Energy-Entanglement Relation for Quantum Energy Teleportation

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Abstract

Protocols of quantum energy teleportation (QET), while retaining causality and local energy conservation, enable the transportation of energy from a subsystem of a many-body quantum system to a distant subsystem by local operations and classical communication through ground-state entanglement. We prove two energy-entanglement inequalities for a minimal QET model. These relations help us to gain a profound understanding of entanglement itself as a physical resource by relating entanglement to energy as an evident physical resource.

1 Introduction

The relationship between energy and information has been investigated extensively in the context of computation energy cost including a modern analysis of Maxwell's demon [1]. In this Letter, we show a new energy-information relation from a different point of view. Recently, it has been reported that energy can be transported by local operations and classical communication while retaining local energy conservation and without breaking causality [2]-[4]. Such protocols are called quantum energy teleportation (QET) and are based on ground-state entanglement of many-body quantum systems. By performing a local measurement on a subsystem A of a many-body system in the ground state, information about the quantum fluctuation of A can be extracted. During this measurement, some amount of energy is infused into A as QET energy input, and the ground-state entanglement gets partially broken. Next, the measurement result is announced to another subsystem B of the many-body system at a speed much faster than the diffusion velocity of the energy infused by the measurement. Soon after the information arrives at B, energy can be extracted from B by performing a local operation on B dependent on the announced measurement data. The root of the protocols is a correlation between the measurement information of A and the quantum fluctuation of B via the ground-state entanglement. Due to the correlation, we are able to estimate the quantum fluctuation of B based on the announced information from A and devise a strategy to control the fluctuation. By the above-mentioned selected local operation on B, the fluctuation of B can be more suppressed than that of the ground state, yielding negative energy density around B in the many-body system. The concept of negative energy density has been investigated in quantum field theory for a long time [5]. Quantum interference among particle-number eigenstates can produce various states containing regions of negative energy density, although the total energy remains nonnegative. The regions of negative energy density can appear in general many-body quantum systems including spin chains [2] and trapped ions [3] by fixing the origin of the energy density such that the expectational value vanishes for the ground state. In spite of the emergence of negative energy density, the total energy also remains nonnegative for the general cases. In the QET protocols, during the generation of negative energy density at B, surplus energy is transferred from B to external systems and can be harnessed as QET energy output. Meanwhile, using the usual protocols of quantum teleportation, quantum states can be transported from one party to another by the consumption of shared entanglement between the two parties [6]. As is well known [7], transfer of a large number of quantum states requires a large amount of consumption of shared entanglement as a physical resource. Taking into account the fact, it seems natural for the QET protocols to expect that a large amount of teleported energy also requests a large amount of consumption of the ground-state entanglement between A and B. If such a non-trivial relation exists between teleported energy and breaking of ground-state entanglement by measurement, the relation may shed new light on the interplay between quantum physics and quantum information theory. In this Letter, the first example of the energy-entanglement relation for a minimal QET model is presented. The minimal QET model is the smallest physical system for which non-trivial QET can be implemented; this model consists of two qubits with an interaction of the Ising spin chain in the presence of a transverse magnetic field. We explicitly show that for the minimal model, the consumption of entanglement between A and B during the measurement of A is lower bounded by a positive value that is proportional to the maximum amount of energy teleported from A to B. In addition, we obtain another inequality in which the maximum amount of energy teleported from A to B is lower bounded by a different positive value that is proportional to the amount of entanglement breaking between A and B by the measurement of A. These energy-entanglement inequalities are of importance because they help in gaining a profound understanding of entanglement itself as a physical resource by relating entanglement to energy as an evident physical resource.

2 Minimal QET Model

First of all, we introduce the minimal QET model. The system consists of two qubits A and B. Its Hamiltonian is the same as that of the Ising spin chain in the presence of a transverse magnetic field as follows: $H = H_A + H_B + V$, where each contribution is given by

$$H_A = h\sigma_A^z + \frac{h^2}{\sqrt{h^2+k^2}}, \tag{1}$$

$$H_B = h\sigma_B^z + \frac{h^2}{\sqrt{h^2+k^2}}, \tag{2}$$

$$V = 2k\sigma_A^x\sigma_B^x + \frac{2k^2}{\sqrt{h^2+k^2}}, \tag{3}$$

and h and k are positive constants with energy dimensions, σ_A^x (σ_B^x) is the x-component of the Pauli operators for the qubit A (B), and σ_A^z (σ_B^z) is the z-component for the qubit A (B). The constant terms in Eqs. (1)-(3) are added in order to make the expectational value of each operator zero for the ground state $|g\rangle$: $\langle g|H_A|g\rangle = \langle g|H_B|g\rangle = \langle g|V|g\rangle = 0$. Because the lowest eigenvalue of the total Hamiltonian H is zero, H is a nonnegative operator. Meanwhile, it should be noticed that H_B has a negative eigenvalue, which can yield negative energy density. The ground state is given by

$$|g\rangle = \frac{1}{\sqrt{2}}\sqrt{1-\frac{h}{\sqrt{h^2+k^2}}}|+\rangle_A|+\rangle_B - \frac{1}{\sqrt{2}}\sqrt{1+\frac{h}{\sqrt{h^2+k^2}}}|-\rangle_A|-\rangle_B,$$

where $|\pm\rangle_A$ ($|\pm\rangle_B$) is the eigenstate of σ_A^z (σ_B^z) with eigenvalue ±1. Let S_{M_A} denote a set of POVM measurements [7] for A which measurement operators $M_A(\mu)$ with measurement output μ commute with the interaction Hamiltonian V. The measurement operator $M_A(\mu)$ takes the form of $M_A(\mu) = m_\mu + e^{i\alpha_\mu} l_\mu \sigma_A^x$, where m_μ, l_μ, and α_μ are real constants which satisfy $\sum_\mu \left(m_\mu^2 + l_\mu^2\right) = 1$ and $\sum_\mu m_\mu l_\mu \cos\alpha_\mu = 0$. The POVM corresponding to $M_A(\mu)$ is defined by $\Pi_A(\mu) = M_A(\mu)^\dagger M_A(\mu)$, which satisfies $\sum_\mu \Pi_A(\mu) = 1_A$. By introducing the emergence probability $p_A(\mu)$ of output μ for the ground state and a real parameter $q_A(\mu)$, the POVM is written as follows:

$$\Pi_A(\mu) = p_A(\mu) + q_A(\mu)\sigma_A^x.$$

By taking suitable values of m_μ, l_μ, and α_μ, all values of $p_A(\mu)$ and $q_A(\mu)$ are permissible as long as they satisfy $\sum_\mu p_A(\mu) = 1$, $\sum_\mu q_A(\mu) = 0$ and $p_A(\mu) \geq |q_A(\mu)|$. The post-measurement state with output μ is given by $|A(\mu)\rangle = \frac{1}{\sqrt{p_A(\mu)}} M_A(\mu)|g\rangle$. This measurement excites the system and the average post-measurement state has a positive expectational value E_A of H, which energy distribution is localized at A. In fact, the value defined by $E_A = \sum_\mu \langle g|M_A(\mu)^\dagger H M_A(\mu)|g\rangle$ is computed as

$$E_A = \sum_\mu \langle g|M_A(\mu)^\dagger H_A M_A(\mu)|g\rangle = \frac{2h^2}{\sqrt{h^2+k^2}} \sum_\mu l_\mu^2. \qquad (4)$$

The key feature of this model is that any measurement of S_{M_A} does not increase the average energy of B at all. By explicit calculations, the average values of the Hamiltonian contributions H_B and V are found to remain zero after the measurement and are the same as those of the ground state. Thus, we cannot extract energy from B by the standard way soon after the measurement. Of course, the infused energy E_A diffuses to B after a while. The time evolution of the expectational values H_B and V of the average post-measurement state is calculated as

$$\langle H_B(t)\rangle = \sum_\mu p_A(\mu)\langle A(\mu)|e^{itH} H_B e^{-itH}|A(\mu)\rangle = \frac{h^2\sum_\mu l_\mu^2}{\sqrt{h^2+k^2}}\left[1-\cos\left(4kt\right)\right],$$

and $\langle V(t)\rangle = 0$. Therefore, energy can be extracted from B after a diffusion time scale of $1/k$; this is just a usual energy transportation from A to B. Amazingly, the QET protocol can transport energy from A to B in a time scale much shorter than that of the usual transportation. In the protocol, the measurement output μ is announced to B. Because the model is non-relativistic, the propagation speed of the announced output can be much faster than the diffusion speed of the infused energy and can be approximated as infinity. Soon after the arrival of the output μ, we perform a local operation $U_B(\mu)$ on B dependent on μ. Then, the average state after the operation is given by

$$\rho = \sum_\mu U_B(\mu) M_A(\mu)|g\rangle\langle g|M_A(\mu)^\dagger U_B(\mu)^\dagger.$$

Of course, if this local operation does not depend on μ, the operation does not extract but instead gives energy to B. However, if a suitable operation $U_B(\mu)$ dependent on μ is selected and performed, the energy of the system decreases by a positive value E_B after the execution of $U_B(\mu)$. Because of local energy conservation, the same amount of energy E_B is transferred from the system to external systems, including the operation device of $U_B(\mu)$, and is considered as the QET energy output. In Figure 1, a schematic diagram of this QET model is presented. The expectational value of the total energy after the operation is given by $\mathrm{Tr}\left[\rho H\right] = \sum_\mu \langle g|M_A(\mu)^\dagger U_B(\mu)^\dagger H U_B(\mu) M_A(\mu)|g\rangle$. On the basis of the fact that $U_B(\mu)$ commutes with H_A and Eq. (4), E_B is computed as $E_B = E_A - \mathrm{Tr}\left[\rho H\right] = -\mathrm{Tr}\left[\rho\left(H_B + V\right)\right]$. Further, on the basis of the fact that $M_A(\mu)$ commutes with $U_B(\mu)$, H_B and V, the energy can be written as $E_B = -\sum_\mu \langle g|\Pi_A(\mu)\left(H_B(\mu) + V(\mu)\right)|g\rangle$, where the μ-dependent operators are given by $H_B(\mu) = U_B(\mu)^\dagger H_B U_B(\mu)$ and $V(\mu) = U_B(\mu)^\dagger V U_B(\mu)$. Here, let us write the general form of $U_B(\mu)$ as follows: $U_B(\mu) = \cos\omega_\mu + i\vec{n}_\mu\cdot\vec{\sigma}_B \sin\omega_\mu$, where ω_μ is a real parameter, $\vec{n}_\mu = (n_{x\mu}, n_{y\mu}, n_{z\mu})$ is a three-dimensional unit real vector and $\vec{\sigma}_B$ is the Pauli spin vector operator of B. Then, an explicit evaluation of E_B becomes possible. The result is expressed as $E_B = \frac{1}{\sqrt{h^2+k^2}}\sum_\mu Q(\mu)$, where $Q(\mu)$ is given by

$$Q(\mu) = X(\mu)\cos\left(2\omega_\mu\right) - hkq_A(\mu)n_{y\mu}\sin(2\omega_\mu) - X(\mu), \tag{5}$$

where $X(\mu)$ is defined by

$$X(\mu) = p_A(\mu)\left[h^2\left(1 - n_{z\mu}^2\right) + 2k^2\left(1 - n_{x\mu}^2\right)\right] - 3hkq_A(\mu)n_{x\mu}n_{z\mu}.$$

In order to maximize the teleported energy E_B for a given POVM measurement of A, let us first maximize $Q(\mu)$ in Eq. (5) by changing the parameter ω_μ. This maximum value is calculated as

$$\max_{\omega_\mu} Q(\mu) = \sqrt{X(\mu)^2 + \left[hkq_A(\mu)n_{y\mu}\right]^2} - X(\mu). \tag{6}$$

Next, let us introduce a parametrization of $n_{x\mu}$ and $n_{z\mu}$ as $n_{x\mu} = \sqrt{z}\cos\psi_\mu$ and $n_{z\mu} = \sqrt{z}\sin\psi_\mu$ for fixed $z = 1 - n_{y\mu}^2$ which runs over $[0, 1]$, where ψ_μ is a real parameter. It is observed that $\max_{\omega_\mu} Q(\mu)$ in Eq. (6) is a monotonically decreasing function of $X(\mu)$. Thus, we must find the minimum value of $X(\mu)$ in terms of ψ_μ. By using the parametrization, we can minimize $X(\mu)$ as

$$\min_{\psi_\mu} X(\mu) = \left(1 - \frac{z}{2}\right)p_A(\mu)\left(h^2 + 2k^2\right) - \frac{z}{2}\sqrt{\left(h^2 - 2k^2\right)^2 p_A(\mu)^2 + 9h^2k^2q_A(\mu)^2}.$$

Therefore, the maximum value of $\max_{\omega_\mu} Q(\mu)$ in terms of ψ_μ is obtained as follows:

$$\max_{\omega_\mu,\psi(\mu)} Q(\mu) = \sqrt{\left(\min_{\psi(\mu)} X(\mu)\right)^2 + h^2k^2q_A(\mu)^2(1-z)} - \min_{\psi(\mu)} X(\mu).$$

Next, in order to maximize $\max_{\omega_\mu,\psi_\mu} Q(\mu)$ in terms of z, let us write it as a function $T(z)$ of z:

$$T(z) = \max_{\omega_\mu,\psi_\mu} Q(\mu) = \sqrt{(a-bz)^2 + c(1-z)} - (a-bz),$$

where a, b and c are positive constants given by

$$\begin{aligned}
a &= p_A(\mu)\left(h^2+2k^2\right), \\
b &= \frac{p_A(\mu)}{2}\left(h^2+2k^2\right) + \frac{1}{2}\sqrt{\left(h^2-2k^2\right)^2 p_A(\mu)^2 + 9h^2k^2 q_A(\mu)^2}, \\
c &= h^2k^2 q_A(\mu)^2.
\end{aligned}$$

The derivative of $T(z)$ can be calculated as $\partial_z T(z) = \frac{t(z)}{2\sqrt{(a-bz)^2+c(1-z)}}$, where $t(z)$ is a function given by $t(z) = -c + 2b\left(\sqrt{(a-bz)^2+c(1-z)} - (a-bz)\right)$. It can be verified that $t(z)$ and $\partial_z T(z)$ are nonpositive for $z \in [0,1]$. This verification can be done as follows. Let us first consider an equation $t(\bar{z}) = 0$. It turns out that, in the transformation of this equation for solving $\bar{z}$, the dependence of $\bar{z}$ gets lost and we get just a constraint condition on $p_A(\mu)$ and $q_A(\mu)$ such that $q_A(\mu)^2\left(p_A(\mu)^2 - q_A(\mu)^2\right) = 0$. Thus, if $p_A(\mu)^2 = q_A(\mu)^2$ or $q_A(\mu)^2 = 0$, the equation $t(z) = 0$ holds for all $z \in [0,1]$. If $p_A(\mu)^2 \neq q_A(\mu)^2$ and $q_A(\mu)^2 \neq 0$, the solution $\bar{z}$ does not exist and $t(z)$ has a definite sign for $z \in [0,1]$. In order to check the sign, let us substitute $z = 1$ into $t(z)$. Then, when $a \geq b$, we get $t(1) = -h^2k^2q_A(\mu)^2 < 0$, and when $a \leq b$, $t(1) = -8h^2k^2(p_A(\mu)^2 - q_A(\mu)^2) < 0$. Thus, it is verified that $t(z)$ and $\partial_z T(z)$ are nonpositive. Therefore, $T(z)$ takes the maximum value at $z = 0$. This implies that $Q(\mu)$ can be maximized as $\max_{U_B(\mu)} Q(\mu) := \max_{\omega_\mu,\psi_\mu,z} Q(\mu) = T(0)$. This leads to our final expression of the maximum teleported energy for the measurement, which is clearly nonnegative, as follows:

$$\max_{U_B(\mu)} E_B = \frac{h^2+2k^2}{\sqrt{h^2+k^2}} \sum_\mu p_A(\mu)\left[\sqrt{1 + \frac{h^2k^2}{\left(h^2+2k^2\right)^2}\frac{q_A(\mu)^2}{p_A(\mu)^2}} - 1\right]. \quad (7)$$

The operation $U_{\max}(\mu)$ which attains the maximum of teleported energy is given by $U_{\max}(\mu) = \cos\Omega_\mu + i\sigma_B^y \sin\Omega_\mu$, where Ω_μ is a real constant which satisfies

$$\cos\left(2\Omega_\mu\right) = \frac{\left(h^2+2k^2\right)p_A(\mu)}{\sqrt{\left(h^2+2k^2\right)^2 p_A(\mu)^2 + h^2k^2 q_A(\mu)^2}},$$
$$\sin\left(2\Omega_\mu\right) = -\frac{hkq_A(\mu)}{\sqrt{\left(h^2+2k^2\right)^2 p_A(\mu)^2 + h^2k^2 q_A(\mu)^2}}.$$

Besides, the teleported energy can be maximized among POVM measurements of S_{M_A}. This is achieved when each POVMs are proportional to projective operators and given by

$$\max_{S_{M_A}, U_B(\mu)} E_B = \frac{h^2+2k^2}{\sqrt{h^2+k^2}}\left[\sqrt{1+\frac{h^2k^2}{\left(h^2+2k^2\right)^2}}-1\right].$$

3 Relation between Entanglement Breaking and Teleported Energy

Next, we analyze entanglement breaking by the POVM measurement of A and show two inequalities between the maximum teleported energy and the entanglement breaking. We adopt entropy of entanglement as a quantitative measure of entanglement. The entropy of a pure state $|\Psi_{AB}\rangle$ of A and B is defined as

$$S_{AB} = -\operatorname*{Tr}_B\left[\operatorname*{Tr}_A\left[|\Psi_{AB}\rangle\langle\Psi_{AB}|\right]\ln \operatorname*{Tr}_A\left[|\Psi_{AB}\rangle\langle\Psi_{AB}|\right]\right].$$

Before the measurement, the total system is prepared to be in the ground state $|g\rangle$. The reduced state of B is given by $\rho_B = \mathrm{Tr}_A\left[|g\rangle\langle g|\right]$. After the POVM measurement outputting μ, the state is transferred into a pure state $|A(\mu)\rangle$. The reduced post-measurement state of B is calculated as $\rho_B(\mu) = \frac{1}{p_A(\mu)}\mathrm{Tr}_A\left[\Pi_A(\mu)|g\rangle\langle g|\right]$. The entropy of entanglement of the ground state is given by $-\mathrm{Tr}_B\left[\rho_B \ln \rho_B\right]$ and that of the post-measurement state with output μ is given by $-\mathrm{Tr}_B\left[\rho_B(\mu)\ln\rho_B(\mu)\right]$. By using these results, we define the consumption of ground-state entanglement by the measurement as the difference between the ground-state entanglement and the averaged post-measurement-state entanglement:

$$\Delta S_{AB} = -\operatorname*{Tr}_{B}\left[\rho_B \ln \rho_B\right] - \sum_{\mu} p_A(\mu)\left(-\operatorname*{Tr}_{B}\left[\rho_B(\mu)\ln \rho_B(\mu)\right]\right).$$

Interestingly, this quantity is tied to the mutual information between the measurement result of A and the post-measurement state of B. Let us introduce a Hilbert space for a measurement pointer system $\bar{A}$ of the POVM measurement, which is spanned by orthonormal states $|\mu_{\bar{A}}\rangle$ corresponding to the output μ satisfying $\langle\mu_{\bar{A}}|\mu'_{\bar{A}}\rangle = \delta_{\mu\mu'}$. Then, the average state of $\bar{A}$ and B after the measurement is given by $\Phi_{\bar{A}B} = \sum_{\mu} p_A(\mu)|\mu_{\bar{A}}\rangle\langle\mu_{\bar{A}}| \otimes \rho_B(\mu)$.By using the reduced operators $\Phi_{\bar{A}} = \mathrm{Tr}_B\left[\Phi_{\bar{A}B}\right]$ and $\Phi_B = \mathrm{Tr}_{\bar{A}}\left[\Phi_{\bar{A}B}\right]$, the mutual information $I_{\bar{A}B}$ is defined as

$$I_{\bar{A}B} = -\operatorname*{Tr}_{\bar{A}}\left[\Phi_{\bar{A}}\ln\Phi_{\bar{A}}\right] - \operatorname*{Tr}_{B}\left[\Phi_B \ln \Phi_B\right] + \operatorname*{Tr}_{\bar{A}B}\left[\Phi_{\bar{A}B}\ln\Phi_{\bar{A}B}\right].$$

By using $\mathrm{Tr}_B\left[\Phi_{\bar{A}B}\right] = \sum_{\mu} p_A(\mu)|\mu_{\bar{A}}\rangle\langle\mu_{\bar{A}}|$ and $\mathrm{Tr}_{\bar{A}}\left[\Phi_{\bar{A}B}\right] = \sum_{\mu} p_A(\mu)\rho_B(\mu) = \rho_B$, it can be straightforwardly proven that $I_{\bar{A}B}$ is equal to ΔS_{AB}. This relation provides another physical interpretation of ΔS_{AB}.

Next, let us calculate ΔS_{AB} explicitly. All the eigenvalues of $\rho_B(\mu)$ are given by

$$\lambda_{\pm}(\mu) = \frac{1}{2}\left[1 \pm \sqrt{\cos^2\varsigma + \sin^2\varsigma\frac{q_A(\mu)^2}{p_A(\mu)^2}}\right], \tag{8}$$

where ς is a real constant which satisfies

$$\cos\varsigma = \frac{h}{\sqrt{h^2+k^2}},\quad \sin\varsigma = \frac{k}{\sqrt{h^2+k^2}}.$$

The eigenvalues of ρ_B are obtained by substituting $q_A(\mu) = 0$ into Eq. (8). By using $\lambda_s(\mu)$, ΔS_{AB} can be evaluated as

$$\Delta S_{AB} = \sum_{\mu} p_A(\mu) f_I\left(\frac{q_A(\mu)^2}{p_A(\mu)^2}\right), \tag{9}$$

where $f_I(x)$ is a monotonically increasing function of $x \in [0,1]$ and is defined by

$$
\begin{aligned}
f_I(x) = &\frac{1}{2}\left(1+\sqrt{\cos^2\varsigma + x\sin^2\varsigma}\right)\ln\left(\frac{1}{2}\left(1+\sqrt{\cos^2\varsigma + x\sin^2\varsigma}\right)\right) \\
&+\frac{1}{2}\left(1-\sqrt{\cos^2\varsigma + x\sin^2\varsigma}\right)\ln\left(\frac{1}{2}\left(1-\sqrt{\cos^2\varsigma + x\sin^2\varsigma}\right)\right) \\
&-\frac{1}{2}\left(1+\cos\varsigma\right)\ln\left(\frac{1}{2}\left(1+\cos\varsigma\right)\right) \\
&-\frac{1}{2}\left(1-\cos\varsigma\right)\ln\left(\frac{1}{2}\left(1-\cos\varsigma\right)\right).
\end{aligned}
$$

It is worth noting that the optimal teleported energy $\max_{U_B(\mu)} E_B$ in Eq. (7) takes a form similar to Eq. (9) as

$$
\max_{U_B(\mu)} E_B = \sum_{\mu} p_A(\mu) f_E\left(\frac{q_A(\mu)^2}{p_A(\mu)^2}\right), \tag{10}
$$

where $f_E(x)$ is a monotonically increasing function of $x \in [0,1]$ and is defined by

$$
f_E(x) = \sqrt{h^2+k^2}\left(1+\sin^2\varsigma\right)\left[\sqrt{1+\frac{\cos^2\varsigma\sin^2\varsigma}{\left(1+\sin^2\varsigma\right)^2}x}-1\right].
$$

Expanding both $f_I(x)$ and $f_E(x)$ around $x=0$ yields

$$
\begin{aligned}
f_I(x) &= \frac{\sin^2\varsigma}{4\cos\varsigma}\ln\frac{1+\cos\varsigma}{1-\cos\varsigma}x + O(x^2), \\
f_E(x) &= \sqrt{h^2+k^2}\frac{\cos^2\varsigma\sin^2\varsigma}{2\left(1+\sin^2\varsigma\right)}x + O(x^2).
\end{aligned}
$$

By deleting x in the above two equations, we obtain the following relation for weak measurements with infinitesimally small $q_A(\mu)$:

$$
\Delta S_{AB} = \frac{1+\sin^2\varsigma}{2\cos^3\varsigma}\ln\frac{1+\cos\varsigma}{1-\cos\varsigma}\frac{\max_{U_B(\mu)} E_B}{\sqrt{h^2+k^2}} + O(q_A(\mu)^4).
$$

It is of great significance that this relation can be extended as the following inequality for general measurements of S_{M_A} :

$$\Delta S_{AB} \geq \frac{1+\sin^2\varsigma}{2\cos^3\varsigma}\ln\frac{1+\cos\varsigma}{1-\cos\varsigma}\frac{\max_{U_B(\mu)} E_B}{\sqrt{h^2+k^2}}. \tag{11}$$

This inequality is one of the main results of this Letter and implies that a large amount of teleported energy really requests a large amount of consumption of the ground-state entanglement between A and B. The proof of Eq. (11) is as follows. Let us introduce two rescaled functions as follows.

$$\bar{f}_I(x) = 4\frac{\cos\varsigma}{\sin^2\varsigma}\left(\ln\frac{1+\cos\varsigma}{1-\cos\varsigma}\right)^{-1} f_I(x),$$
$$\bar{f}_E(x) = \frac{1}{\sqrt{h^2+k^2}}\frac{2\left(1+\sin^2\varsigma\right)}{\cos^2\varsigma\sin^2\varsigma} f_E(x).$$

It can be easily shown that $\bar{f}_E(x)$ is a convex function: $\partial_x^2\bar{f}_E(x) < 0$. From this convexity and $\bar{f}_E(x) = x + O(x^2)$, the function $\bar{f}_E(x)$ satisfies

$$\bar{f}_E(x) \leq x \tag{12}$$

for $0 \leq x \leq 1$. On the other hand, it is observed that $\bar{f}_I(x)$ is a concave function, as shown below. The derivative of $\bar{f}_I(x)$ is computed as $\partial_x\bar{f}_I(x) = \frac{2\cos\varsigma}{\ln\frac{1+\cos\varsigma}{1-\cos\varsigma}} g_I(y(x))$, where $y(x) = \sqrt{\cos^2\varsigma + x\sin^2\varsigma}$ and $g_I(y)$ is a positive function of y defined as $g_I(y) = \frac{1}{y}\ln\frac{1+y}{1-y}$. It should be noted that $y(x)$ is a monotonically increasing function of x. The derivative of $g_I(y)$ is calculated as $\partial_y g_I(y) = \frac{s_I(y)}{y^2}$, where $s_I(y)$ is a function of y given by $s_I(y) = \frac{2y}{1-y^2} - \ln\left(\frac{1+y}{1-y}\right)$, and satisfies a boundary condition as $s_I(0) = 0$. It is also easy to show that the derivative of $s_I(y)$ is positive for $y > 0$: $\partial_y s_I(y) = \frac{4y^2}{(1-y^2)^2} > 0$. Thus, $s_I(y)$ and $\partial_y g_I(y)$ are positive for $y \in [\cos\varsigma, 1]$. From these results, it has been proven that $\bar{f}_I(x)$ is concave for $x \in [0,1]$: $\partial_x^2\bar{f}_I(x) > 0$. From this concavity and $\bar{f}_I(x) = x + O(x^2)$, it is shown that $\bar{f}_I(x)$ satisfies the relation

$$\bar{f}_I(x) \geq x. \tag{13}$$

Because of Eqs. (12) and (13), we obtain the following inequality:

$$\bar{f}_I(x) \geq x \geq \bar{f}_E(x).$$

This implies that the energy-entanglement inequality in Eq. (11) holds. In addition, we can prove another inequality between energy and entanglement breaking. Because the convex function $\bar{f}_E(x)$ and the concave function $\bar{f}_I(x)$ are monotonically increasing functions of $x \in [0,1]$ which satisfy $\bar{f}_E(0) = \bar{f}_I(0) = 0$, we have the following relation: $\bar{f}_E(x) \geq \frac{\bar{f}_E(1)}{\bar{f}_I(1)}\bar{f}_I(x)$. Consequently, the following inequality, which is another main result of this Letter, is obtained for all measurements of S_{M_A}:

$$\max_{U_B(\mu)} E_B \geq \frac{2\sqrt{h^2+k^2}\left[\sqrt{4-3\cos^2\varsigma}-2+\cos^2\varsigma\right]}{(1+\cos\varsigma)\ln\left(\frac{2}{1+\cos\varsigma}\right)+(1-\cos\varsigma)\ln\left(\frac{2}{1-\cos\varsigma}\right)}\Delta S_{AB}. \tag{14}$$

This ensures that if we have consumption of ground-state entanglement ΔS_{AB} for a measurement of S_{M_A}, we can in principle teleport energy from A to B, where the energy amount is greater than the value of the right-hand-side term of Eq. (14). This bound is achieved for non-zero energy transfer by measurements with $q_A(\mu) = \pm p_A(\mu)$. The inequalities in Eq. (11) and Eq. (14) help us to gain a deeper understanding of entanglement as a physical resource because they show that the entanglement decrease by the measurement of A is directly related to the increase of the available energy at B as an evident physical resource.

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Figure Caption

Figure 1: Schematic diagram of the minimal QET model. A POVM measurement is performed on A with infusion of energy E_A. The measurement result μ is announced to B through a classical channel. After the arrival of μ, a unitary operation dependent on μ is performed on B with extraction of energy E_B.

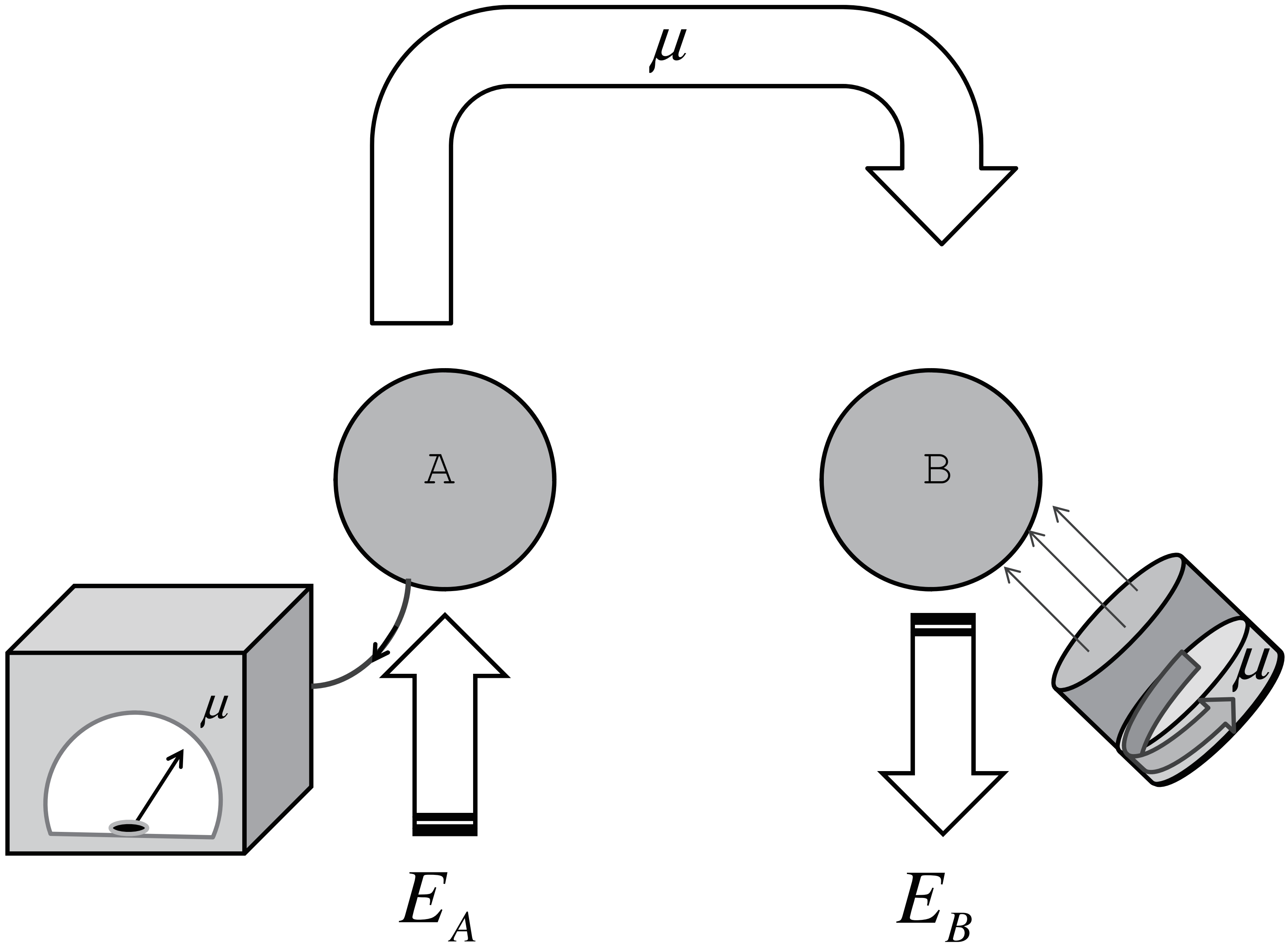

Figure 1