogy with the term "proper time" in STR, we will introduce the term "proper time" for the case $k \geq 1$, which we will denote with $\mathbf{T}_0$.

Since time has k dimensions, the proper time $d\mathbf{T}_0$ will be a k-dimensional vector. We have $d\mathbf{T}_0 = (dt_{01}, dt_{02}, \ldots, dt_{0k})$, where dt_{01}, dt_{02}, …, dt_{0k} are the projections of the proper time on the different time axes. Let us set

$$dT_0 = ||d\mathbf{T}_0|| = \sqrt{\sum_{\theta=1}^{k} dt_{0\theta}^2} \geq 0.$$

Let us consider the motion of a particle in relation to the frame K. The $(n + k)$-dimensional interval, being invariant, is given in this case by the expression $ds_{n,k}^2 = c^2 dt_1^2 + \cdots + c^2 dt_k^2 - dx_{k+1}^2 - \cdots - dx_{k+n}^2 \geq 0$.

By analogy with STR, we have to assume that the proper time is invariant. If we consider the quotient of the two invariant quantities $ds_{n,k}$ and c, we obtain an invariant value having the physical dimension of time. We assume that this value is equal to the length of the vector $d\mathbf{T}_0$. In the general case ($k \geq 1$), the obtained value will be the proper time. We will have $dT_0 = |ds_{n,k}|/c$. Hence, the length of the vector $d\mathbf{T}_0$ is invariant.

In order to understand the physical meaning of proper time, we will make the following considerations: We will have $ds_{n,k}^2 = c^2(dt_1^2 + dt_2^2 + \cdots + dt_k^2)(1 - \beta^2) = c^2 dT_0^2$, where

$$\beta = \frac{1}{\sqrt{\sum_{\varsigma=1}^{k} \frac{c^2}{V_\varsigma^2}}},$$

$$V_\varsigma = \frac{\sqrt{\sum_{\pi=k+1}^{k+n} dx_\pi^2}}{|dt_\varsigma|} > 0.$$

Let us assume that the $(n + k)$-dimensional vector of the location of the particle $\mathbf{R} = (ct_1, \ldots, ct_k, x_{k+1}, \ldots, x_{k+n})$ is a function of the proper time $\mathbf{T}_0 = (t_{01}, t_{02}, \ldots, t_{0k})$—that is, $x_\mu = x_\mu(\mathbf{T}_0)$, where $\mu = 1, 2, \ldots, k + n$; here $x_\mu = ct_\mu$ at $\mu = 1, 2, \ldots, k$. Let us consider the motion of a particle in the inertial frame of reference K_0, the velocity of which at a given moment $\mathbf{T}_0 = (t_{01}, t_{02}, \ldots, t_{0k})$ coincides with the velocity of the particle. Let the coordinates of the particle at this moment be $x_{0\mu} = x_{0\mu}(\mathbf{T}_0)$. Since at the considered moment the particle is at rest relative to K_0, at $\mu = k+1, k+2, \ldots, k+n$ we will have $x_{0\mu} = x_{0\mu}(\mathbf{T}_0 + d\mathbf{T}_0)$. So we will obtain $x_{01}(\mathbf{T}_0 + d\mathbf{T}_0) - x_{01}(\mathbf{T}_0) = dx_{01}$, $x_{02}(\mathbf{T}_0 + d\mathbf{T}_0) - x_{02}(\mathbf{T}_0) = dx_{02}$, ⋯, $x_{0k}(\mathbf{T}_0 + d\mathbf{T}_0) - x_{0k}(\mathbf{T}_0) = dx_{0k}$.

Let us set $dx_{0\theta} = c\,dt_{0\theta}$, where $\theta = 1, 2, \ldots, k$. Here $t_{0\theta} = x_{0\theta}/c$ are the time axes of the frame K_0. Hence we have $c\,dT_0 = ds_{n,k} = ds_{n,k}^0 = c\,dT_0^0$, where $dT_0^0 = \sqrt{dt_{01}^2 + dt_{02}^2 + \cdots + dt_{0k}^2}$. We have $dT_0 = dT_0^0$—that is,

$$dT_0^2 = dt_{01}^2 + \cdots + dt_{0k}^2 = (dt_1^2 + \cdots + dt_k^2)(1 - \beta^2). \tag{35}$$

The obtained formula coincides with Eq. (28) (see Section V). We can see that the values $dt_{01}^2, dt_{02}^2, \cdots, dt_{0k}^2$ are transferred according to the same law as $(dt_1'')^2, (dt_2'')^2, \cdots, (dt_k'')^2$ in Eq. (28)—which had to be expected. The frame K_0 we will call the proper frame of reference. Taking into consideration Eqs. (28) and (35), we can understand the physical meaning of proper time: It is the time measured in the proper frame of reference K_0 where the particle is at rest. The values dt_{01}, dt_{02}, …, dt_{0k} are projections of the proper time $d\mathbf{T}_0$ on the axes t_{01}, t_{02}, …, t_{0k}, respectively, of K_0.

We note that if $\sum_{\varsigma=1}^{k}(c^2/V_\varsigma^2) = 1$—that is, $ds_{n,k}^2 = 0$ and accordingly $dT_0^2 = 0$—then the particle will not move in time (for the observed particle, time will not exist). Indeed, in this case we have $dT_0^2 = dt_{01}^2 + dt_{02}^2 + \cdots + dt_{0k}^2 = 0$—that is, $dt_{01}^2 = dt_{02}^2 = \cdots = dt_{0k}^2 = 0$. Thus, for the observed particle the time axes t_{01}, t_{02}, …, t_{0k} do not exist.

Let us set $dt_{0\theta} = \alpha_{0\theta} dT_0$. We have $\sum_{\theta=1}^{k} \alpha_{0\theta}^2 = 1$—that is, $|\alpha_{0\theta}| \leq 1$. Let us set $dX = \sqrt{\sum_{\eta=k+1}^{k+n} dx_\eta^2} > 0$, $dT = \sqrt{\sum_{\sigma=1}^{k} dt_\sigma^2} > 0$. From Eq. (35) we obtain $dT_0 = dT\sqrt{1 - \beta^2}$.

If we use the equalities $ds_{n,k}^2 - c^2 dT_0^2 = 0$, $dT_0^2 = \sum_{\theta=1}^{k} dt_{0\theta}^2 = (dt_{0\theta}/\alpha_{0\theta})^2$, we will have

$$\begin{aligned} &c^2 dt_1^2 + \cdots + c^2 dt_k^2 - dx_{k+1}^2 - \cdots - dx_{k+n}^2 - c^2 dt_{01}^2 \\ &\quad - \cdots - c^2 dt_{0k}^2 \\ &\quad = 0, \end{aligned} \tag{36}$$

$$\begin{aligned} &c^2 dt_1^2 + \cdots + c^2 dt_k^2 - dx_{k+1}^2 - \cdots - dx_{k+n}^2 - c^2 dT_0^2 \\ &\quad = 0, \end{aligned} \tag{37}$$

$$\begin{aligned} &c^2 dt_1^2 + \cdots + c^2 dt_k^2 - dx_{k+1}^2 - \cdots - dx_{k+n}^2 \\ &\quad - c^2 \left(\frac{dt_{0\theta}}{\alpha_{0\theta}}\right)^2 \\ &\quad = 0, \end{aligned} \tag{38}$$

$$c^2 dT^2 - dX^2 - c^2 dT_0^2 = 0. \tag{39}$$

According to Eq. (36), instead of $(n + k)$-dimensional space-time, we can consider a generalized $[(n + k) + k]$-dimensional space-time, where k dimensions are related to the projections of the proper time. In the generalized space-time, k dimensions are timelike (ct_1, ct_2, …, ct_k) and $(n + k)$ dimensions are spacelike (x_{k+1}, x_{k+2}, …, x_{k+n}, ct_{01}, ct_{02}, …, ct_{0k}). In accordance with Eqs. (37) and (38), instead of $(n + k)$-dimensional space-time, we can consider a generalized $[(n + 1) + k]$-dimensional space-time, where the additional dimension is related to the proper time cT_0 or to the proper time $(ct_{0\theta}/\alpha_{0\theta})$. In this case, k dimensions are timelike (ct_1, ct_2, …, ct_k) and $(n + 1)$ dimensions are spacelike (x_{k+1}, x_{k+2}, …, x_{k+n}, cT_0 or x_{k+1}, x_{k+2}, …, x_{k+n}, $ct_{0\theta}/\alpha_{0\theta}$). According to Eq. (39), instead of $(n + k)$-dimensional space-time, we can consider a generalized (2 + 1)-dimensional space-time. In this case, one of the dimensions is timelike (cT) and two of the dimensions are spacelike (X, cT_0). Similar considerations are valid in STR (see Section II).

By analogy with STR, in the case of $(n + k)$-dimensional space-time we can define the generalized velocity $\mathbf{U}$. Let us assume that the location of an observed particle in the frame K_0 is given by the $(n+k)$-dimensional vector $\mathbf{R}_0 = (ct_{01}, \ldots, ct_{0k}, x_{0(k+1)}, \ldots, x_{0(k+n)})$; then the location of the particle in K is given by the $(n + k)$-dimensional vector $\mathbf{R} = (ct_1, \ldots, ct_k, x_{k+1}, \ldots, x_{k+n})$. If the location of the particle in K_0 is given by the $(n + k)$-dimensional vector $\mathbf{R}_0 + d\mathbf{R}_0 = [ct_{01} + cdt_{01}, \ldots, ct_{0k} + cdt_{0k}, x_{0(k+1)}, x_{0(k+2)}, \ldots, x_{0(k+n)}]$, then the location of the particle in K is given by the $(n+k)$-dimensional vector $\mathbf{R} + d\mathbf{R} = (ct_1 + cdt_1, \ldots, ct_k + cdt_k, x_{k+1} + dx_{k+1}, \ldots, x_{k+n} + dx_{k+n})$. For the case of multidimensional time we have $d\mathbf{T}_0 \times \mathbf{U} = d\mathbf{R}$, where $d\mathbf{T}_0 = (dt_{01}, dt_{02}, \ldots, dt_{0k})$ and $d\mathbf{R} = (cdt_1, cdt_2, \ldots, cdt_k, dx_{k+1}, dx_{k+2}, \ldots, dx_{k+n})$—see also Section II. It is easy to prove that the generalized velocity $\mathbf{U}$ is a $k \times (k+n)$ matrix with elements $u_{\theta\mu}$ $(\theta = 1, 2, \ldots, k;$ $\mu = 1, 2, \ldots, k+n)$—that is, $\mathbf{U} = [u_{\theta\mu}]_{k\times(k+n)}$. Let us denote

$$\mathbf{u}_\mu = \begin{pmatrix} u_{1\mu} \\ u_{2\mu} \\ \vdots \\ u_{k\mu} \end{pmatrix}.$$

Then we have $d\mathbf{T}_0 \times \mathbf{u}_\sigma = cdt_\sigma, d\mathbf{T}_0 \times \mathbf{u}_\eta = dx_\eta$ $(\sigma = 1, 2, \ldots, k;\ \eta = k+1, \ldots, k+n)$—that is,

$$\frac{dt_{01}}{cdt_\sigma} u_{1\sigma} + \frac{dt_{02}}{cdt_\sigma} u_{2\sigma} + \cdots + \frac{dt_{0k}}{cdt_\sigma} u_{k\sigma} = 1, \tag{40}$$

$$\frac{dt_{01}}{dx_\eta} u_{1\eta} + \frac{dt_{02}}{dx_\eta} u_{2\eta} + \cdots + \frac{dt_{0k}}{dx_\eta} u_{k\eta} = 1. \tag{41}$$

It is clear that if for a given δ $(1 \le \delta \le k)$ it is true that $dt_{0\delta} = 0$, then the components of the velocity $u_{\delta 1}, u_{\delta 2}, \ldots, u_{\delta(k+n)}$ will be undefined values. If for a given φ it is true that $dt_\varphi = 0$ (where $1 \le \varphi \le k$) or $dx_\varphi = 0$ (where $k+1 \le \varphi \le k+n$), then $u_{1\varphi} = u_{2\varphi} = \cdots = u_{k\varphi} = 0$.

Let us assume that $u_{\theta\sigma} = \lambda_{\theta\sigma}(cdt_\sigma/dt_{0\theta}), u_{\theta\eta} = \lambda_{\theta\eta}(dx_\eta/dt_{0\theta})$, where $\theta, \sigma = 1, 2, \ldots, k$; $\eta = k+1, k+2, \ldots, k+n$. Then we have $\sum_{\theta=1}^{k} \lambda_{\theta\mu} = 1$, where $\mu = 1, 2, \ldots, k+n$.

Let us set $dt_\sigma = \chi_\sigma dT$ and $dx_\eta = \chi_\eta dX$ $(\sigma = 1, 2, \ldots, k;\ \eta = k+1, k+2, \ldots, k+n)$. We then have $\sum_{\sigma=1}^{k} \chi_\sigma^2 = 1$ and $\sum_{\eta=k+1}^{k+n} \chi_\eta^2 = 1$—that is, $|\chi_\sigma| \le 1$ and $|\chi_\eta| \le 1$.

Let us set

$$U_{t\sigma} = \sqrt{\sum_{\theta=1}^{k} u_{\theta\sigma}^2} = \frac{c|dt_\sigma|}{dT_0} \sqrt{\sum_{\theta=1}^{k} \left(\frac{\lambda_{\theta\sigma}}{\alpha_{0\theta}}\right)^2},$$

$$U_{t0\theta} = \sqrt{\sum_{\sigma=1}^{k} u_{\theta\sigma}^2} = \frac{cdT}{|dt_{0\theta}|} \sqrt{\sum_{\sigma=1}^{k} (\lambda_{\theta\sigma}\chi_\sigma)^2},$$

$$U_t = \sqrt{\sum_{\sigma=1}^{k} \sum_{\theta=1}^{k} u_{\theta\sigma}^2}.$$

Then for the velocity U_t we have

$$U_t = \sqrt{\sum_{\sigma=1}^{k} U_{t\sigma}^2} = \sqrt{\sum_{\theta=1}^{k} U_{t0\theta}^2} = \frac{cdT}{dT_0} \sqrt{\sum_{\sigma=1}^{k} \sum_{\theta=1}^{k} \left(\frac{\lambda_{\theta\sigma}\chi_\sigma}{\alpha_{0\theta}}\right)^2}. \tag{42}$$

Let us set

$$U_{s\eta} = \sqrt{\sum_{\theta=1}^{k} u_{\theta\eta}^2} = \frac{|dx_\eta|}{dT_0} \sqrt{\sum_{\theta=1}^{k} \left(\frac{\lambda_{\theta\eta}}{\alpha_{0\theta}}\right)^2},$$

$$U_{s\theta} = \sqrt{\sum_{\eta=k+1}^{k+n} u_{\theta\eta}^2} = \frac{dX}{|dt_{0\theta}|} \sqrt{\sum_{\eta=k+1}^{k+n} (\lambda_{\theta\eta}\chi_\eta)^2},$$

$$U_s = \sqrt{\sum_{\eta=k+1}^{k+n} \sum_{\theta=1}^{k} u_{\theta\eta}^2}.$$

Then for the velocity U_s we have

$$U_s = \sqrt{\sum_{\eta=k+1}^{k+n} U_{s\eta}^2} = \sqrt{\sum_{\theta=1}^{k} U_{s\theta}^2} = \frac{dX}{dT} \sqrt{\sum_{\eta=k+1}^{k+n} \sum_{\theta=1}^{k} \left(\frac{\lambda_{\theta\eta}\chi_\eta}{\alpha_{0\theta}}\right)^2}. \tag{43}$$

We are going to show the conditions for the velocities $U_{t\sigma}, U_{t0\theta}, U_t$ and $U_{s\eta}, U_{s\theta}, U_s$. These conditions are imposed by physical considerations (see also Sections II and VII).

Since during a change of the basis of the time axes of K_0 (the so-called passive linear transformation) the value of dT_0^2 does not change, the values dX^2 and dT^2 remain the same as well [see Eq. (39)]. Likewise, during a change of the basis of the spatial or the temporal axes of K, the values dX^2, dT^2, and dT_0^2 remain constant. It is clear that the temporal axes of the frame K_0 and the spatial and temporal axes of frame K can be chosen independently from each other (see Sections II and VII). Therefore, when applying these transformations, some physical values (in this case velocities and consequently energy or momentum—see further) must remain invariant.

In the expression for the velocity $U_{t\sigma}$, all time axes of K_0 are included in the denominator $(dT_0 = \sqrt{\sum_{\theta=1}^{k} dt_{0\theta}^2})$; we thus assume that the velocity $U_{t\sigma}$ is invariant during a change of the basis of the time axes of K_0. Therefore, the velocity $U_{t\sigma}$ can be presented in the form $U_{t\sigma} = \gamma_{t\sigma}(|cdt_\sigma|/dT_0)$, where $\gamma_{t\sigma} > 0$ is a parameter that does not depend on $dt_{0\theta}$ or, consequently, on the numbers $\alpha_{0\theta}$ $(\theta = 1, 2, \ldots, k)$.

In the expression for the velocity $U_{t0\theta}$, all time axes of K are included in the numerator $(dT = \sqrt{\sum_{\sigma=1}^{k} dt_\sigma^2})$; we thus assume that the velocity $U_{t0\theta}$ is invariant during a change of the basis of the time axes of K. Therefore, the velocity $U_{t0\theta}$ can be presented in the form $U_{t0\theta} = \gamma_{t0\theta}(cdT/|dt_{0\theta}|)$, where $\gamma_{t0\theta} > 0$ is a parameter that does not depend on dt_σ or, consequently, on the numbers χ_σ (σ = 1, 2, ..., k).

From here we can conclude that the velocity U_t is invariant during a change of the basis of the time axes of K_0 and during a change of the basis of time axes of K—i.e., the velocity U_t can be presented in the form $U_t = \gamma_t(cdT/dT_0)$, where $\gamma_t > 0$ is a parameter that does not depend on $dt_{0\theta}$ and dt_σ or, consequently, on the numbers $\alpha_{0\theta}$ and χ_σ (σ, $\theta = 1, 2, \ldots, k$)—see Eq. (42). Indeed,

$$U_t = \sqrt{\sum_{\sigma=1}^{k} U_{t\sigma}^2} = \frac{cdT}{dT_0}\sqrt{\sum_{\sigma=1}^{k} \chi_\sigma^2\gamma_{t\sigma}^2} = \sqrt{\sum_{\theta=1}^{k} U_{t0\theta}^2} = \frac{cdT}{dT_0}\sqrt{\sum_{\theta=1}^{k}\frac{\gamma_{t0\theta}^2}{\alpha_{0\theta}^2}}.$$

Let us set

$$\gamma_t = \sqrt{\sum_{\sigma=1}^{k} \chi_\sigma^2\gamma_{t\sigma}^2} = \sqrt{\sum_{\theta=1}^{k}\frac{\gamma_{t0\theta}^2}{\alpha_{0\theta}^2}}.$$

Since the values χ_σ and $\gamma_{t\sigma}$ do not depend on the numbers $\alpha_{0\theta}$, the parameter $\gamma_t = \sqrt{\sum_{\sigma=1}^{k}\chi_\sigma^2\gamma_{t\sigma}^2}$ will not either. Since the values $\alpha_{0\theta}$ and $\gamma_{t0\theta}$ do not depend on the numbers χ_σ, the parameter

$$\gamma_t = \sqrt{\sum_{\theta=1}^{k}\frac{\gamma_{t0\theta}^2}{\alpha_{0\theta}^2}}$$

will not either. Therefore, the parameter γ_t does not depend on the numbers $\alpha_{0\theta}$ and χ_σ or, consequently, on $dt_{0\theta}$ and dt_σ.

In the expression for the velocity $U_{s\eta}$, all the time axes of K_0 are included in the denominator ($dT_0 = \sqrt{\sum_{\theta=1}^{k} dt_{0\theta}^2}$); we thus assume that the velocity $U_{s\eta}$ is invariant during a change of the basis of the time axes of K_0. Therefore, the velocity $U_{s\eta}$ can be presented in the form $U_{s\eta} = \gamma_{s\eta}(|dx_\eta|/dT_0)$, where $\gamma_{s\eta} > 0$ is a parameter that does not depend on $dt_{0\theta}$ or, consequently, on the numbers $\alpha_{0\theta}$ ($\theta = 1, 2, \ldots, k$)—see Eq. (4).

In the expression for the velocity $U_{s\theta}$, all the space axes of the frame K are included in the numerator ($dX = \sqrt{\sum_{\eta=k+1}^{k+n} dx_\eta^2}$); we thus assume that the velocity $U_{s\theta}$ is invariant during a change of the basis of the space axes of K. Therefore, the velocity $U_{s\theta}$ can be presented in the form $U_{s\theta} = \gamma_{s\theta}(dX/|dT_{0\theta}|)$, where $\gamma_{s\theta} > 0$ is a parameter that does not depend on dx_η or, consequently, on the numbers χ_η ($\eta = k+1, k+2, \ldots, k+n$)—see Eq. (8).

We can prove by analogy that the velocity U_s is invariant during a change of the basis of the time axes of K_0 and during a change of the basis of the space axes of K—i.e., the velocity U_s can be presented in the form $U_s = \gamma_s(dX/dT_0)$, where $\gamma_s > 0$ is a parameter that does not depend on $dt_{0\theta}$ and dx_η or, consequently, on the numbers $\alpha_{0\theta}$ and χ_η ($\theta = 1, 2, \ldots, k$; $\eta = k+1, k+2, \ldots, k+n$)—see Section II and Eq. (43).

Let us set

$$U = \sqrt{\sum_{\mu=1}^{k+n}\sum_{\theta=1}^{k} u_{0\mu}^2} = \sqrt{U_t^2 + U_s^2}.$$

Since $U_t = \gamma_t(cdT/dT_0)$ and $U_s = \gamma_s(dX/dT_0)$, we obtain

$$U^2 = \gamma_t^2\left(\frac{cdT}{dT_0}\right)^2 + \gamma_s^2\left(\frac{dX}{dT_0}\right)^2.$$

In support of these assumptions we can present one more argument. In STR, the real space-time is compared to the dual space of the energy-momentum. These considerations must apply also for the case of multidimensional time. Therefore, between the total energy and total momentum in the multidimensional time there should be a relation similar to the one in Eq. (39). The velocities U_t and U_s participate accordingly in the formulas for the total energy and for the total momentum—see Eqs. (61), (71), and (73). If the equalities $U_t = \gamma_t(cdT/dT_0)$ and $U_s = \gamma_s(dX/dT_0)$ are valid, then between the total energy and the total momentum there will be a dependence which is similar to the relation in Eq. (39)—see Eqs. (73) and (74).

Let us assume that $\alpha_{01} = \alpha_{02} = \cdots = \alpha_{0k} = 1/\sqrt{k}$ (that is, $dt_{01} = dt_{02} = \cdots = dt_{0k} = dT_0/\sqrt{k} > 0$), $\chi_1 = \chi_2 = \cdots = \chi_k = 1/\sqrt{k}$ (that is, $dt_1 = dt_2 = \cdots = dt_k = dT/\sqrt{k} > 0$), and $\chi_{k+1} = \chi_{k+2} = \cdots = \chi_{k+n} = 1/\sqrt{n}$, (that is, $dx_{k+1} = dx_{k+2} = \cdots = dx_{k+n} = dX/\sqrt{n} > 0$). Let us apply a proper or improper rotation of the time axes of the frame K_0 in the hyperplane of the time. The transformation under consideration can be presented through the orthogonal matrix $\mathbf{A} = [a_{\theta\varsigma}]_{k\times k}$, belonging to the orthogonal group O(k, $\mathbb{R}$), where $\mathbb{R}$ denotes the real numbers field (see also Section VII). The new time axes obtained after applying this transformation we will denote with $t'_{0\theta}$ (θ = 1, 2, ..., k). Since $\sum_{\theta=1}^{k}(\alpha'_{0\theta})^2 = \sum_{\theta=1}^{k}\alpha_{0\theta}^2 = 1$, we will have

$$\begin{pmatrix}\alpha'_{01}\\ \alpha'_{02}\\ \vdots\\ \alpha'_{0k}\end{pmatrix} = \mathbf{A}\times\begin{pmatrix}\alpha_{01}\\ \alpha_{02}\\ \vdots\\ \alpha_{0k}\end{pmatrix},$$

that is,

$$\alpha'_{0\theta} = \sum_{\varsigma=1}^{k} a_{\theta\varsigma}\alpha_{0\varsigma} = \frac{1}{\sqrt{k}}\sum_{\varsigma=1}^{k} a_{\theta\varsigma}. \tag{44}$$

Since the velocity $U_{t\sigma}$ is invariant during a change of the basis of the time axes of K_0, the expression

$$\sum_{\theta=1}^{k}\left(\frac{\lambda_{\theta\sigma}}{\alpha_{0\theta}}\right)^2$$

remains constant during a change of the numbers $\alpha_{0\theta}$—that is,

$$\sum_{\theta=1}^{k}\left(\frac{\lambda'_{\theta\sigma}}{\alpha'_{0\theta}}\right)^2 = \sum_{\theta=1}^{k}\left(\frac{\lambda_{\theta\sigma}}{\alpha_{0\theta}}\right)^2 = k\sum_{\theta=1}^{k}\lambda_{\theta\sigma}^2 = \gamma_{t\sigma}^2, \tag{45}$$

where $\gamma_{t\sigma}$ is a parameter that does not depend on the numbers $\alpha_{0\theta}$ ($\theta = 1, 2, \ldots, k$).

Let us consider the orthogonal matrix $\mathbf{B} = [b_{\theta v}]_{k\times k}$, belonging to the orthogonal group O(k, $\mathbb{R}$). Taking into account Eq. (45), we likewise obtain

$$\frac{\lambda'_{\theta\sigma}}{\alpha'_{0\theta}} = \sum_{v=1}^{k} \frac{b_{\theta v}\lambda_{v\sigma}}{\alpha_{0v}} = \sqrt{k}\sum_{v=1}^{k} b_{\theta v}\lambda_{v\sigma}. \tag{46}$$

From Eqs. (44) and (46) we can define the quantities $\lambda'_{\theta\sigma}$:

$$\lambda'_{\theta\sigma} = \sum_{v=1}^{k}\sum_{\varsigma=1}^{k} \frac{a_{\theta\varsigma}\alpha_{0\varsigma}b_{\theta v}\lambda_{v\sigma}}{\alpha_{0v}} = \sum_{v=1}^{k}\sum_{\varsigma=1}^{k} a_{\theta\varsigma}b_{\theta v}\lambda_{v\sigma}. \tag{47}$$

Let us apply a proper or improper rotation of the time axes of the frame K in the hyperplane of the time. The transformation under consideration can be presented through the orthogonal matrix $\mathbf{J} = [j_{\delta\sigma}]_{k\times k}$, belonging to the orthogonal group O(k, $\mathbb{R}$). The new time axes obtained applying this transformation we will denote with t''_σ ($\sigma = 1, 2, \ldots, k$). Since $\sum_{\sigma=1}^{k}(\chi''_\sigma)^2 = \sum_{\sigma=1}^{k}\chi_\sigma^2 = 1$, we will have $(\chi''_1, \chi''_2, \cdots, \chi''_k) = (\chi_1, \chi_2, \cdots, \chi_k)\times\mathbf{J}$—that is,

$$\chi''_\sigma = \sum_{\delta=1}^{k}\chi_\delta j_{\delta\sigma} = \frac{1}{\sqrt{k}}\sum_{\delta=1}^{k} j_{\delta\sigma}. \tag{48}$$

Since the velocity $U_{t0\theta}$ is invariant during a change of the basis of the temporal axes in K, the expression

$$\sum_{\sigma=1}^{k}(\lambda'_{\theta\sigma}\chi_\sigma)^2$$

remains constant during change of the numbers χ_σ—that is,

$$\sum_{\sigma=1}^{k}(\lambda''_{\theta\sigma}\chi''_\sigma)^2 = \sum_{\sigma=1}^{k}(\lambda'_{\theta\sigma}\chi_\sigma)^2 = \frac{1}{k}\sum_{\sigma=1}^{k}(\lambda'_{\theta\sigma})^2 = \gamma_{t0\theta}^2, \tag{49}$$

where $\gamma_{t0\theta}$ is a parameter that does not depend on χ_σ ($\sigma = 1, 2, \ldots, k$).

Let us consider the orthogonal matrix $\mathbf{L} = [l_{\vartheta\sigma}]_{k\times k}$, belonging to the orthogonal group O(k, $\mathbb{R}$). Taking into account Eq. (49), we likewise obtain

$$\lambda''_{\theta\sigma}\chi''_\sigma = \sum_{\vartheta=1}^{k}\lambda'_{\theta\vartheta}\chi_\vartheta l_{\vartheta\sigma} = \frac{1}{\sqrt{k}}\sum_{\vartheta=1}^{k}\lambda'_{\theta\vartheta}l_{\vartheta\sigma}. \tag{50}$$

From Eqs. (47), (48), and (50) we can define the quantities $\lambda''_{\theta\sigma}$:

$$\lambda''_{\theta\sigma} = \frac{1}{\sum_{\delta=1}^{k}\chi_\delta j_{\delta\sigma}}\left(\sum_{\vartheta=1}^{k}\sum_{v=1}^{k}\sum_{\varsigma=1}^{k}\frac{a_{\theta\varsigma}\alpha_{0\varsigma}b_{\theta v}\lambda_{v\vartheta}\chi_\vartheta l_{\vartheta\sigma}}{\alpha_{0v}}\right) = \frac{\sum_{\vartheta=1}^{k}\sum_{v=1}^{k}\sum_{\varsigma=1}^{k} a_{\theta\varsigma}b_{\theta v}\lambda_{v\vartheta}l_{\vartheta\sigma}}{\sum_{\delta=1}^{k} j_{\delta\sigma}}, \tag{51}$$

where the numbers $\lambda_{v\vartheta}$ are defined provided that $\alpha_{0\theta} = \chi_\sigma = 1/\sqrt{k}$. Since the parameter γ_t does not depend on the numbers $\alpha_{0\theta}$ and χ_σ, we can set $\alpha_{0\theta} = \chi_\sigma = 1/\sqrt{k}$. In this case we have

$$\gamma_t = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}\left(\frac{\lambda_{\theta\sigma}\chi_\sigma}{\alpha_{0\theta}}\right)^2} = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}\lambda_{\theta\sigma}^2}. \tag{52}$$

[See Eq. (42).]

Similar considerations apply for the velocities $U_{s\eta}$ and $U_{s\theta}$ (see Section II). For example, if the quantities $\lambda''_{\theta\eta}$ are defined at arbitrary values of $\alpha'_{0\theta}$ and χ''_η, and the quantities $\lambda_{v\rho}$ are defined provided that $\alpha_{0\theta} = 1/\sqrt{k}$, $\chi_\eta = 1/\sqrt{n}$, then we have

$$\lambda''_{\theta\eta} = \frac{1}{\sum_{\pi=k+1}^{k+n}\chi_\pi h_{\pi\eta}}\left(\sum_{\rho=k+1}^{k+n}\sum_{v=1}^{k}\sum_{\varsigma=1}^{k}\frac{a_{\theta\varsigma}\alpha_{0\varsigma}b_{\theta v}\lambda_{v\rho}\chi_\rho q_{\rho\eta}}{\alpha_{0v}}\right) = \frac{\sum_{\rho=k+1}^{k+n}\sum_{v=1}^{k}\sum_{\varsigma=1}^{k} a_{\theta\varsigma}b_{\theta v}\lambda_{v\rho}q_{\rho\eta}}{\sum_{\pi=k+1}^{k+n} h_{\pi\eta}}. \tag{53}$$

[See Eq. (10).] Here $a_{\theta\varsigma}$, $b_{\theta v}$, $h_{\pi\eta}$, $q_{\rho\eta}$ are elements of the orthogonal matrices $\mathbf{A} = [a_{\theta\varsigma}]_{k\times k}$, $\mathbf{B} = [b_{\theta v}]_{k\times k}$, $\mathbf{H} = [h_{\pi\eta}]_{n\times n}$, $\mathbf{Q} = [q_{\rho\eta}]_{n\times n}$, respectively. [The matrices **A** and **B** belong to the orthogonal group O(k, $\mathbb{R}$), and the matrices **H** and **Q** belong to the orthogonal group P(n, $\mathbb{R}$).] For the parameter γ_s we have

$$\gamma_s = \sqrt{\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}\left(\frac{\lambda_{\theta\eta}\chi_\eta}{\alpha_{0\theta}}\right)^2} = \sqrt{\frac{k}{n}}\sqrt{\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}\lambda_{\theta\eta}^2}. \tag{54}$$

[See Eqs. (11) and (43).] In Eq. (54) the quantities $\lambda_{\theta\eta}$ are determined provided that $\alpha_{0\theta} = 1/\sqrt{k}, \chi_\eta = 1/\sqrt{n}$.

Let us set $\Lambda''_\mu = \sum_{\theta=1}^{k}\lambda''_{\theta\mu}$, $\mu = 1, 2, \ldots, k+n$. [See Eqs. (51) and (53).] In the general case (i.e., at arbitrary values of $\alpha'_{0\theta}$, χ''_σ, and χ''_η), we will have

$$d\mathbf{T}_0 \times \mathbf{U}^{\Lambda''} = d\mathbf{R},$$

where

$$\mathbf{U}^{\Lambda''} = \left[u^{\Lambda''}_{\theta\mu}\right]_{k\times(k+n)}, \qquad u^{\Lambda''}_{\theta\sigma} = \frac{\lambda''_{\theta\sigma}\chi''_\sigma c\,dT}{\Lambda''_\sigma\alpha'_{0\theta}dT_0},$$

$$u^{\Lambda''}_{\theta\eta} = \frac{\lambda''_{\theta\eta}\chi''_\eta dX}{\Lambda''_\eta\alpha'_{0\theta}dT_0}.$$

[See Eqs. (40), (41), (44), (48), (51), and (53).] If $\sum_{\theta=1}^{k}\lambda''_{\theta\mu} \neq \sum_{\theta=1}^{k}\lambda_{\theta\mu}$ (that is, $\Lambda''_\mu \neq 1$, $\mu = 1, 2, \ldots, k+n$), then

$$u^{\Lambda''}_{\theta\sigma} = \frac{\lambda''_{\theta\sigma}\chi''_\sigma c\,dT}{\Lambda''_\sigma\alpha'_{0\theta}dT_0} \neq u''_{\theta\sigma} = \frac{\lambda''_{\theta\sigma}\chi''_\sigma c\,dT}{\alpha'_{0\theta}dT_0}$$

and

$$u_{\theta\eta}^{\Lambda''} = \frac{\lambda''_{\theta\eta}\chi''_{\eta}dX}{\Lambda''_{\eta}\alpha'_{0\theta}dT_0} \neq u''_{\theta\eta} = \frac{\lambda''_{\theta\eta}\chi''_{\eta}dX}{\alpha'_{0\theta}dT_0}.$$

Let us set $u_s = (dX/dT_0) = (U_s/\gamma_s)$. We will say that u_s is the total proper velocity of the particle under consideration. The relation between the total coordinate velocity $u_c = (dX/dT) = (U_c/\gamma)$—see Section II—and the total proper velocity u_s is given through the expression $u_s = u_c\sqrt{1-\beta^2}$. We have $U_s = \gamma_s U_c/(\gamma\sqrt{1-\beta^2})$—see Eqs. (2), (11), (43), and (54). If $dt_{0\theta} = dt_\theta\sqrt{1-\beta^2}$ ($\theta = 1, 2, \ldots, k$), then $\gamma_s = \gamma$ (see Sections II and V).

Let us consider the frame K''', moving uniformly and rectilinearly in relation to K. It is clear that in the frame K''' a particle will have a different generalized velocity $\mathbf{U}'''$, given by the matrix $\mathrm{U}''' = [u'''_{\theta\mu}]_{k\times(k+n)}$. Let us choose the frames K_0 and K''' in such a way that $\alpha_{0\theta} = dt_{0\theta}/dT_0 = 1/\sqrt{k}$, $\chi'''_\sigma = dT'''_\sigma/dT''' = 1/\sqrt{k}$, $\chi'''_\eta = dx'''_\eta/dX''' = 1/\sqrt{n}$, where θ, $\sigma = 1, 2, \ldots, k$; $\eta = k+1, k+2, \ldots, k+n$. In this case the quantities $\lambda'''_{\theta\mu}$,defined in the frame K'''—as with $\lambda_{\theta\mu}$, defined in K—can take arbitrary (real) values. The only condition for this is defined with the equality $\sum_{\theta=1}^{k}\lambda'''_{\theta\mu} = 1$, where $\mu = 1, 2, \ldots, k+n$. (For the quantities $\lambda_{\theta\mu}$, the similar equality $\sum_{\theta=1}^{k}\lambda_{\theta\mu} = 1$ applies.) Due to this fact we can assume that the values of $\lambda'''_{\theta\mu}$ in the frame K''' coincide with the corresponding values of $\lambda_{\theta\mu}$ in the frame K—that is, $\lambda'''_{\theta\mu} = \lambda_{\theta\mu}$, where $\theta = 1, 2, \ldots, k$; $\mu = 1, 2, \ldots, k+n$. (The values $\lambda'''_{\theta\mu}$ are determined provided that $\alpha_{0\theta} = \chi'''_\sigma = 1/\sqrt{k}$, $\chi'''_\eta = 1/\sqrt{n}$, and the values $\lambda_{\theta\mu}$ are determined provided that $\alpha_{0\theta} = \chi_\sigma = 1/\sqrt{k}$, $\chi_\eta = 1/\sqrt{n}$.)

For the velocities $u_{\theta\mu}$ ($\theta = 1, 2, \ldots, k$; $\mu = 1, 2, \ldots, k+n$) we have

$$u_{\theta\sigma} = \lambda_{\theta\sigma}\frac{cdt_\sigma}{dt_{0\theta}} = \frac{\lambda_{\theta\sigma}\chi_\sigma c}{\alpha_{0\theta}\sqrt{1-\beta^2}}, \tag{55}$$

$$u_{\theta\eta} = \lambda_{\theta\eta}\frac{dx_\eta}{dt_{0\theta}} = \frac{\lambda_{\theta\eta}\chi_\eta c\beta}{\alpha_{0\theta}\sqrt{1-\beta^2}}, \tag{56}$$

where $\sigma = 1, 2, \ldots, k$; $\eta = k+1, k+2, \ldots, k+n$.

B. Energy and momentum of a particle moving in *k*-dimensional time

Let us consider the motion of a particle in relation to the frame K. In the case of the one-dimensional time of STR, a (3 + 1)-dimensional energy-momentum vector is defined. Likewise, in the case of multidimensional time we can define a $k \times (k+n)$ energy-momentum matrix $\mathbf{p} = m_0\mathbf{U}$, where $m_0 > 0, \mathbf{U} = [u_{0\mu}]_{k\times(k+n)}$. The physical meaning of the constant m_0 will be clarified later.

Let us denote $\mathbf{p}_\mu = m_0\mathbf{u}_\mu$ and $p_{\theta\mu} = m_0u_{\theta\mu}$ ($\theta = 1, 2, \ldots, k$; $\mu = 1, 2, \ldots, k+n$). First, let us consider the components $p_{\theta\sigma}$, where $\sigma,\theta = 1, 2, \ldots, k$. By analogy with STR, if we multiply by the velocity of light in a vacuum c, we will obtain the components of the energy $e_{\theta\sigma}$. The energy of the particle defined in relation to the time dimension t_σ has k components:

$$\mathbf{E}_\sigma = \begin{pmatrix} e_{1\sigma} \\ e_{2\sigma} \\ \vdots \\ e_{k\sigma} \end{pmatrix},$$

where

$$e_{\theta\sigma} = m_0u_{\theta\sigma}c = \frac{\lambda_{\theta\sigma}\chi_\sigma m_0c^2}{\alpha_{0\theta}\sqrt{1-\beta^2}}, \tag{57}$$

σ, $\theta = 1, 2, \ldots, k$ [see also Eq. (55)]. The energy of the particle will have k^2 total components. If for a given δ ($1 \le \delta \le k$) the condition $dt_{0\delta} = 0$ is fulfilled, then the velocity components $u_{\delta1}, u_{\delta2}, \ldots, u_{\delta k}$ will be undefined values and thus the energy components $e_{\delta1}, e_{\delta2}, \ldots, e_{\delta k}$ will also be undefined values. If for a given δ ($1 \le \delta \le k$) the condition $dt_\delta = 0$ is fulfilled—i.e., a given particle is not moving in the time dimension t_δ—then we have $u_{1\delta} = u_{2\delta} = \cdots = u_{k\delta} = 0$ and therefore $e_{1\delta} = e_{2\delta} = \cdots = e_{k\delta} = 0$.

From Eqs. (40) and (57) the following important equalities are obtained:

$$\frac{dt_{01}e_{1\sigma} + dt_{02}e_{2\sigma} + \cdots + dt_{0k}w_{k\sigma}}{m_0c} = cdt_\sigma,$$

that is,

$$\frac{\alpha_{01}e_{1\sigma} + \alpha_{02}e_{2\sigma} + \cdots + \alpha_{0k}e_{k\sigma}}{\chi_\sigma} = \frac{m_0c^2}{\sqrt{1-\beta^2}} > 0. \tag{58}$$

Let us assume that for a given δ ($1 \le \delta \le k$) we have $\chi_\delta < 0$—that is, the particle under consideration is moving backward in the time dimension t_δ ($dt_\delta < 0$). Let us simultaneously multiply the numbers $\chi_\delta, \alpha_{01}, \alpha_{02}, \ldots, \alpha_{0k}$ by –1. This operation corresponds to an inversion of the time dimension $dt_\delta = \chi_\delta dT$ and an inversion of the proper time $d\mathbf{T}_0 = (dt_{01}, dt_{02}, \ldots, dt_{0k}) = (\alpha_{01}dT_0, \alpha_{02}dT_0, \ldots, \alpha_{0k}dT_0)$. According to Eq. (58), this operation will not change the components of the energy $e_{1\delta}, e_{2\delta}, \ldots, e_{k\delta}$. (If $\sigma \neq \delta$, during this operation the number χ_σ does not change its sign but the numbers $\alpha_{01}, \alpha_{02}, \ldots, \alpha_{0k}$ change their signs, and therefore the energy components $e_{1\sigma}, e_{2\sigma}, \ldots, e_{k\sigma}$ change their signs, as well.) Therefore, if a given particle is moving backward in the time dimension t_δ (that is, $dt_\delta < 0$) and the projections of the proper time on the time axes are $dt_{01}, dt_{02}, \ldots, dt_{0k}$, then we can assume that the particle is actually moving forward in the time dimension t_δ (that is, $dt_\delta > 0$) but the projections of the proper time on the time axes are $-dt_{01}, -dt_{02}, \ldots, -dt_{0k}$. These results have important consequences for antiparticles in multidimensional time (see Section X).

If in Eq. (57) we set $\alpha_{0\theta} = \chi_\sigma = 1/\sqrt{k}$, then we will have $e_{\theta\sigma} = \lambda_{\theta\sigma}m_0c^2/\sqrt{1-\beta^2}$. From here we can define the quantities $\lambda_{\theta\sigma}$:

$$\lambda_{\theta\sigma} = \frac{e_{\theta\sigma}\sqrt{1-\beta^2}}{m_0c^2}. \tag{59}$$

In this case, if $\lambda_{\theta\sigma} > 0$ then we have $e_{\theta\sigma} > 0$, and vice versa. If $\lambda_{\theta\sigma} < 0$ then we will have $e_{\theta\sigma} < 0$, and vice versa.

As pointed out in Subection IX.A, if the frame K''' is moving uniformly and rectilinearly in relation to K, then for an appropriate choice of the axes of K_0 and K''' ($\alpha_{0\theta} = \chi'''_\sigma = 1/\sqrt{k}$), the quantities $\lambda_{\theta\sigma}$ will conserve their value during the transfer from K to K'''.

Let us denote $E_\sigma = ||\mathbf{E}_\sigma|| \geq 0$. The aggregate energy, defined in relation to the time dimension t_σ, is equal to

$$E_\sigma = \sqrt{\sum_{\theta=1}^{k} e_{\theta\sigma}^2} = \frac{m_0 c^2 |\chi_\sigma|}{\sqrt{1-\beta^2}} \sqrt{\sum_{\theta=1}^{k} \left(\frac{\lambda_{\theta\sigma}}{\alpha_{0\theta}}\right)^2} = \frac{m_0 c^2 |\chi_\sigma| \gamma_{t\sigma}}{\sqrt{1-\beta^2}}, \tag{60}$$

where $\gamma_{t\sigma} > 0$ is a parameter that does not depend on the numbers $\alpha_{0\theta}$ ($\theta = 1, 2, \ldots, k$)—see Subsection IX.A and more precisely Eq. (45). [If we assume that the given particle does not move in the time dimension t_δ ($1 \leq \delta \leq k$), then we will have $\chi_\delta = 0$ and accordingly $E_\delta = 0$.]

The total energy of the particle which is moving in k time dimensions will be defined through the expression $E = \sqrt{\sum_{\sigma=1}^{k} E_\sigma^2} > 0$—that is,

$$E = \frac{m_0 c^2}{\sqrt{1-\beta^2}} \sqrt{\sum_{\sigma=1}^{k} \chi_\sigma^2 \gamma_{t\sigma}^2} = \frac{m_0 c^2 \gamma_t}{\sqrt{1-\beta^2}}, \tag{61}$$

where $\gamma_t > 0$ is a parameter that does not depend on the numbers χ_σ and $\alpha_{0\theta}$ [see Subsection IX.A and more precisely Eqs. (42) and (52)].

Let us assume that the frame K coincides with the proper frame of reference K_0 (see Subsection IX.A). In this case we have $dt_\sigma \equiv dt_{0\sigma}$ (that is, $dT = dT_0, \chi_\sigma = \alpha_{0\sigma}$),$dx_\eta = 0$, where $\sigma = 1, 2, \ldots, k$; $\eta = k+1, k+2, \ldots, k+n$. The proper energy or the rest energy of the particle defined in relation to the time dimension $t_\sigma \equiv t_{0\sigma}$ has k components:

$$\mathbf{E}_{0\sigma} = \begin{pmatrix} e_{01\sigma} \\ e_{02\sigma} \\ \vdots \\ e_{0k\sigma} \end{pmatrix},$$

where $e_{00\theta\sigma} = \lambda_{\theta\sigma}\alpha_{0\sigma} m_0 c^2/\alpha_{0\theta},\sigma$, $\theta = 1, 2, \ldots, k$. The aggregate proper energy defined in relation to the time dimension t_σ is equal to $E_{0\sigma} = m_0 c^2 |\chi_\sigma| \gamma_{t\sigma}$.

The total proper energy of the particle moving in k dimensions is defined by the expression $E_0 = \sqrt{\sum_{\sigma=1}^{k} E_{0\sigma}^2} > 0$—that is,

$$E_0 = m_0 c^2 \gamma_t. \tag{62}$$

It is clear that if $k = 1$ (and accordingly $\gamma_t = 1$), then $E = m_0 c^2$. This is the well-known formula obtained in STR. Hence, the constant $m_0 > 0$ is equal to the proper mass or to the rest mass of the particle (at $V_1 = V_2 = \cdots = V_k = 0$).

By analogy with the case $k = 1$ (i.e., STR), we will assume that if a particle has rest mass m_0 and is moving with velocities $V_1, V_2, V_3, \ldots, V_{\mathrm{k}}$ (defined in relation to the time dimensions $t_1, t_2, \ldots, t_k$) in relation to a given frame of reference K, this particle will have relativistic mass $m = m_0/\sqrt{1-\beta^2}$ defined in K. Let us set $V_1 = V_2 = \cdots = V_{\mathrm{k}} = V$. (This can be obtained through an appropriate rotation in the hyperplane of time—see Section VII.) Then we have

$$m = \frac{m_0}{\sqrt{1-\beta^2}} = \frac{m_0}{\sqrt{1-\frac{V^2}{kc^2}}}. \tag{63}$$

Equation (63) gives the relationship between relativistic mass m, rest mass m_0, velocity V of a particle, and the number of time dimensions k in which the particle is moving. According to Eq. (63), a given particle has zero rest mass if $\beta = 1$—that is,

$$\sum_{\varsigma=1}^{k} \frac{c^2}{V_\varsigma^2} = 1$$

(and accordingly $V = c\sqrt{k}$). At $k = 1$ we obtain the well-known case of STR. From Eq. (63) it follows that at constant velocity V ($V = \text{const}$), as the number of time dimensions increases, the relativistic mass m decreases.

If in Eq. (61) we set $V_1 = V_2 = \cdots = V_k = V$, then we obtain

$$E = \frac{m_0 c^2 \gamma_t}{\sqrt{1-\frac{V^2}{kc^2}}} = mc^2\gamma_t. \tag{64}$$

Through Eq. (64) we determine the relation between total energy E, rest mass m_0, relativistic mass m, velocity V of a particle, and number of time dimensions k in which the particle is moving.

As can be seen from Eq. (64), only for the case $k = 1$ do we have following peculiarity: If $V \to c$, then $E \to \infty$—that is, the energy of the particle is an infinite quantity (see Figs. 4 and 5). It is clear that if $k > 1$ and $V = c$, then $E = m_0 c^2 \gamma_t \sqrt{k}/\sqrt{k-1}$. Therefore, if a particle with rest mass (rest energy) differing from 0 and moving in a number $k > 1$ of time dimensions with the same velocity—which is equal to the speed of light in vacuum—then its energy is not infinite, but will have a finite value. However, if $V \to c\sqrt{k}$, then $E \to \infty$—that is, the energy of the particle is an infinite value. (As noted in Section VIII, the velocity $V = c\sqrt{k}$ is a constant, invariant value in relation to all inertial frames of reference.)

Let us consider two particles L and N, moving uniformly and rectilinearly to each other. Let us assume that particle N is moving in only one time dimension (t_1) and that particle L is moving in k time dimensions ($t_1, t_2, \ldots, t_k$), where $k > 1$. We assume that particle N is not a luxon—i.e., the rest mass of the particle N differs from 0. Let us assume, as well, that particle L is a luxon—i.e., the velocity of L in relation to N defined according to the different temporal dimensions $t_1, t_2, \ldots, t_k$ is the same and is equal to the speed of light in a vacuum. According to Eq. (64), the total energy of particle L defined in relation to particle N is equal to $E = m_0\ c^2 \gamma_t \sqrt{k}/\sqrt{k-1}$ (where $k > 1$). Although particle L is moving in relation to particle N with a velocity equal to the speed of light in a vacuum, the rest mass of particle L differs from 0: $m_0 =$

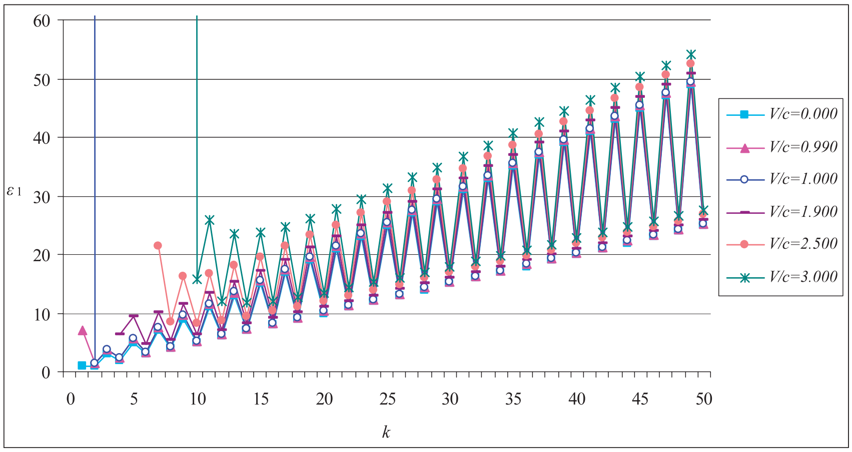

FIG. 4. (Color online) Relation between the quantity ε_1 and the number of time dimensions k.

$E\sqrt{k-1}/c^2\gamma_t\sqrt{k} \neq 0$. The reason for this result is that particle L is moving in more than one time dimension ($k > 1$). Therefore, the particles which are moving with a velocity equal to the speed of light in a vacuum (i.e., the luxons) can have nonzero rest mass, but on the condition that they move in two, three, or more time dimensions.

Let us assume $\lambda_{1\mu}^2 = \lambda_{2\mu}^2 = \cdots = \lambda_{k\mu}^2$—that is, $|\lambda_{1\mu}| = |\lambda_{2\mu}| = \cdots = |\lambda_{k\mu}|$, $\mu = 1, 2, \ldots, k+n$. Since $\sum_{\theta=1}^{k} \lambda_{\theta\mu} = 1$, the minimum value which the quantity $|\lambda_{\theta\mu}|$ can take in this case is $1/k$ (here $\theta = 1, 2, \ldots, k$). If k is an odd number, then the maximum value which the quantity $|\lambda_{\theta\mu}|$ can take is 1; and if k is an even number, then the maximum value which $|\lambda_{\theta\mu}|$ can take is 1/2. If we take into account Eqs. (52) and (54), then for odd values of k we will obtain $1 \leq \gamma_t \leq k$ and accordingly $1 \leq \gamma_s \leq k$. For even values of k we will have $1 \leq \gamma_t \leq k/2$ and accordingly $1 \leq \gamma_s \leq k/2$. We note that if $k = 2$, then $\gamma_t = 1$ and thus $\gamma_s = 1$.

Since $1 \leq \gamma_t \leq k$, according to Eq. (64) we have

$$\frac{m_0 c^2}{\sqrt{1-\frac{V^2}{kc^2}}} \leq E \leq \frac{m_o c^2 k}{\sqrt{1-\frac{V^2}{kc^2}}}. \tag{65}$$

If $k = 2$, then $\gamma_t = 1$—that is,

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{V^2}{2c^2}}}.$$

If we set $V = 0$, then according to Eq. (65) we obtain the following expression for the proper energy of the particle:

$$m_0 c^2 \leq E_0 \leq m_0 c^2 k. \tag{66}$$

From Eq. (64) it follows that if $V = \text{const}$ and $\gamma_t = k$ (k is an odd number) or $\gamma_t = k/2$ (k is an even number), then as the number of time dimensions increases, so does the total energy (see Fig. 4). In this case, the additional time dimensions "add" additional energy to the energy of the particle. Let us set $\gamma_t = k$ at $k = 2k_1 + 1$ and $\gamma_t = k/2$ at $k = 2k_1$, where $k_1 = 0, 1, 2, 3, \ldots$. Let us denote $\varepsilon_1 = E/m_0 c^2$. According to Eq. (64) we obtain

$$\varepsilon_1 = \frac{k}{\sqrt{1-\frac{V^2}{kc^2}}}$$

at $k = 2k_1 + 1$ and

$$\varepsilon_1 = \frac{k}{2\sqrt{1-\frac{V^2}{kc^2}}}$$

at $k = 2k_1$. Figure 4 gives the relation between the quantity

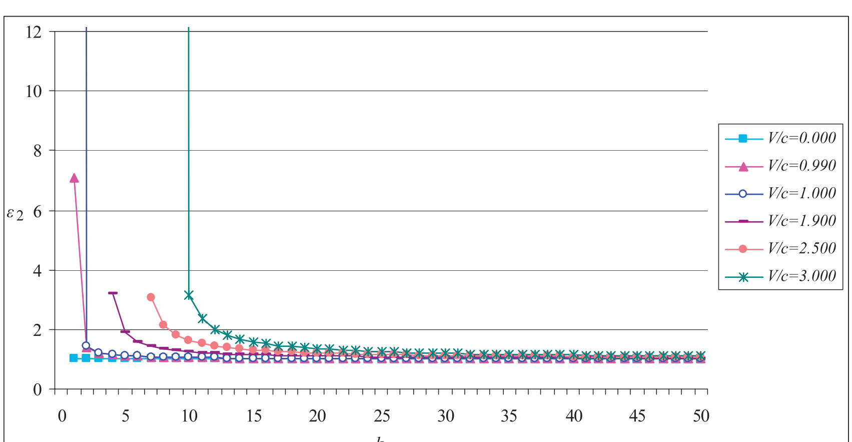

FIG. 5. (Color online) Relation between the quantity ε_2 and the number of time dimensions k.

ε_1 and the number of time dimensions k at different values of the ratio V/c (0.000, 0.990, 1.000, 1.900, 2.500, 3.000).

According to Eq. (64), if $V = \text{const}$ and $\gamma_t = 1$, then as the number of time dimensions increases, the total energy decreases (see Fig. 5). In this case, the additional time dimensions "subtract" from the energy of the particle. Let us set $\gamma_t = 1$ and $\varepsilon_2 = E/m_0c^2$. According to Eq. (64) we obtain

$$\varepsilon_2 = \frac{1}{\sqrt{1 - \frac{V^2}{kc^2}}}$$

Figure 5 gives the relation between the quantity ε_2 and the number of time dimensions k at different values of the ratio V/c.

As can be seen from Figs. 4 and 5, only in the case $k = 1$ do we have following peculiarity: If $V \to c$, then $\varepsilon_{1,2} \to \infty$ and accordingly $E \to \infty$. Further, if $k = 9$ and $V \to 3c$, then $\varepsilon_{1,2} \to \infty$ and accordingly $E \to \infty$.

In the case of multidimensional time, the momentum of the particle defined in relation to the space dimension x_η has k components:

$$\mathbf{p}_\eta = \begin{pmatrix} p_{1\eta} \\ p_{2\eta} \\ \vdots \\ p_{k\eta} \end{pmatrix},$$

where

$$p_{\theta\eta} = m_0 u_{\theta\eta} = \frac{m_0 \lambda_{\theta\eta} \chi_\eta c\beta}{\alpha_{0\theta}\sqrt{1-\beta^2}}. \tag{67}$$

[Here $\theta = 1, 2, \ldots, k$; $\eta = k+1, k+2, \ldots, k+n$—see also Eq. (56).] The momentum of the particle has nk total components. If for a given δ ($1 \le \delta \le k$) the condition $dt_{0\delta} = 0$ is fulfilled, then the velocity components $u_{\delta(k+1)}$, $u_{\delta(k+2)}, \ldots, u_{\delta(k+n)}$ will be undefined values and therefore the momentum components $p_{\delta(k+1)}, p_{\delta(k+2)}, \ldots, p_{\delta(k+n)}$ will also be undefined. If for a given φ ($k+1 \le \varphi \le k+n$) the condition $dx_\varphi = 0$ is fulfilled—i.e., the particle does not move in the space dimension x_φ—then we will have $\chi_\varphi = 0$ and therefore $p_{1\varphi} = p_{2\varphi} = \cdots = p_{k\varphi} = 0$. It is seen that if for a given ϑ ($1 \le \vartheta \le k$) the condition $V_\vartheta = 0$ is fulfilled, then $\beta = 0$ and therefore $p_{\theta\eta} = 0$ for all values of θ and η.

From Eqs. (41) and (67) we obtain the following:

$$\frac{\alpha_{01}p_{1\eta} + \alpha_{02}p_{2\eta} + \cdots + \alpha_{0k}p_{k\eta}}{\chi_\eta} = \frac{m_0 c\beta}{\sqrt{1-\beta^2}} > 0. \tag{68}$$

From Eq. (68) it follows that if we simultaneously multiply the numbers $\alpha_{01}, \alpha_{02}, \ldots, \alpha_{0k}, \chi_{k+1}, \chi_{k+2}, \ldots, \chi_{k+n}$ by the number –1, then in this operation the momentum components $p_{\theta\eta}$ will not change.

According to Eqs. (58) and (68), the multiplication of the value m_0 by the number –1 is equivalent to the multiplication of all numbers $\alpha_{01}, \alpha_{02}, \ldots, \alpha_{0k}$ by –1—that is, the rest-mass sign inversion is equivalent to the proper time inversion $d\mathbf{T}_0 = (dt_{01}, dt_{02}, \ldots, dt_{0k})$. The application of these two operations leads to identical results, namely changing the signs of all components of the energy and of the momentum. A similar statement is valid in STR. These results have important significance in relation to antiparticles in multidimensional time (see Section X).

If in Eq. (67) we set $\alpha_{0\theta} = 1/\sqrt{k}$, $\chi_\eta = 1/\sqrt{n}$, then we obtain

$$p_{\theta\eta} = \frac{m_0 \lambda_{\theta\eta} c\beta\sqrt{k}}{\sqrt{n}\sqrt{1-\beta^2}}.$$

From here we can define the quantities $\lambda_{\theta\eta}$:

$$\lambda_{\theta\eta} = \frac{p_{\theta\eta}\sqrt{n}\sqrt{1-\beta^2}}{m_0 c\beta\sqrt{k}}. \tag{69}$$

In this case, if $\lambda_{\theta\eta} > 0$ then we have $p_{\theta\eta} > 0$, and vice versa. If $\lambda_{\theta\eta} < 0$ then we have $p_{\theta\eta} < 0$, and vice versa. As already pointed out in Subsection IX.A, if the frame K'''' is moving uniformly and rectilinearly in relation to K, then for appropriate choice of the axes of K_0 and K'''' ($\alpha_{0\theta} = 1/\sqrt{k}$, $\chi''''_\eta = 1/\sqrt{n}$), the quantities $\lambda_{\theta\eta}$ conserve their value during transfer from K to K''''.

Let us denote $p_\eta = \|\mathbf{p}_\eta\| > 0$. The momentum of the particle defined in relation to the space dimension x_η is given through the following formula:

$$p_\eta = \sqrt{\sum_{\theta=1}^{k} p_{\theta\eta}^2} = \frac{m_0|\chi_\eta|\gamma_{s\eta}c\beta}{\sqrt{1-\beta^2}}. \tag{70}$$

Here

$$\gamma_{s\eta} = \sqrt{\sum_{\theta=1}^{k}\left(\frac{\lambda_{\theta\eta}}{\alpha_{0\theta}}\right)^2} > 0$$

is a parameter which does not depend on the numbers $\alpha_{0\theta}$ (see Subsection IX.A).

The total momentum is defined through the following formulas:

$$p_s = \sqrt{\sum_{\eta=k+1}^{k+n} p_\eta^2} \ge 0,$$

that is,

$$p_s = \frac{m_0\gamma_s c\beta}{\sqrt{1-\beta^2}} = m\gamma_s c\beta, \tag{71}$$

where $\gamma_s = \sqrt{\sum_{\eta=k+1}^{k+n}\chi_\eta^2\gamma_{s\eta}^2} > 0$ is a parameter which does not depend on the numbers χ_η and $\alpha_{0\theta}$ [see Subsection IX.A and more precisely Eqs. (43) and (54)].

Let us set $V_1 = V_2 = \cdots = V_k = V$. (This can be obtained through an appropriate rotation in the hyperplane of time—see Section VII.) We have

$$p_s = \frac{m_0\gamma_s V}{\sqrt{k - \frac{V^2}{c^2}}} = \frac{m\gamma_s V}{\sqrt{k}}. \tag{72}$$

Equation (72) gives the relation between total momentum p_s, rest mass m_0, relativistic mass m, velocity V of a particle, and number of the time dimensions k in which the particle is moving.

As is easy to see, following equalities are fulfilled:

$$\frac{cdT}{dT_0} = \frac{c}{\sqrt{1-\beta^2}} = \frac{E}{m_0 c \gamma_t}; \quad \frac{dX}{dT_0} = \frac{c\beta}{\sqrt{1-\beta^2}} = \frac{p_s}{m_0 \gamma_s}. \tag{73}$$

[See Eqs. (61) and (71).] Taking into consideration Eqs. (39) and (73), we obtain the following important equality:

$$\frac{E^2}{\gamma_t^2} - \frac{p_s^2 c^2}{\gamma_s^2} = m_0^2 c^4. \tag{74}$$

Equation (74) gives the relation between total energy E, total momentum p_s, and rest mass m_0. If $k = 1$, then from Eq. (74) we obtain the well-known equality derived in STR: $E^2 - c^2 p_s^2 = m_0^2 c^4$.

C. Energy-momentum conservation law

In STR, the energy-momentum conservation law is derived as a consequence of continuous space-time symmetry (Noether's theorem). For example, let us consider the process of decay of a particle. Applied to a decay process, energy-momentum conservation states that the vector sum of the energy-momentum four-vectors of the decay products should equal the energy-momentum four-vector of the original particle. Some aspects of the problem for conservation of energy and momentum in multidimensional time are discussed by Dorling.[3] Until now it was accepted that in the case of k-dimensional time, the energy is a k-dimensional vector[3–5]. But according to the previous considerations, in the case of k-dimensional time and n-dimensional space, the energy is a $k \times k$ matrix and the momentum is a $k \times n$ matrix. In this case the energy-momentum conservation law applied to the process of decay of a particle will state following: The matrix entrywise sum of the energy-momentum $k \times (k + n)$ matrices of the decay products is equal to the energy-momentum $k \times (k + n)$ matrix of the original particle. In particular, the matrix entrywise sum of the energy $k \times k$ matrices of the decay products is equal to the energy $k \times k$ matrix of the original particle. Moreover, the matrix entrywise sum of the momentum $k \times n$ matrices of the decay products is equal to the momentum $k \times n$ matrix of the original particle. For example, if we denote with $\mathbf{p}^H[p^H_{\theta\mu}]_{k\times(k+n)}$ the energy-momentum matrix of the original particle (particle H) and with $\mathbf{p}^A = [p^A_{\theta\mu}]_{k\times(k+n)}$, $\mathbf{p}^B = [p^B_{\theta\mu}]_{k\times(k+n)}$, $\cdots$, $\mathbf{p}^D = [p^D_{\theta\mu}]_{k\times(k+n)}$ the energy-momentum matrices of the decay products (of the particles A, B, …, D, respectively), then we have

$$\mathbf{p}^H = \mathbf{p}^A + \mathbf{p}^B + \cdots + \mathbf{p}^D,$$

that is,

$$p^H_{\theta\mu} = p^A_{\theta\mu} + p^B_{\theta\mu} + \cdots + p^D_{\theta\mu}, \tag{75}$$

($\theta = 1, 2, \ldots, k$; $\mu = 1, 2, \ldots, k + n$). For the energy and the momentum, respectively, we have $\mathbf{E}^H = \mathbf{E}^A + \mathbf{E}^B + \cdots + \mathbf{E}^D$—that is, $e^H_{\theta\sigma} = e^A_{\theta\sigma} + e^B_{\theta\sigma} + \cdots + e^D_{\theta\sigma}$—and $\mathbf{p}^H_s = \mathbf{p}^A_s + \mathbf{p}^B_s + \cdots + \mathbf{p}^D_s$—that is, $p^H_{\theta\eta} = p^A_{\theta\eta} + p^B_{\theta\eta} + \cdots + p^D_{\theta\eta}$ ($\sigma = 1, 2, \ldots, k$; $\eta = k+1, \ldots, k+n$). For the total energies of the considered particles, we have

$$E^H = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^H_{\theta\sigma})^2} = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^A_{\theta\sigma} + e^B_{\theta\sigma} + \cdots + e^D_{\theta\sigma})^2},$$

$$E^A = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^A_{\theta\sigma})^2},$$

$$E^B = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^B_{\theta\sigma})^2}, \ldots, E^D = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^D_{\theta\sigma})^2}.$$

The total energies E^H, E^A, E^B, …, E^D are the Frobenius norms of the matrices $\mathbf{E}^H = [e^H_{\theta\sigma}]_{k\times k}$, $\mathbf{E}^A = [e^A_{\theta\sigma}]_{k\times k}$, $\mathbf{E}^B = [e^B_{\theta\sigma}]_{k\times k}$, $\cdots$ $\mathbf{E}^D = [e^D_{\theta\sigma}]_{k\times k}$, respectively—i.e., $\mathrm{E}^H = ||\mathbf{E}^H||_\mathrm{F} = ||\mathbf{E}^A + \mathbf{E}^B + \cdots + \mathbf{E}^D||_\mathrm{F}$, $E^A = ||\mathbf{E}^A||_\mathrm{F}$, $E^B = ||\mathbf{E}^B||_\mathrm{F}$, $\cdots$, $E^D = ||\mathbf{E}^D||_\mathrm{F}$. (The total momentums p^H_s, p^A_s, p^B_s, $\cdots$, p^D_s are the Frobenius norms of the matrices $\mathbf{p}^H_s = [p^H_{\theta\eta}]_{k\times n}$, $\mathbf{p}^A_s = [p^A_{\theta\eta}]_{k\times n}$, $\mathbf{p}^B_s = [p^B_{\theta\eta}]_{k\times n}$, $\cdots$, $\mathbf{p}^D_s = [p^D_{\theta\eta}]_{k\times n}$, respectively.) The Frobenius norm possesses the following property: $||\mathbf{E}^A + \mathbf{E}^B + \cdots + \mathbf{E}^D||_\mathrm{F} \le ||\mathbf{E}^A||_\mathrm{F} + ||\mathbf{E}^B + \cdots + \mathbf{E}^D||_\mathrm{F} \le \cdots \le ||\mathbf{E}^A||_\mathrm{F} + ||\mathbf{E}^B||_\mathrm{F} + \cdots + ||\mathbf{E}^D||_\mathrm{F}$ (the triangle inequality property). Therefore, we have

$$E^H \le E^A + E^B + \cdots + E^D. \tag{76}$$

We obtained the result that in the case of multidimensional time, the energy conservation law as defined in STR will be violated—the magnitude of the total energy of the original particle H is less than the sum of the magnitudes of the total energies of the decay products (the particles A, B, …, D). The same applies also for the momentum of the considered particles—the momentum conservation law for the case of multidimensional time differs from the momentum conservation law for the case of one-dimensional time (STR).

Let us assume that the particle H is moving only in one time dimension t_1 and that the remaining particles A, B, …, D are moving in $k > 1$ time dimensions t_1, …, t_k. According to the energy conservation law, the equality $E^H \equiv e^H_{11} = e^A_{11} + e^B_{11} + \cdots + e^D_{11}$ will be fulfilled. Further, at $\sigma > 1$ or $\theta > 1$ the equality $e^H_{\theta\sigma} = 0$,will be fulfilled—i.e., $e^A_{\theta\sigma} + e^B_{\theta\sigma} + \cdots + e^D_{\theta\sigma} = 0$.

Up to now, it has been accepted that particles moving in multidimensional time are more unstable and decay more easily than those moving in one-dimensional time.[3] However, this is only valid if we suppose that in the case of k-dimensional time and n-dimensional space the energy is a k-dimensional vector and the momentum is an n-dimensional vector. According to our obtained results, in the case of k-dimensional time and n-dimensional space the energy is a $k \times k$ matrix and the momentum is a $k \times n$ matrix. We are going to prove that in multidimensional time, like in one-dimensional time, the sum of the rest masses of the decay products is always less than or equal to the rest mass of the original particle. Therefore, particles moving in multidimensional time are as stable as those moving in one-dimensional time. Indeed, let us consider the process of decay of the particle H into the particles A, B, …, D. Let us choose the values of $\lambda_{\theta\mu}$ in such a way that for a particular choice of time axes in the frame K, of some space axes in K, and of some time axes in K_0 (i.e., at a given χ_σ, χ_η, $\alpha_{0\theta}$), the following equalities are fulfilled:

$$\frac{\lambda^H_{\theta\mu}\chi^H_\mu}{\alpha^H_{0\theta}} = \frac{\lambda^A_{\theta\mu}\chi^A_\mu}{\alpha^A_{0\theta}} = \frac{\lambda^B_{\theta\mu}\chi^B_\mu}{\alpha^B_{0\theta}} = \cdots = \frac{\lambda^D_{\theta\mu}\chi^D_\mu}{\alpha^D_{0\theta}}. \tag{77}$$

(Here θ, $\sigma = 1, 2, \ldots, k$; $\eta = k+1, k+2, \ldots, k+n$; $\mu = 1, 2, \ldots, k+n$—see Subsection IX.A.) It is obvious that in this case the parameters γ_t, γ_s will be the same for all particles H, A, B, …, D—i.e., $\gamma^H_t = \gamma^A_t = \gamma^B_t = \cdots = \gamma^D_t = \gamma_t > 0$, $\gamma^H_s = \gamma^A_s + \gamma^B_s = \cdots = \gamma^D_s = \gamma_s > 0$ [see Eqs. (52) and (54)]. For the total energies and the total momentums of the particles we have

$$E^H = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^H_{\theta\sigma})^2} = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^A_{\theta\sigma} + e^B_{\theta\sigma} + \cdots + e^D_{\theta\sigma})^2},$$

$$E^A = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^A_{\theta\sigma})^2}, \ldots,$$

$$E^D = \sqrt{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^D_{\theta\sigma})^2},$$

$$p^H_s = \sqrt{\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}(p^H_{\theta\eta})^2} = \sqrt{\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}(p^A_{\theta\eta} + p^B_{\theta\eta} + \cdots + p^D_{\theta\eta})^2},$$

$$p^A_s = \sqrt{\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}(p^A_{\theta\eta})^2}, \ldots,$$

$$p^D_s = \sqrt{\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}(p^D_{\theta\eta})^2}.$$

[See Eqs. (60), (61), (70), and (71)]. According to Eq. (74), we have

$$\frac{(E^H)^2}{\gamma_t^2} - \frac{(p^H_s)^2c^2}{\gamma_s^2} = (m^H_0)^2c^4, \frac{(E^A)^2}{\gamma_t^2} - \frac{(p^A_s)^2c^2}{\gamma_s^2} = (m^A_0)^2c^4, \ldots,$$

$$\frac{(E^D)^2}{\gamma_t^2} - \frac{(p^D_s)^2c^2}{\gamma_s^2} = (m^D_0)^2c^4.$$

It is clear that if the inequality

$$\sqrt{\frac{(E^H)^2}{\gamma_t^2} - \frac{(p^H_s)^2c^2}{\gamma_s^2}} \geq \sqrt{\frac{(E^A)^2}{\gamma_t^2} - \frac{(p^A_s)^2c^2}{\gamma_s^2}} + \cdots + \sqrt{\frac{(E^D)^2}{\gamma_t^2} - \frac{(p^D_s)^2c^2}{\gamma_s^2}}$$

is fulfilled, then the inequality $m^H_0 \geq m^A_0 + m^B_0 + \cdots + m^D_0$ will be valid as well. However, the first inequality is equivalent to the following inequality:

$$\frac{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^A_{\theta\sigma} + \cdots + e^D_{\theta\sigma})^2}{\gamma_t^2} - \frac{c^2\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}(p^A_{\theta\eta} + \cdots + p^D_{\theta\eta})^2}{\gamma_s^2}$$
$$\geq \frac{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^A_{\theta\sigma})^2}{\gamma_t^2} - \frac{c^2\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}(p^A_{\theta\eta})^2}{\gamma_s^2} + \cdots + \frac{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}(e^D_{\theta\sigma})^2}{\gamma_t^2} - \frac{c^2\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}(p^D_{\theta\eta})^2}{\gamma_s^2}$$
$$+ 2m^A_0m^B_0c^4 + \cdots + 2m^A_0m^D_0c^4 + \cdots.$$

From this inequality we obtain

$$\frac{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}e^A_{\theta\sigma}e^B_{\theta\sigma}}{\gamma_t^2} - \frac{c^2\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}p^A_{\theta\eta}p^B_{\theta\eta}}{\gamma_s^2} + \cdots + \frac{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}e^A_{\theta\sigma}e^D_{\theta\sigma}}{\gamma_t^2} - \frac{c^2\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}p^A_{\theta\eta}p^D_{\theta\eta}}{\gamma_s^2} + \cdots$$
$$\geq m^A_0m^B_0c^4 + \cdots + m^A_0m^D_0c^4 + \cdots.$$

In order to find whether the obtained inequality is fulfilled, we have to compare the expressions on each side of the inequality concerning the respective pairs of particles. For example, let us consider the expressions concerning the pair A and B—these are the expressions

$$\frac{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}e^A_{\theta\sigma}e^B_{\theta\sigma}}{\gamma_t^2} - \frac{c^2\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}p^A_{\theta\eta}p^B_{\theta\eta}}{\gamma_s^2}$$

and $m^A_0m^B_0c^4$. (The comparison for the remaining pairs of particles is done similarly.) Taking into account Eqs. (57), (67), (52), and (54), we have

$$\frac{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k} e^A_{\theta\sigma}e^B_{\theta\sigma}}{\gamma_t^2} - \frac{c^2\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k} p^A_{\theta\eta}p^B_{\theta\eta}}{\gamma_s^2} = \frac{m_0^A m_0^B c^4(1-\beta^A\beta^B)}{\sqrt{1-(\beta^A)^2}\sqrt{1-(\beta^B)^2}}.$$

[We have accepted that $(\lambda^A_{0\mu}\chi^A_\mu/\alpha^A_{00}) = (\lambda^B_{0\mu}\chi^B_\mu/\alpha^B_{00})$; therefore we have

$$\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}\frac{\lambda^A_{0\sigma}\chi^A_\sigma\lambda^B_{0\sigma}\chi^B_\sigma}{\alpha^A_{00}\alpha^B_{00}} = \sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}\left(\frac{\lambda^A_{0\sigma}\chi^A_\sigma}{\alpha^A_{00}}\right)^2 = \sum_{\sigma=1}^{k}\sum_{\theta=1}^{k}\left(\frac{\lambda^B_{0\sigma}\chi^B_\sigma}{\alpha^B_{00}}\right)^2 = \gamma_t^2$$

and

$$\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}\frac{\lambda^A_{0\eta}\chi^A_\eta\lambda^B_{0\eta}\chi^B_\eta}{\alpha^A_{00}\alpha^B_{00}} = \sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}\left(\frac{\lambda^A_{0\eta}\chi^A_\eta}{\alpha^A_{00}}\right)^2 = \sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k}\left(\frac{\lambda^B_{0\eta}\chi^B_\eta}{\alpha^B_{00}}\right)^2 = \gamma_s^2.$$

See Eqs. (52), (54), and (77).] We have obtained the expressions

$$\frac{m_0^A m_0^B c^4(1-\beta^A\beta^B)}{\sqrt{1-(\beta^A)^2}\sqrt{1-(\beta^B)^2}}$$

and $m_0^A m_0^B c^4$, which have to be compared. Because β^A and β^B are real numbers, we have $0 \geq -(\beta^A - \beta^B)^2$ and

$$\frac{1-\beta^A\beta^B}{\sqrt{1-(\beta^A)^2}\sqrt{1-(\beta^B)^2}} \geq 1,$$

respectively. (Both are equivalent.) From here it follows that

$$\frac{\sum_{\sigma=1}^{k}\sum_{\theta=1}^{k} e^A_{\theta\sigma}e^B_{\theta\sigma}}{\gamma_t^2} - \frac{c^2\sum_{\eta=k+1}^{k+n}\sum_{\theta=1}^{k} p^A_{\theta\eta}p^B_{\theta\eta}}{\gamma_s^2} \geq m_0^A m_0^B c^4.$$

It is clear that the same inequality will be valid also for the remaining pairs of particles. Therefore, if Eq. (77) is fulfilled for particles undergoing decay (i.e., the values of $\lambda_{0\mu}$ are chosen in an appropriate way), then the inequality $m_0^H \geq m_0^A + m_0^B + \cdots + m_0^D$ will be fulfilled. (If $\beta^A = \beta^B = \cdots = \beta^D$ then this inequality becomes an equality.) The obtained inequality is similar to that derived in STR.

X. ANTIPARTICLES IN MULTIDIMENSIONAL TIME

In relativistic quantum mechanics, an antiparticle is attached to every particle. The so-called switching principle (SP) (or reinterpretation principle) was formulated by Stückelberg, Feynman, Sudarshan, and Recami.[15–27] According to this principle,

> positive energy objects traveling backwards in time do not exist; and any negative energy particle travelling backwards in time can and must be described as its antiparticle, endowed with positive energy and motion forward in time (but going the opposite way in space).[21]

Thus the antiparticle moving forward in time and possessing positive energy in fact can be regarded as a particle moving backward in time and possessing negative energy.[7]

This principle can be generalized for the case of k-dimensional time $t_1, t_2, \ldots, t_k$. A particle moving backward in a given time dimension t_δ (that is, $dt_\delta < 0$, where $1 \leq \delta \leq k$) and possessing energy

$$-\mathbf{E}_\delta = \begin{pmatrix} -e_{1\delta} \\ -e_{2\delta} \\ \vdots \\ -e_{k\delta} \end{pmatrix}$$

defined in relation to t_δ can be described as an antiparticle moving forward in the time dimension t_δ (that is, $dt_\delta > 0$) and possessing energy

$$\mathbf{E}_\delta = \begin{pmatrix} e_{1\delta} \\ e_{2\delta} \\ \vdots \\ e_{k\delta} \end{pmatrix}$$

defined in relation to t_δ. In the case of multidimensional time, the components of energy $e_{\theta\sigma}$ ($\sigma, \theta = 1, 2, \ldots, k$) can accept not only positive but also negative values (unlike the case of one-dimensional time).

First, we will consider the case $(n, k) = (3, 1)$. Some authors[11,21,23,25,28–33] have shown that the nonorthochronous, proper Lorentz transformations (i.e., transformations which include "total inversion" $-\mathbf{1}$) can be connected with the existence of antimatter. The assertion[24] is reasonable that the term "antimatter" is strictly relativistic, and that on the ground of the double sign in the formula for energy, the existence of antiparticles could be predicted right after 1905, provided that the switching principle is applied.

It is important to note that the full group of Lorentz transformations acts as the four-dimensional vector on the position of a given object, and also on the other four-dimensional vectors (four-momentum, four-current, etc.) which are connected with the object. We will introduce the new notations $\overline{P}$ for strong parity and $\overline{T}$ for strong time reversal, in order to denote the inversion of the sign of the first component and of the next three components of all four-vectors. The operation strong reflection, $\overline{P}\,\overline{T}$, which changes the sign of the three-vector $\mathbf{x}$ and of the time t, will change the sign of the three-momentum $\mathbf{p}$ and of the energy E as well. We can write $\overline{P} \equiv P\hat{\mathbf{p}}$ and $\overline{T} \equiv T\hat{E}$, where $\hat{\mathbf{p}}$ and $\hat{E}$ are operators respectively changing the signs of the three-momentum **p** and of the energy E (operations of three-momentum and energy reversal). In this new formalism, the operation $\bar{P}$ is in fact equivalent to the standard operation P (parity), but the operation $\bar{T}$ is not equivalent to the standard operation T (time reversal transformation) because $\bar{T}$ does not contain the operation X, which performs the exchange of emission and absorption.

It can be proven[29,30,34] that the strong reflection PT is equivalent to the normal CPT operation: $\bar{P}\bar{T}\xrightarrow{SP}CPT$. Indeed, applying the switching principle (SP), we obtain[33,34]

$$SP \equiv \mathrm{C}\hat{\mathrm{E}}\hat{\mathbf{p}}, \tag{78}$$

where C is the conjugation of all conserved additive charges. Since $\bar{P} \equiv P\hat{\mathbf{p}}$ and $\bar{T} \equiv T\hat{E}$, we have $\bar{P}\bar{T} \equiv P\hat{\mathbf{p}}T\hat{E}$. Thus we have[34]

$$SP\bar{P}\,\bar{T} \equiv (C\hat{E}\hat{\mathbf{p}})P\hat{\mathbf{p}}T\hat{E} \equiv CPT. \tag{79}$$

If we apply the switching principle, then we have[11]

$$SP \equiv CC_m X \equiv \bar{C}X, \tag{80}$$

where $\bar{C} \equiv CC_m$; here C_m is the rest-mass sign inversion and X operates the exchange of absorption and emission. The operation $\bar{C}$ will be called strong conjugation.[h] One can easily realize that in the frame of quantum mechanics (in the case of states with definite parity), $\bar{C} \equiv P_5$, where P_5 is the chirality operator ($\bar{C}^{-1}\psi\bar{C} \equiv \gamma^5\psi = P_5^{-1}\psi P_5$)—see Ref. 11. Let us now consider generalized five-dimensional space-time instead of the four-dimensional of space-time of Minkowski, where the fifth dimension corresponds to proper time and consequently is connected with the rest mass.[11,35–46] It turns out that from the geometric point of view, chirality P_5 in fact means inversion of the fifth axis (i.e., inversion of the proper time or accordingly of the rest mass). Taking into account Eq. (80), we can write $SP = P_5X$ (see Ref. 11). Therefore, SP is a combination of inversion in relation to the fifth axis (proper time and accordingly rest mass) and application of the operator X.

Since $\bar{P}\bar{T}$ means the sign inversion of all components of all four-vectors, for getting such an effect it is enough to write $\bar{P}\bar{T} \equiv PT\hat{\mathbf{u}}X^{-1}$, where $X^{-1} \equiv X$ has been introduced because ordinary T contains the exchange X of emission and absorption exchange (different from $\bar{T}$)—see Ref. 33. With $\hat{\mathbf{u}}$ here is denoted the four-velocity inversion. We obtain

$$SP\bar{P}\,\bar{T} \equiv \bar{C}X(PT\hat{\mathbf{u}}X^{-1}) \equiv CPT. \tag{81}$$

The switching principle changes the sign of the three-momentum but does not change the sign of the three-velocity—i.e., it changes the sign of the rest mass. In fact, the four-velocity inversion is equivalent to the rest-mass inversion: $\hat{\mathbf{u}} \equiv C_m$.

If we denote with $L_+^\uparrow$ the proper orthochronous Lorentz transformations (i.e., transformations including the unit matrix **1**) and with $L_+^\downarrow$ the proper nonorthochronous Lorentz transformations (i.e., transformations including total inversion –**1**), then we will have[11,21]

$$L_+^\downarrow \equiv (-\mathbf{1})L_+^\uparrow \equiv \bar{P}\,\bar{T}L_+^\uparrow \xrightarrow{SP} CPTL_+^\uparrow. \tag{82}$$

[Here the operation (–**1**) means total inversion, as in (–1) $\equiv \bar{P}\bar{T} \equiv CPT$—see Refs. 11, 29, 30, and 34.[i]]

According to the results obtained in Section IX.B, in the case of one-dimensional time the formula $E \equiv e_{11} = m_0u_{11}c$ is valid for all free particles, where u_{11} is the time component of the four-velocity. According to Eq. (58), at $k = 1$ we have

$$E \equiv e_{11} = \frac{\chi_1 m_0 c^2}{\alpha_{01}\sqrt{1-\beta^2}}.$$

(In the case $dT = |dt_1| > 0$, $dT_{01} = |dt_{01}| > 0$, $\lambda_{11} = 1$—see Subsection IX.B.) Let us assume that $\chi_1 > 0$ and $\alpha_{01} > 0$—that is, the energy of the particle is a positive quantity: $E \equiv e_{11} > 0$. Let us consider the proper nonorthochronous Lorentz transformations $L_+^\downarrow$, which lead to the exchange of the signs of all time components:

$$E' = -E = m_0(-u_{11})c = \frac{(-\chi_1)m_0c^2}{\alpha_{01}\sqrt{1-\beta^2}} < 0.$$

Let us apply the switching principle, which is expressed in the inversion of the fifth axis (i.e., inversion of the proper time $dt_{01} = \alpha_{01}dT_0$ or, accordingly, of the rest mass). The switching principle will mean multiplying the number $\alpha_{01} = dt_{01}/dT_0$ by the number –1. We obtain the related antiparticle:

$$E'' = -E' = \frac{(-\chi_1)m_0c^2}{(-\alpha_{01})\sqrt{1-\beta^2}} = \frac{(-\chi_1)(-m_0)c^2}{\alpha_{01}\sqrt{1-\beta^2}} > 0.$$

We determined that for the antiparticle (moving in one-dimensional time) we have to attach negative proper time or, accordingly, negative rest mass, but of course positive full relativistic mass and energy.[11]

Recmi and Ziino[11] formulated so-called strong $\bar{C}\bar{P}\bar{T}$-symmetry: The physical world is symmetric (i.e., physical laws are invariant) during total five-dimensional inversion of the axes x, y, z, ct, ct_{01} (where t_{01} is the proper time).

If the number of time dimensions in which a particle is moving is greater than one, then the particle will have more than one antiparticle. There will be a violation of Lorentz covariance and therefore of CPT symmetry. In the case of multidimensional time, CPT symmetry must be exchanged with another generalized symmetry. Now we are going discuss this issue.

Here we will consider the case $(n, k) = (3, 2)$. The time dimensions we will denote with t, τ and the space dimensions with x, y, z. In the case of two-dimensional

[h] We point out that in the formalism used here, the strong conjugation is a unitary operator when acting on the state space, as well as strong time reversal.

[i] In the formalism used here, CPT is a linear operator in pseudo-Euclidean space.

time the term "switching principle" will mean inversion of the proper time $d\mathbf{T}_0 = (dt_{01}, dt_{02}) = (\alpha_{01} dT_0, \alpha_{02} dT_0)$—that is, multiplication of the numbers α_{01} and α_{02} by -1. (See Subsection IX.A.) This is equivalent to rest-mass sign inversion. Therefore, every object moving backward in the time dimension t (that is, $dt < 0$) and having energy

$$-\mathbf{E}_1 = \begin{pmatrix} -e_{11} \\ -e_{21} \end{pmatrix}$$

defined in relation to t can be described as the corresponding antiobject moving forward in the time dimension t (that is, $dt > 0$) and having energy

$$\mathbf{E}_1 = \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix}$$

defined in relation to t. The same considerations apply to the time dimension τ as well.

Let us denote with A_{-+}, A_{-0}, A_{--}, A_{0-}, A_{+-} the different kinds of antiparticles in two-dimensional time. The antiparticle A_{-+} is moving backward in the time dimension t and forward in the time dimension τ (that is, $dt < 0$ and $d\tau > 0$). The antiparticle A_{-0} is moving backward in the time dimension t but does not move in the time dimension τ (that is, $dt < 0$ and $d\tau = 0$). The antiparticle A_{--} moves backward in the time dimensions t and τ (that is, $dt < 0$ and $d\tau < 0$). The antiparticle A_{0-} moves backward in the time dimension τ but does not move in the time dimension t (that is, $dt = 0$ and $d\tau < 0$). The antiparticle A_{+-} moves forward in the time dimension t and backward in the time dimension τ (that is, $dt > 0$ and $d\tau < 0$).

By analogy with Eq. (78), we have the equality

$$SP_{t\tau} \equiv C\hat{E}\hat{\mathbf{p}}, \tag{83}$$

where $\hat{E}$ is an operator changing the signs of all components of the energy $\mathbf{E}$ [the quantities $e_{\theta\sigma}$ (σ, $\theta = 1, 2$)], $\hat{\mathbf{p}}$ is an operator changing the sign of all components of the momentum $\mathbf{p}$ [the quantities $p_{\theta x}$, $p_{\theta y}$, $p_{\theta z}$ ($\theta = 1, 2$)], and C is the conjugation of all conserved additive charges.

By analogy with Eq. (80), we have the equality

$$SP_{t\tau} \equiv CC_m X, \tag{84}$$

where the operation C_m is the rest-mass sign inversion and the operator X performs the exchange of emission and absorption (and vice versa).

If in Eq. (36) we set $(n, k) = (3, 2)$, then we obtain

$$c^2dt^2 + c^2d\tau^2 - dx^2 - dy^2 - dz^2 - c^2dt_{01}^2 - c^2dt_{02}^2 = 0. \tag{85}$$

By means of Eqs. (84) and (85) we can interpret the meaning of the operation $SP_{t\tau}$ from the geometrical point of view. According to Eq. (85), we could consider a generalized [(3 + 2) + 2]-dimensional space-time, where the two additional dimensions are the projections of the proper time t_{01} and t_{02}. Therefore, $SP_{t\tau}$ is a combination of inversion in relation to the axes ct_{01} and ct_{02} and application of the operator X.

By analogy with Eq. (82), in the case of two dimensional time we have the following three formulas:

$$\Lambda_{-}^{\downarrow\leftarrow} \equiv (-\mathbf{1})\Lambda_{+}^{\uparrow\rightarrow} \equiv \Pi\Gamma\Lambda_{+}^{\uparrow\rightarrow} \xrightarrow{SP_{t\tau}} CP\hat{t}\hat{\tau}\Lambda_{+}^{\uparrow\rightarrow}, \tag{86}$$

$$\Lambda_{+}^{\uparrow\leftarrow} \equiv \Pi\Omega\Lambda_{+}^{\uparrow\rightarrow} \xrightarrow{SP_{t\tau}} CP\hat{t}\Lambda_{+}^{\uparrow\rightarrow}, \tag{87}$$

$$\Lambda_{+}^{\downarrow\rightarrow} \equiv \Pi\Xi\Lambda_{+}^{\uparrow\rightarrow} \xrightarrow{SP_{t\tau}} CP\hat{\tau}\Lambda_{+}^{\uparrow\rightarrow}, \tag{88}$$

where the operators $\hat{t}$ and $\hat{\tau}$ correspond to the standard t-reversal transformation and τ-reversal transformation operations, and P is the standard operation parity (see Subsection III.C and more precisely Table II). The transformations $\Lambda_{-}^{\downarrow\leftarrow}$ lead to a change of the signs of the numbers $\chi_1 = dt/dT$ and $\chi_2 = d\tau/dT$ and therefore to a change of the signs of all components of the energy $e_{\theta\sigma}$ (σ, $\theta = 1, 2$)—see Eq. (58). The transformations $\Lambda_{+}^{\uparrow\leftarrow}$ lead to a change of the sign of the number χ_1 and therefore to a change of the signs only of the t-components of the energy $e_{\theta 1}$ ($\theta = 1, 2$). The transformations $\Lambda_{+}^{\downarrow\rightarrow}$ lead to a change of the sign of the number χ_2 and therefore to a change of the signs only of the τ-components of the energy $e_{\theta 2}$ ($\theta = 1, 2$). The transformations $\Lambda_{-}^{\downarrow\leftarrow}$, $\Lambda_{+}^{\uparrow\leftarrow}$, and $\Lambda_{+}^{\downarrow\rightarrow}$ lead to a sign change of the numbers $\chi_3 = dx/dX$, $\chi_4 = dy/dX$, $\chi_5 = dz/dX$ and therefore to a sign change of all momentum components $p_{\theta x}$, $p_{\theta y}$, $p_{\theta z}$ ($\theta = 1, 2$)—see Eq. (68). The same is valid also for the respective components in the dual spaces.

Let particle M be moving forward in the time dimensions t and τ. Application of the proper orthochronous transformations $\Lambda_{+}^{\uparrow\rightarrow}$ in some cases leads to a direction change of the time dimensions t or τ (see Subsection III.B). Therefore, as a result of the transformations $\Lambda_{+}^{\uparrow\rightarrow}$ applied on the particle M, in these cases we obtain some of the listed antiparticles: A_{-+}, A_{-0}, A_{--}, A_{0-}, or A_{+-}.

Let us assume, for example, that as a result of the transformations $\Lambda_{+}^{\uparrow\rightarrow}$ applied on the particle M, the antiparticle A_{--} ($dt < 0, d\tau < 0$) has been created. In this case, when applying the operations in Eqs. (86), (87), and (88) we will have the respective antiparticles of the particle A_{--}. For example, if to the obtained antiparticle A_{--} we apply the operations in Eq. (86), then we will obtain the given particle M. Therefore, with application of the operations in Eqs. (86), (87), and (88) the directions of t or τ are reversed, but it is possible to obtain antiparticles moving backward in the time dimensions t or τ as well as particles moving forward in the time dimensions t or τ (depending on the transformations $\Lambda_{+}^{\uparrow\rightarrow}$).

Let us assume that the proper orthochronous transformations $\Lambda_{+}^{\uparrow\rightarrow}$ have not changed the directions of the time dimensions t and τ—that is, $dt > 0, d\tau > 0$.

The transformations $\Lambda_{-}^{\downarrow\leftarrow}$ lead to a sign change of all energy components $e_{\theta\sigma}$ (σ, $\theta = 1, 2$). Therefore, for the energy components (after application of $\Lambda_{-}^{\downarrow\leftarrow}$) we will have

$$\mathbf{E}'_t = -\mathbf{E}_t = \begin{pmatrix} -e_{11} \\ -e_{21} \end{pmatrix},$$

where

$$-e_{\theta 1} = m_0(-u_{\theta 1})c = \frac{\lambda_{\theta 1}(-\chi_1)m_0c^2}{\alpha_{0\theta}\sqrt{1-\beta^2}}, \qquad \theta = 1,2;$$

$$\mathrm{E}'_\tau = -\mathbf{E}_\tau = \begin{pmatrix} -e_{12} \\ -e_{22} \end{pmatrix},$$

where

$$-e_{\theta 2} = m_0(-u_{\theta 2})c = \frac{\lambda_{\theta 2}(-\chi_2)m_0c^2}{\alpha_{0\theta}\sqrt{1-\beta^2}}, \qquad \theta = 1,2.$$

If we apply $SP_{t\tau}$ we obtain $dt''_{01} = -dt_{01}$ (that is, $\alpha''_{01} = -\alpha_{01}$) and $dt''_{02} = -dt_{02}$ (that is, $\alpha''_{02} = -\alpha_{02}$. According to Eq. (58), we will have

$$\mathbf{E}''_t = -\mathbf{E}'_t = \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix}$$

and

$$\mathbf{E}''_\tau = -\mathbf{E}'_\tau = \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix},$$

where $e_{\theta 1} = (-m_0)(-u_{\theta 1})c$ and $e_{\theta 2} = (-m_0)(-u_{\theta 2})c$. As a result of these operations we obtain the antiparticle A_{--}.

The transformations $\Lambda_+^{\uparrow\leftarrow}$ lead to a sign change only of the t-components of the energy $e_{\theta 1}$ ($\theta = 1, 2$). Therefore, for the energy components (after application of $\Lambda_+^{\uparrow\leftarrow}$) we will have

$$\mathrm{E}'_t = -\mathbf{E}_t = \begin{pmatrix} -e_{11} \\ -e_{21} \end{pmatrix},$$

where

$$-e_{\theta 1} = m_0(-u_{\theta 1})c = \frac{\lambda_{\theta 1}(-\chi_1)m_0c^2}{\alpha_{0\theta}\sqrt{1-\beta^2}}, \qquad \theta = 1,2;$$

$$\mathbf{E}'_\tau = \mathbf{E}_\tau = \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix},$$

where

$$e_{\theta 2} = m_0u_{\theta 2}c = \frac{\lambda_{\theta 2}\chi_2m_0c^2}{\alpha_{0\theta}\sqrt{1-\beta^2}}, \qquad \theta = 1,2.$$

If we apply $SP_{t\tau}$ we obtain $dt''_{01} = -dt_{01}$ (that is, $\alpha''_{01} = -\alpha_{01}$) and $dt''_{02} = -dt_{02}$ (that is, $\alpha''_{02} = -\alpha_{02}$). According to Eq. (58), we will have

$$\mathbf{E}''_t = -\mathbf{E}'_t = \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix}$$

and

$$\mathbf{E}''_\tau = -\mathbf{E}'_\tau = \begin{pmatrix} -e_{12} \\ -e_{22} \end{pmatrix},$$

where $e_{\theta 1} = (-m_0)(-u_{\theta 1})c$ and $-e_{\theta 2} = (-m_0)u_{\theta 2}c$. As a result of these operations we obtain the antiparticle A_{-+} or the antiparticle A_{-0}.

The transformations $\Lambda_+^{\downarrow\rightarrow}$ lead to a sign change only of the τ-components of the energy $e_{\theta 2}$ ($\theta = 1, 2$). Therefore, for the energy components (after application of $\Lambda_+^{\downarrow\rightarrow}$) we will have

$$\mathbf{E}'_t = \mathbf{E}_t = \begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix},$$

where

$$e_{\theta 1} = m_0u_{\theta 1}c = \frac{\lambda_{\theta 1}\chi_1m_0c^2}{\alpha_{0\theta}\sqrt{1-\beta^2}}, \qquad \theta = 1,2;$$

$$\mathbf{E}'_\tau = -\mathbf{E}_\tau = \begin{pmatrix} -e_{12} \\ -e_{22} \end{pmatrix},$$

where

$$-e_{\theta 2} = m_0(-u_{\theta 2})c = \frac{\lambda_{\theta 2}(-\chi_2)m_0c^2}{\alpha_{0\theta}\sqrt{1-\beta^2}}, \qquad \theta = 1,2.$$

If we apply $SP_{t\tau}$ we obtain $dt''_{01} = -dt_{01}$ (that is, $\alpha''_{01} = -\alpha_{01}$) and $dt''_{02} = -dt_{02}$ (that is, $\alpha''_{02} = -\alpha_{02}$). According to Eq. (58), we will have

$$\mathbf{E}''_t = -\mathbf{E}'_t = \begin{pmatrix} -e_{11} \\ -e_{21} \end{pmatrix}$$

and

$$\mathbf{E}''_\tau = -\mathbf{E}'_\tau = \begin{pmatrix} e_{12} \\ e_{22} \end{pmatrix},$$

where $-e_{\theta 1} = (-m_0)u_{\theta 1}c$ and $e_{\theta 2} = (-m_0)(-u_{\theta 2})c$. As a result of these operations we obtain the antiparticle A_{+-} or the antiparticle A_{0-}.

According to these considerations, a negative rest mass equal to $-m_0$ must be attached to the antiparticles A_{-+}, A_{-0}, A_{--}, A_{0-}, A_{+-}.

On the ground of the obtained results, it is possible to make a generalization of the strong CPT symmetry formulated by Recami and Ziino.[11] In the case of two-dimensional time the physical world is symmetric (i.e., the physical laws are invariant) for inversion of the axes x, y, z, ct, $c\tau$, ct_{01}, ct_{02} [see Eqs. (84) and (85)].

Let us summarize the obtained results: If $k = 1$, then the number of antiparticles is 1; if $k = 2$, then the number of antiparticles is 5. With similar considerations, one can find that if the number of time dimensions is equal to k, then the number of the different antiparticles is equal to $3^k - 2^k$. For example, for the case $k = 3$ we obtain $3^3 - 2^3 = 19$ different antiparticles. Indeed, let us denote with t_1, t_2, …, t_k the time dimensions. Since we consider antiparticles, it must be true that $dt_\delta < 0$ for at least one of the time dimensions t_δ ($1 \le \delta \le k$). Obviously, for each of the quantities dt_σ ($\sigma = 1, 2, \ldots, k$) there are three possibilities: $dt_\sigma > 0$ or $dt_\sigma < 0$ or $dt_\sigma = 0$. If we take into consideration these three possibilities and the circumstance that the number of time dimensions is equal to k, then we have a total of $\bar{V}(3, k) = 3^k$ cases. (Here $\bar{V}(3, k)$ denotes the respective variations with repetition.) From these cases we have to exclude those in which all quantities dt_1, dt_2, ..., dt_k are non-negative. These are the cases where for all values of σ ($\sigma = 1, 2, \ldots, k$), either $dt_\sigma > 0$ or $dt_\sigma = 0$ is true. This comes to $\bar{V}(2, k) = 2^k$ cases. Therefore, the total number of different kinds of antiparticles is equal to $3^k - 2^k$.

As we pointed out, the antiparticles must move backward in at least one of the time dimensions; in the remaining $k - 1$ time dimensions they can move forward, backward, or not at all. It is easy to prove that there exist

$$\binom{k}{q}(2^{k-q} - 1)$$

different antiparticles which do not move in q of a total k time dimensions. Here $1 \le q \le k$ and

$$\binom{k}{q}$$

is the binomial coefficient. [If $q = 0$, then for each σ ($\sigma = 1, 2, \ldots, k$) the condition $dt_\sigma \neq 0$ is fulfilled.]

Let us have an antiparticle A moving in k-dimensional time. We can prove that in the case of k-dimensional time there exist $2^k - 1$ different particles M corresponding to the antiparticle A. [Concerning the movement of the particle M in the hyperplane of time, the condition $dt_\sigma \ge 0$ is fulfilled for all values of σ ($\sigma = 1, 2, \ldots, k$) and the condition $dt_\sigma > 0$ is fulfilled for at least one value of σ.) For example, in the case $k = 2$ we obtain $2^2 - 1 = 3$ different particles M corresponding to the antiparticle A: M_{++} ($dt > 0$, $d\tau > 0$), M_{+0} ($dt > 0$, $d\tau = 0$), M_{0+} ($dt = 0$, $d\tau > 0$).

XI. DISTINCTION BETWEEN TACHYONS AND PARTICLES MOVING IN MULTIDIMENSIONAL TIME

Recami pointed out that in the course of any generalization of STR in such a way that tachyons are included, it will turn out that these particles move in three time dimensions and one space dimension.[7]

Tegmark[5] pointed out that the case $(n, k) = (1, 3)$ is mathematically equivalent to the case $(n, k) = (3, 1)$, so that "all particles are tachyons with imaginary rest mass."[47]

However, from the physical point of view these two cases are not equivalent. It is necessary to distinguish between the transformations bradyon → tachyon, where a particle is created moving with a velocity greater than the speed of light in a vacuum,[j] and the transformations $(n, k) = (3, 1) \to (n, k) = (1, 3)$, where a particle is created moving in three-dimensional time and one-dimensional space.

It has been proven[48–56] that if both postulates of STR are fulfilled (namely the principle of invariance of the speed of light and the principle of relativity), then the transformations describing the transfer between the two inertial reference frames must be such that

$$\begin{aligned}(x^{1\prime})^2 - (x^{2\prime})^2 - (x^{3\prime})^2 - (x^{4\prime})^2 \\ = \pm\left[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2\right]\end{aligned} \tag{89}$$

for each four-vector $\mathbf{x} = (x^1, x^2, x^3, x^4)$, which can be a four-dimensional vector on the position of a given particle, a four-momentum, four-velocity, four-current, etc. In the special case where space-time coordinates are considered, Eq. (89) takes the following form:

$$\begin{aligned}(ct')^2 - (x')^2 - (y')^2 - (z')^2 \\ = \pm\left[(ct)^2 - (x)^2 - (y)^2 - (z)^2\right].\end{aligned} \tag{90}$$

The plus sign in the right-hand side of Eqs. (89) and (90) corresponds to the standard case of subluminal relative velocities, i.e., concerns bradyons [here $(n, k) = (3, 1)$]; the minus sign must be chosen in the case of superluminal relative velocities, i.e., concerns tachyons (see, for example, Refs. 57 and 58).

The difference between the transformations bradyon → tachyon and the transformations $(n, k) = (3, 1) \to (n, k) = (1, 3)$ can be understood best if, instead of four-dimensional Minkowski space-time, we consider a generalized five-dimensional space-time (x, y, z, ct, ct_{01}), where the fifth dimension corresponds to proper time t_{01} and therefore is related to the rest mass (see Section X). If we set $(s)^2 = (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2$ and $(s')^2 = (x^{1\prime})^2 - (x^{2\prime})^2 - (x^3)^2 - (x^{4\prime})^2$, then according to Eq. (89) we will have $(s')^2 = \pm(s)^2 (s')^2 = \pm(s)^2$, where $(s)^2 > 0$. Therefore, Eqs. (89) and (90) can be written as

$$\begin{aligned}(x^{1\prime})^2 - (x^{2\prime})^2 - (x^{3\prime})^2 - (x^{4\prime})^2 - (s')^2 = 0 \\ = \pm\left[(x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 - (s)^2\right],\end{aligned} \tag{91}$$

$$\begin{aligned}(ct')^2 - (x')^2 - (y')^2 - (z')^2 \mp (ct_{01})^2 = 0 \\ = \pm\left[(ct)^2 - (x)^2 - (y)^2 - (z)^2 - (ct_{01})^2\right].\end{aligned} \tag{92}$$

[Here, $s = ct_{01}$—see Eq. (36).] Since in the case of superluminal relative velocities one should choose the minus sign in the right-hand side of Eq. (91), during the transformations bradyon → tachyon the signs in the expressions $(x^1)^2$, $(x^2)^2$, $(x^3)^2$, $(x^4)^2$, $(s)^2$ will change. This is equivalent to a multiplication of the row matrix $\mathbf{S} = (x^1, x^2, x^3, x^4, s)$ by the complex diagonal matrix $\mathbf{D} = \mathrm{diag}(i, i, i, i, i)$, where $i = \sqrt{-1}$—that is, the product $\mathbf{S} \times \mathbf{D}$. The multiplication of the row matrix $\mathbf{S}$ by the matrix $\mathbf{D}$ corresponds to rotation of all axes x^1, x^2, x^3, x^4, s through an angle of $\pi/2$ (arg $i = \pi/2$).

According to Eq. (92), in the case of superluminal relative velocities we have $(ct')^2 - (x')^2 - (y')^2 - (z')^2 + (ct_{01})^2$. Therefore, instead of (3 + 1)-dimensional space-time, we could consider a generalized [3 + (1 + 1)]-dimensional space-time, where the additional dimension corresponds to proper time ct_{01}. In the generalized space-time, 1 + 1 dimensions are timelike (ct', ct_{01}) and three dimensions are spacelike (x', y', z'). These considerations also concern the dual space: In the generalized dual space,

[j] We point out that the velocities of the bradyon and of the tachyon are defined from the point of view of an observer situated in $(n, k) = (3, 1)$ space-time.

$1+1$ dimensions are timelike ($x^{1\prime}=E'$, $s'=m_0c^2$) and three dimensions are spacelike ($x^{2\prime}=p'_x c$, $x^{3\prime}=p'_y c$, $x^{4\prime}=p_z^{\prime C}$). (Here $m_0^2>0$.) The energy E' of the tachyon will have only one component, and the momentum $\mathbf{p}'_s$ of the tachyon will have three components. The expression $(E')^2-(\mathbf{p}'_s)^2c^2=-m_0^2c^2<0$ will be valid. Obviously, the tachyon will have imaginary rest mass: $(im_0)^2=-m_0^2<0$.

As we know, in the case of $(n, k)=(1, 3)$ the following equality is fulfilled for each four-vector $\mathbf{x}=(x^1, x^2, x^3, x^4)$:

$$\begin{aligned}&-(x^{1\prime})^2+(x^{2\prime})^2+(x^{3\prime})^2+(x^{4\prime})^2\\&=-(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2.\end{aligned}\tag{93}$$

(Here x_1 is spacelike dimension and x_2, x_3, x_4 are timelike dimensions.) If we set $(s)^2=-[(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2]$ and $(s')^2=-(x^{1\prime})^2+(x^{2\prime})^2+(x^{3\prime})^2+(x^{4\prime})^2$, then in accordance with Eq. (93) we will have $(s')^2=(s)^2>0$. Therefore, Eq. (93) can be written in the form

$$\begin{aligned}&-(x^{1\prime})^2+(x^{2\prime})^2+(x^{3\prime})^2+(x^{4\prime})^2-(s')^2=0\\&=-\Big[(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2\Big]-(s)^2.\end{aligned}\tag{94}$$

It is clear that if $(n, k)=(3, 1)$, then we have $(s)^2=(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2>0$. [See Eq. (91) for the case of subluminal relative velocities, i.e., those with the plus sign in the right-hand side of Eq. (91).] If $(n, k)=(1, 3)$, then we have $(s)^2=-[(x^1)^2-(x^2)^2-(x^3)^2-(x^4)^2]>0$ [see Eq. (94)]. Therefore, in the transformations $(n, k)=(3, 1)\to(n, k)=(1, 3)$ the signs are changed in front of the expressions $(x^1)^2$, $(x^2)^2$, $(x^3)^2$, $(x^4)^2$, but unlike in the transformations bradyon $\to$ tachyon, the sign is not changed in front of the expression $(s)^2$. This is equivalent to multiplication of the row matrix $\mathbf{S}=(x^1, x^2, x^3, x^4, s)$ by the complex diagonal matrix $\mathbf{J}=\mathrm{diag}(i, i, i, i, 1)$, where $i=\sqrt{-1}$—that is, the product $\mathbf{S}\times\mathbf{J}$. The multiplication of $\mathbf{S}$ by $\mathbf{J}$ reflects rotation of the axes x^1, x^2, x^3, x^4 through angle $\pi/2$ around the axis s (the axis s is invariant in relation to the applied operation).

According to Eq. (94), in the special case of space-time coordinates we have

$$\begin{aligned}&-(x')^2+(ct'_1)^2+(ct'_2)^2+(ct'_3)^2-(ct_{01})^2-(ct_{02})^2\\&-(ct_{03})^2=0.\end{aligned}\tag{95}$$

Here t_{01}, t_{02}, t_{03} are the projections of the proper time $\mathbf{T}_0$ [see Eq. (36)]. According to Eq. (95), instead of $(1+3)$-dimensional space-time we can consider a generalized $[(1+3)+3]$-dimensional space-time, where three dimensions correspond to the projections of proper time (ct_{01}, ct_{02}, ct_{03}). In the generalized space-time, three dimensions are timelike (ct'_1, ct'_2, ct'_3) and $1+3$ dimensions are spacelike (x', ct_{01}, ct_{02}, ct_{03}).

These considerations also concern the dual space: In the generalized dual space, three dimensions are timelike ($x^{2\prime}=E'_1/\gamma_t$, $x^{3\prime}=E'_2/\gamma_t$, $x^{4\prime}=E'_3/\gamma_t$) and $1+3$ dimensions are spacelike ($x^{1\prime}=p'_x c/\gamma_s$, $s^{1\prime}=E_{01}/\gamma_t$, $s^{2\prime}=E_{02}/\gamma_t$, $s^{3\prime}=E_{03}/\gamma_t$)—see Eqs. (62) and (74). (Here $p'_s\equiv p'_x$.)

According to the considerations in Subsection IX.B, the total energy of the particle will have nine components $e'_{\theta s}=m_0u'_{\theta\sigma}c$ (σ, $\theta=1, 2, 3$)—see Eq. (57). The particle's momentum will have three components $p'_{\theta x}=m_0u'_{\theta x}$ ($\theta=1, 2, 3$)—see Eq. (67). The following relation will be fulfilled [see Eq. (74)]:

$$\frac{(E')^2}{\gamma_t^2}-\frac{(p'_x)^2c^2}{\gamma_s^2}=\frac{E_0^2}{\gamma_t^2}=m_0^2c^4>0.$$

The particle will have real rest mass ($m_0^2>0$).

According to these considerations (see Section IV), in the case $(n, k)=(1, 3)$ the particle can have velocity (defined in relation to one of the time dimensions) which is greater than, less than, or equal to the speed of light in a vacuum. Until now it was accepted that if a particle moves according to one observer with a velocity which is greater than the speed of light in a vacuum, then the particle has imaginary rest mass (and accordingly it has real mass measured by the observer). However, this is not fulfilled if the particle moves in multidimensional time—in this case it will have real rest mass.

If the particle moves in the space-time $(n, k)=(1, 3)$ and the condition

$$\sum_{\varsigma=1}^{3}\frac{c^2}{(V'_\varsigma)^2}<1$$

is fulfilled, where $V'_\varsigma=|\mathrm{d}x'|/|dt'_\varsigma|$ (see Sections IV and VIII), then by analogy with Eqs. (91) and (92) we have

$$\begin{aligned}&-(x^{1\prime})^2+(x^{2\prime})^2+(x^{3\prime})^2+(x^{4\prime})^2-(s')^2=0\\&=-\Big[-(x^1)^2+(x^2)^2+(x^3)^2+(x^4)^2-(s)^2\Big]\end{aligned}$$

and

$$\begin{aligned}&-(x')^2+(ct'_1)^2+(ct'_2)^2+(ct'_3)^2+(ct_{01})^2+(ct_{02})^2\\&+(ct_{03})^2=0=-\Big[-(x)^2+(ct_1)^2+(ct_2)^2+(ct_3)^2\\&-(ct_{01})^2-(ct_{02})^2-(ct_{03})^2\Big].\end{aligned}$$

Here $(s')^2=-(s)^2<0$.

XII. CONCLUSION

The simplest way of thinking about and considering multidimensional time is using a branching or train-track model. Meiland proposed a more formal model of multidimensional time, where the past can be changed.[59] Despite the changes, however, there is only one past. According to Meiland, his two-dimensional time model is not radically different from our ordinary, one-dimensional perception of time. He treats "the past as a continuant, as existing at each of several times."[59]

In multidimensional time, like in one-dimensional time, every localized object is moving along a one-dimensional timelike world line.[5] Therefore, even in two or more dimensions time will look one-dimensional, because all physical processes (including thinking) will run in a linear sequence, which is characteristic of the perception of reality. Clocks will work in their usual manner. Every localized object will have one single "history" in the multidimensional time. In this sense, the notion an observer will build of multidimensional time will not differ very much from the well-known notion of time. However, in the case of multidimensional time there are problems concerning well-posed causality[4,5,13,14] (see Section IV).

As stated in Section IV, for multidimensional time the notions of past, present, and future can be defined as well. If time is one-dimensional, then one point (the present moment) divides the time axis into two separate regions: past and future. If the number of time dimensions is equal to k, then time can be divided into two k-dimensional regions from one $(k-1)$-dimensional hypersurface—more precisely, from the $(k-1)$-dimensional hypersphere (see Section IV). The two obtained k-dimensional regions can conditionally be called past and future, and the border region—i.e., the $(k-1)$-dimensional hypersphere—can be called present (see Section IV).

Let us consider two nonrelativistic observers moving in different time directions. (In this consideration relativistic effects are neglected.) These observers can meet in the space-time and can synchronize their clocks only if their directions of movement are crossing in time. Let us assume that the observers meet at point O on the hyperplane of time. Then these observers will separate again and will continue to move in their time directions, without any opportunity to meet.[5] Let us assume that, according to the one of the observers, from the moment of their meeting (point O) a period $\Delta T > 0$ has passed. Because in the case of k-dimensional time the present is a $(k-1)$-dimensional hypersphere, in this case both observers will be at points (moments) which lie on the $(k-1)$-dimensional hypersphere with center O and radius ΔT (see Section IV).

In a universe of multidimensional time many other strange things can happen. If two observers are not moving against each other (in space) but they move in different directions in the hyperplane of time, then according to their opinion the same physical process will run with different speeds (see, e.g., Subsection III.A). If two observers move in orthogonal directions to each other in the hyperplane of time, then according to each one of them time for the other observer will "stand still," i.e., will not run. If these observers move relative to each other in space, then according to each one of them the other will move with infinitely high velocity (see Section IV).

In this study, there are results which can be proven experimentally. If there exist particles moving in multidimensional time, the following physical effects can be found which are different from the effects derived from STR:

- The transformations derived in Sections III and V will be valid for transfer from one inertial frame of reference to another.
- The law derived in Section VI will be valid for addition of velocities.
- The Doppler effect derived in Subsection III.D will be valid.
- The principle of invariance of the speed $c\sqrt{k}$ will be valid instead of the principle of the invariance of the speed of light c in STR (Section VIII).
- The causal region in multidimensional time (which is described in Section IV) will differ from the causal region in STR.
- The relations derived in Subsection IX.B will be valid between total energy E, total momentum p_s, rest mass m_0, relativistic mass m, and velocity V of a particle. They differ from the relations in the STR.
- The energy-momentum conservation law derived in Subsection IX.C will be valid. This law differs from the energy-momentum conservation law of STR.
- A new, different CPT symmetry will be valid (Section X).
- In the case of k-dimensional time, there will exist $3^k - 2^k$ different antiparticles (Section X).

According to the results obtained in the study:

- It is proven that in multidimensional time, particles can move in the causal region with a velocity which is greater than, less than, or equal to the speed of light in a vacuum (Section IV). For the case of multidimensional time, it is possible that a particle can move simultaneously faster that the speed of light in a vacuum and forward in the time dimensions, which is not true for the case of one-dimensional time (STR; Subsection III.B). Thus, if the results for the superluminal neutrinos are confirmed,[60] this can be explained with the existence of additional time dimensions in which these neutrinos are moving.
- In the case of multidimensional time, application of the proper orthochronous transformations at certain conditions leads to movement backward in the time dimensions (Subsection III.B).
- It is shown that particles moving faster than the speed of light in a vacuum can have a real rest mass (unlike tachyons), provided that they move in multidimensional time (Subsection IX.B).
- Thus the existence of particles moving in multidimensional time can be proven or rejected through proper experiments.

As a conclusion, we want to point out that if in our universe exist particles moving in two, three, or more time dimensions, then the relations between energy, mass, velocity, and momentum of these particles will be expressed through the formulas derived in this article.

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