

MIT OpenCourseWare
<http://ocw.mit.edu>

16.346 Astrodynamics
Fall 2008

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.

Lecture 1 The Two Body Problem

Newton's Two-Body Equations of Motion 1687

#3.1, #3.3

Force = Mass × Acceleration

$$\begin{aligned} \frac{Gm_1m_2}{r^2} \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{r} = m_1 \frac{d^2\mathbf{r}_1}{dt^2} & \implies \frac{d^2}{dt^2}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2) = \mathbf{0} \\ \frac{Gm_2m_1}{r^2} \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{r} = m_2 \frac{d^2\mathbf{r}_2}{dt^2} & \implies -\frac{G(m_1 + m_2)}{r^2} \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{r} = \frac{d^2}{dt^2}(\mathbf{r}_2 - \mathbf{r}_1) \end{aligned}$$

Conservation of Total Linear Momentum Page 96

$$\frac{d^2\mathbf{r}_{cm}}{dt^2} = \mathbf{0} \implies \mathbf{r}_{cm} = \mathbf{c}_1 t + \mathbf{c}_2 \quad \text{where} \quad \mathbf{r}_{cm} \stackrel{\text{def}}{=} \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Two-Body Equation of Relative Motion Page 108

$$\boxed{\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}} \quad \text{or} \quad \frac{d\mathbf{v}}{dt} = -\frac{\mu}{r^3}\mathbf{r} \quad \text{where} \quad \begin{aligned} \mathbf{r} &= \mathbf{r}_2 - \mathbf{r}_1 \\ r &= |\mathbf{r}| = |\mathbf{r}_2 - \mathbf{r}_1| \\ \mu &= G(m_1 + m_2) \end{aligned}$$

Vector Notation

- Position Vectors

$$\begin{aligned} \mathbf{r}_1 &= x_1 \mathbf{i}_x + y_1 \mathbf{i}_y + z_1 \mathbf{i}_z \\ \mathbf{r}_2 &= x_2 \mathbf{i}_x + y_2 \mathbf{i}_y + z_2 \mathbf{i}_z \\ \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 &= x \mathbf{i}_x + y \mathbf{i}_y + z \mathbf{i}_z \end{aligned} \quad \mathbf{r}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \mathbf{r}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Two-Body Equations of Motion in Rectangular Coordinates

$$\frac{d^2x}{dt^2} + \frac{\mu}{r^3}x = 0 \quad \frac{d^2y}{dt^2} + \frac{\mu}{r^3}y = 0 \quad \frac{d^2z}{dt^2} + \frac{\mu}{r^3}z = 0$$

- Velocity Vectors

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt} \mathbf{i}_x + \frac{dy}{dt} \mathbf{i}_y + \frac{dz}{dt} \mathbf{i}_z = \begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix}$$

- Polar Coordinates

$$\begin{aligned} \mathbf{r} &= r \mathbf{i}_r & \mathbf{i}_r &= \cos \theta \mathbf{i}_x + \sin \theta \mathbf{i}_y & \mathbf{i}_\theta &= -\sin \theta \mathbf{i}_x + \cos \theta \mathbf{i}_y = \frac{d\mathbf{i}_r}{d\theta} \\ \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{dr}{dt} \mathbf{i}_r + r \frac{d\mathbf{i}_r}{d\theta} \frac{d\theta}{dt} = \frac{dr}{dt} \mathbf{i}_r + r \frac{d\theta}{dt} \mathbf{i}_\theta = v_r \mathbf{i}_r + v_\theta \mathbf{i}_\theta \end{aligned}$$

Kepler's Second Law 1609 Equal Areas Swept Out in Equal Times

Assume $z = 0$ so that the motion is confined to the x-y plane

$$0 = x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} = \frac{d}{dt} \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) \implies x \frac{dy}{dt} - y \frac{dx}{dt} = \text{Constant}$$

Using polar coordinates

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \implies x \frac{dy}{dt} - y \frac{dx}{dt} = r^2 \frac{d\theta}{dt} = \text{Constant} \equiv h = 2 \times \frac{d}{dt} \text{Area}$$

Josiah Willard Gibbs (1839–1908) Vector Analysis for the Engineer

Image removed due to copyright restrictions.

Appendix B-1

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 = r_1 r_2 \cos \angle$$

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = r_1 r_2 \sin \angle \mathbf{i}_n$$

$$\mathbf{r}_1 \times \mathbf{r}_2 \cdot \mathbf{r}_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{r}_3 = (\mathbf{r}_1 \cdot \mathbf{r}_3) \mathbf{r}_2 - (\mathbf{r}_2 \cdot \mathbf{r}_3) \mathbf{r}_1$$

$$\mathbf{r}_1 \times (\mathbf{r}_2 \times \mathbf{r}_3) = (\mathbf{r}_1 \cdot \mathbf{r}_3) \mathbf{r}_2 - (\mathbf{r}_1 \cdot \mathbf{r}_2) \mathbf{r}_3$$

Kepler's Second Law 1609 Conservation of Angular Momentum

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{0} \implies \boxed{\mathbf{h} = \mathbf{r} \times \mathbf{v}} = \text{Constant}$$

Motion takes place in a plane and angular momentum is conserved

In polar coordinates

$$\mathbf{r} = r \mathbf{i}_r \quad \frac{d\mathbf{r}}{dt} = \mathbf{v} = \frac{dr}{dt} \mathbf{i}_r + r \frac{d\theta}{dt} \mathbf{i}_\theta = v_r \mathbf{i}_r + v_\theta \mathbf{i}_\theta$$

so that the angular momentum of m_2 with respect to m_1 is

$$m_2 r v_\theta = m_2 r^2 \frac{d\theta}{dt} \stackrel{\text{def}}{=} m_2 h = \text{Constant}$$

- **Rectilinear Motion:** For $\mathbf{r} \parallel \mathbf{v}$, then $\boxed{h = 0}$.

The quantity h is called the angular momentum but is actually the *massless* angular momentum. In vector form $\mathbf{h} = h \mathbf{i}_z$ so that $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ and is a constant in both magnitude and direction. This is called Kepler's second law even though it was really his first major result. As Kepler expressed it, the radius vector sweeps out equal areas in equal time since

$$\boxed{\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{h}{2}} = \text{Constant}$$

Kepler's Law is a direct consequence of radial acceleration!

Eccentricity Vector

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{h}) = \frac{d\mathbf{v}}{dt} \times \mathbf{h} = -\frac{\mu}{r^3} \mathbf{r} \times \mathbf{h} = -\frac{\mu h}{r^2} \mathbf{i}_r \times \mathbf{i}_h = \frac{\mu h}{r^2} \mathbf{i}_\theta = \mu \frac{d\theta}{dt} \mathbf{i}_\theta = \mu \frac{d\mathbf{i}_r}{dt}$$

Hence

$$\boxed{\mu \mathbf{e} = \mathbf{v} \times \mathbf{h} - \frac{\mu}{r} \mathbf{r}} = \text{Constant}$$

The vector quantity $\mu \mathbf{e}$ is often referred to as the Laplace Vector.

We will call the vector \mathbf{e} the eccentricity vector because its magnitude e is the eccentricity of the orbit.

Kepler's First Law 1609 The Equation of Orbit

If we take the scalar product of the Laplace vector and the position vector, we have

$$\mu \mathbf{e} \cdot \mathbf{r} = \mathbf{v} \times \mathbf{h} \cdot \mathbf{r} - \frac{\mu}{r} \mathbf{r} \cdot \mathbf{r} = \mathbf{r} \times \mathbf{v} \cdot \mathbf{h} - \mu r = \mathbf{h} \cdot \mathbf{h} - \mu r = h^2 - \mu r$$

Also $\mu \mathbf{e} \cdot \mathbf{r} = \mu r e \cos f$ where f is the angle between \mathbf{r} and \mathbf{e} so that

$$\boxed{r = \frac{p}{1 + e \cos f}} \quad \text{or} \quad r = p - ex \quad \text{where} \quad \boxed{p \stackrel{\text{def}}{=} \frac{h^2}{\mu}}$$

is the Equation of Orbit in polar coordinates. (Note that $r \cos f = x$.)

The angle f is the **true anomaly** and p , called the **parameter**, is the value of the radius r for $f = \pm 90^\circ$.

The **pericenter** ($f = 0$) and **apocenter** ($f = \pi$) radii are

$$r_p = \frac{p}{1 + e} \quad \text{and} \quad r_a = \frac{p}{1 - e}$$

If $2a$ is the length of the major axis, then $r_p + r_a = 2a \implies \boxed{p = a(1 - e^2)}$

Kepler's Third Law 1619 The Harmony of the World

Archimedes was the first to discover that the area of an ellipse is πab where a and b are the semimajor and semiminor axes of the ellipse.

Since the radius vector sweeps out equal areas in equal times, then the entire area will be swept out in the time interval called the period P . Therefore, from Kepler's Second Law

$$\frac{\pi ab}{P} = \frac{h}{2} = \frac{\sqrt{\mu p}}{2} = \frac{\sqrt{\mu a(1-e^2)}}{2}$$

Also, from the elementary properties of an ellipse, we have $b = a\sqrt{1-e^2}$ so that the **Period** of the ellipse is

$$P = 2\pi \sqrt{\frac{a^3}{\mu}}$$

Other expressions and terminology are used

$$\text{Mean Motion} \quad n = \frac{2\pi}{P} = \sqrt{\frac{\mu}{a^3}} \quad \text{or} \quad \mu = n^2 a^3 \quad \text{or} \quad \frac{a^3}{P^2} = \text{Constant}$$

The last of these is known as Kepler's third law.

- Kepler made the false assumption that μ is the same for all planets.

Units for Numerical Calculations

A convenient choice of units is

Length	The astronomical unit (Mean distance from Earth to the Sun)
Time	The year (the Earth's period)
Mass	The Sun's mass (Ignore other masses compared to Sun's mass)

Then

$$\mu = G(m_1 + m_2) = G(m_{sun} + m_{planet}) = G(m_{sun}) = G$$

so that, from Kepler's Third Law, we have

$$\mu = G = 4\pi^2 \quad \text{or} \quad k = \sqrt{G} = 2\pi$$

where G is the **Universal Gravitation Constant**.



Josiah Willard Gibbs (1839–1908) was a professor of mathematical physics at Yale College where he inaugurated the new subject — three dimensional vector analysis. He had printed for private distribution to his students a small pamphlet on the “Elements of Vector Analysis” in 1881 and 1884.

Gibbs’ pamphlet became widely known and was finally incorporated in the book “Vector Analysis” by J. W. Gibbs and E. B. Wilson and published in 1901.

Gibb’s Method of Orbit Determination Pages 131–133

- Given $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ with $\mathbf{r}_1 \times \mathbf{r}_2 \cdot \mathbf{r}_3 = 0$
- To determine the eccentricity vector \mathbf{e} and the parameter p

$$\mathbf{r}_2 = \alpha\mathbf{r}_1 + \beta\mathbf{r}_3 \quad \text{with} \quad \mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_3 \quad \implies \quad \alpha = \frac{\mathbf{r}_2 \times \mathbf{r}_3 \cdot \mathbf{n}}{n^2} \quad \text{and} \quad \beta = \frac{\mathbf{r}_1 \times \mathbf{r}_2 \cdot \mathbf{n}}{n^2}$$

$$0 = \mathbf{e} \cdot (\mathbf{r}_2 - \alpha\mathbf{r}_1 - \beta\mathbf{r}_3) = p - r_2 - \alpha(p - r_1) - \beta(p - r_3) \quad \implies \quad \boxed{p = \frac{\alpha r_1 - r_2 + \beta r_3}{\alpha - 1 + \beta}}$$

- To determine the eccentricity vector, we observe that:

$$\mathbf{n} \times \mathbf{e} = (\mathbf{r}_1 \times \mathbf{r}_3) \times \mathbf{e} = (\mathbf{e} \cdot \mathbf{r}_1)\mathbf{r}_3 - (\mathbf{e} \cdot \mathbf{r}_3)\mathbf{r}_1 = (p - r_1)\mathbf{r}_3 - (p - r_3)\mathbf{r}_1$$

Then, since $(\mathbf{n} \times \mathbf{e}) \times \mathbf{n} = n^2\mathbf{e}$ it follows that

$$\boxed{\mathbf{e} = \frac{1}{n^2}[(p - r_1)\mathbf{r}_3 \times \mathbf{n} - (p - r_3)\mathbf{r}_1 \times \mathbf{n}]}$$