

# Feynman Cylinder Paradox

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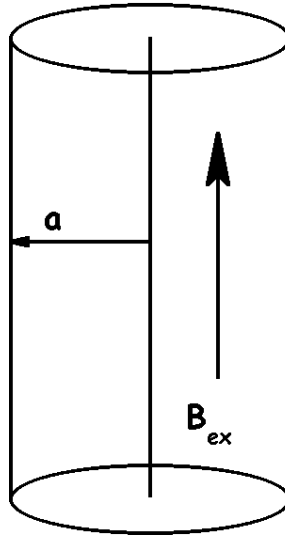
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(1983; updated April 2002)

## 1 Problem

An infinitely long wire with linear charge density  $-\lambda$  lies along the  $z$  axis. An insulating cylindrical shell of radius  $a$  and moment of inertia  $I$  per unit length is concentric with the wire, and can rotate freely about the  $z$  axis. The areal charge density on the cylinder is  $\sigma = \lambda/2\pi a$  and is uniformly distributed.



The cylinder is immersed in an external magnetic field  $B_{ex}\hat{z}$ , and is initially at rest.

Starting at  $t = 0$  the external magnetic field is slowly reduced to zero over a time  $T \gg a/c$ , where  $c$  is the speed of light. What is the final angular velocity  $\omega$  of the cylinder?

## 2 Solution

This problem is a version of the Feynman disk paradox [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] that is particularly easy to analyze. However, it avoids a subtle point related to the return flux of the external magnetic field, as discussed in sec. 2.4. This problem is based on earlier discussions by McKenna [14] and by Romer [3].

## 2.1 Solution Via Conservation of Angular Momentum

The initial angular momentum  $\mathbf{L}_i$  (per unit length) of the system is entirely due to the electromagnetic field,

$$\mathbf{L}_{i,\text{field}} = \int \mathbf{r} \times \mathbf{p} \, d\text{Area} = 2\pi \int_0^\infty \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} r \, dr, \quad (1)$$

recalling that the field momentum density is the Poynting vector  $\mathbf{S} = c\mathbf{E} \times \mathbf{B}/4\pi$  (in Gaussian units) divided by  $c^2$ , where  $c$  is the speed of light.

In the present problem, an electric field exists only for  $r < a$ , since the charge density  $\sigma$  on the cylinder has been chosen to cancel the field from the charged wire for  $r > a$ . From Gauss' law we obtain

$$\mathbf{E} = -\frac{2\lambda}{r} \hat{\mathbf{r}} \quad (r < a), \quad (2)$$

and hence the field momentum density in a cylindrical coordinate system  $(r, \phi, z)$  is

$$\mathbf{p} = \frac{\lambda B_{\text{ex}}}{2\pi c r} \hat{\phi} \quad (r < a). \quad (3)$$

The initial angular momentum is therefore

$$\mathbf{L}_{i,\text{field}} = 2\pi \frac{\lambda B_{\text{ex}}}{2\pi c} \int_0^a \frac{\mathbf{r} \times \hat{\phi}}{r} r \, dr = \frac{\lambda a^2 B_{\text{ex}}}{2c} \hat{\mathbf{z}}. \quad (4)$$

The angular momentum when the external magnetic field is zero is due to the rotation of the cylinder at angular velocity  $\omega$ . There is now the mechanical angular momentum  $I\omega$  as well as the field angular momentum due to the solenoidal magnetic field inside the rotating, charged cylinder. The azimuthal current (per unit length) is

$$j_\phi = \frac{Q}{T} = 2\pi a \sigma \frac{\omega}{2\pi} = \frac{\lambda \omega}{2\pi}. \quad (5)$$

The resulting final magnetic field is along the  $z$  axis, with strength

$$B_f = \frac{4\pi j_\phi}{c} = \frac{2\lambda \omega}{c} \quad (r < a), \quad (6)$$

independent of radius for  $r < a$  according to Ampere's law. Since this field is in the same sense as the original field, we can immediately use eq. (4) to find the final field angular momentum:

$$\mathbf{L}_{f,\text{field}} = \frac{\lambda a^2 B_f}{2c} \hat{\mathbf{z}} = \frac{\lambda^2 a^2 \omega}{c^2} \hat{\mathbf{z}}. \quad (7)$$

The total angular momentum in the final state is therefore

$$\mathbf{L}_f = \mathbf{L}_{f,\text{mechanical}} + \mathbf{L}_{f,\text{field}} = \left( I + \frac{\lambda^2 a^2}{c^2} \right) \omega \hat{\mathbf{z}}. \quad (8)$$

Since there is no frictional torque in this problem (and we ignore radiation), angular momentum is conserved. Hence,

$$\omega = \frac{\lambda a^2 B_{\text{ex}}}{2cI(1 + \lambda^2 a^2/c^2 I)} \approx \frac{\lambda a^2 B_{\text{ex}}}{2cI} \left( 1 - \frac{\lambda^2 a^2}{c^2 I} \right). \quad (9)$$

The presence of  $c^2$  in the denominator of the last term of eq. (9) indicates the presence of relativistic effects in this problem.

## 2.2 Solution Via Faraday's Law

As the magnetic field drops, its time derivative  $\dot{\mathbf{B}}$  results in an induced electric field in the azimuthal direction. According to Faraday's law, we have

$$E_\phi(r) = -\frac{r\dot{B}_z}{2c}. \quad (10)$$

This field acts on the charged cylindrical shell to produce an azimuthal torque (per unit length) of

$$N_\phi = aE_\phi(a)2\pi a\sigma = -\frac{\lambda a^2\dot{B}_z}{2c} = \frac{dL_{\text{mechanical}}}{dt} = I\frac{d\omega}{dt}. \quad (11)$$

We integrate to find the final angular velocity:

$$\omega = \int_0^\infty \frac{d\omega}{dt} dt = -\frac{\lambda a^2}{2cI} \int_0^\infty \dot{B}_z dt = \frac{\lambda a^2(B_{\text{ex}} - B_f)}{2cI}, \quad (12)$$

Again, we must note that the final magnetic field is not zero, but is given by eq. (6). With this, eq. (12) becomes

$$\omega = \frac{\lambda a^2(B_{\text{ex}} - 2\lambda\omega/c)}{2cI}, \quad (13)$$

which again leads to eq. (9).

## 2.3 Another Relativistic Correction

*[This and the following section were written April, 2002.]*

In addition to the above accounting of angular momentum, there is a small amount of initial angular momentum associated with the motion of the conduction current that produces the field  $B_{\text{ex}}$ . Furthermore, in the final state the cylinder of radius  $a$  is rotating angular velocity  $\omega$ , so its moment of inertia increases by the factor  $\gamma = 1/\sqrt{1 - a^2\omega^2/c^2}$  due to the relativistic increase of mass.

These small effects are related to the so-called hidden mechanical momentum associated with electrical currents [15, 16].

To characterize the initial mechanical angular momentum, we suppose the magnetic field  $\mathbf{B}_{\text{ex}}$  is produced by a long cylinder of radius  $b > a$ , which must therefore carry azimuthal current (per unit length along the  $z$  axis)

$$I_{\text{ex}} = \frac{c}{4\pi} B_{\text{ex}}. \quad (14)$$

This current is due to an areal number density  $n_e$  of conduction electrons that we take to have velocity  $v_e$ . Then, the current  $I_{\text{ex}}$  is also related by

$$I_{\text{ex}} = -en_e v_e, \quad (15)$$

writing  $e > 0$  as the magnitude of the charge of the electron. Hence,

$$n_e v_e = -\frac{c}{4\pi} \frac{B_{\text{ex}}}{e}. \quad (16)$$

The initial mechanical angular momentum (per unit length) associated with conduction electrons is

$$\mathbf{L}_{i,\text{mech}} = 2\pi b n_e \gamma_e m_e v_e b \hat{\mathbf{z}} = -\gamma_e \frac{m_e c}{2e} b^2 B_{\text{ex}} \hat{\mathbf{z}}, \quad (17)$$

where the total number of conduction electrons per unit length is  $2\pi b n_e$ ,  $m_e$  is the rest mass of the electron, and  $\gamma_e = 1/\sqrt{1 - v_e^2/c^2} \approx 1$ . Combining this with eq. (4), the total initial angular momentum is

$$\mathbf{L}_i = \frac{\lambda a^2 B_{\text{ex}}}{2c} \left(1 - \gamma_e \frac{m_e c^2 b^2}{\lambda e a^2}\right) \hat{\mathbf{z}} = \frac{\lambda a^2 B_{\text{ex}}}{2c} \left(1 - \gamma_e \frac{e b^2}{\lambda r_e a^2}\right) \hat{\mathbf{z}}, \quad (18)$$

where  $r_e = e^2/m_e c^2$  is the classical electron radius. The last term in eq. (18) is not necessarily small, since  $e/r_e$  corresponds to  $\approx 10^{13}$  electrons/cm.

Reviewing the argument of sec. 2.2, we see that in eq. (11) the derivative  $d\omega/dt$  should really be  $d\gamma\omega/dt$ , with the moment of inertia  $I$  being calculated using the rest mass of the cylinder. However, eq. (6) for the final magnetic field remains the same, so eq. (13) becomes

$$\gamma\omega = \frac{\lambda a^2 (B_{\text{ex}} - 2\lambda\omega/c)}{2cI}, \quad (19)$$

Expanding  $\gamma$  as approximately  $1 + a^2\omega^2/2c^2$ , we find

$$\omega \approx \frac{\lambda a^2 B_{\text{ex}}}{2cI} \left(1 - \frac{\lambda^2 a^2}{c^2 I} - \frac{\lambda^2 a^6 B_{\text{ex}}^2}{2c^4 I^2}\right). \quad (20)$$

## 2.4 A Subtle Point

This example, and near equivalents [14, 3, 6], are crafted so as to avoid a complication associated with the return flux of the magnetic field.

To see the difficulty, suppose instead that the linear charge density on the central wire were  $\lambda_0$ , while that on the cylinder of radius  $a$  is still called  $\lambda$ . Then, the initial field angular momentum would be

$$\mathbf{L}_{i,\text{field}} = -\frac{B_{\text{ex}}}{2c} [\lambda_0 b^2 + \lambda(b^2 - a^2)] \hat{\mathbf{z}} = \frac{B_{\text{ex}}}{2c} [\lambda a^2 - (\lambda + \lambda_0) b^2] \hat{\mathbf{z}}. \quad (21)$$

where  $b$  is the radius of the solenoid that provides the external field. Here, we make the usual (but as we will see, unwarranted) assumption that the field of a long solenoid is essentially zero outside the solenoid.

The final magnetic field is still given by eq. (6), so the final field angular momentum would be

$$\mathbf{L}_{f,\text{field}} = \frac{\lambda\omega}{c^2} [\lambda a^2 - (\lambda + \lambda_0) b^2] \hat{\mathbf{z}}. \quad (22)$$

The total final angular momentum would be

$$\mathbf{L}_f = \left(I + \frac{\lambda}{c^2} [\lambda a^2 - (\lambda + \lambda_0) b^2]\right) \omega \hat{\mathbf{z}}. \quad (23)$$

Equating (21) and (23) the final angular velocity would be

$$\omega = \frac{B_{\text{ex}}[\lambda a^2 - (\lambda + \lambda_0)b^2]}{2c\{I + [\lambda a^2 - (\lambda + \lambda_0)b^2]/c^2\}}. \quad (24)$$

However, the argument in sec. 2.2 based on Faraday's law is exactly the same as before, which again implies that the final angular velocity is given by eq. (9).

The argument based on Faraday's law seems the more robust, so I conclude that eq. (9) is correct for any value of the charge density  $\lambda_0$  on the central wire.

The field angular momentum calculations must be in error. Real solenoids have only finite length, and the magnetic field is not quite zero outside the solenoid since all of the magnetic flux inside the solenoid must be returned on paths outside the solenoid. Hence, the field angular momentum calculations (21) and (22) are missing pieces that are hard to calculate directly, but which must be  $B_z b^2(\lambda + \lambda_0)\hat{\mathbf{z}}/2c$ , yielding exactly the same result for the field angular momentum as if  $\lambda_0 = -\lambda$ .

While this resolution of the Feynman disk paradox is ultimately satisfactory, we see that a model based on an infinite solenoid does not in general permit explicit agreement between a torque analysis and a field angular momentum analysis. It is perhaps more reassuring to consider examples in which the source of the magnetic field has only a finite extent so that an analysis in spherical coordinates is possible [1, 13].

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