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Finite energy superluminal solutions of Maxwell equations

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Abstract

We exhibit exact finite energy superluminal solutions of Maxwell equations in vacuum and discuss the physical meaning of these solutions. © 2001 Elsevier Science B.V. All rights reserved.

Recently, some papers [1,2] have appeared in the literature showing that in some hypothetical media there is the possibility of the existence of superluminal electromagnetic pulses (solutions of Maxwell equations) such their *fronts* travel in the media with *superluminal* velocities. Now, the solutions discovered in [1,2], despite their theoretical interest have *infinite* energy and as such cannot be produced in the physical world. Only finite aperture approximations to these waves can eventually be produced (supposing the existence of the special media). The objective of this Letter is to show that in contrast to the solutions discovered in [1,2] (that, as already said have infinite energy), there exist vacuum solutions of Maxwell equations which are *finite energy superluminal solutions*. These new solutions, as we shall see, appear when we solve some Sommerfeld like problems [3,4] to be reported below. We discuss if the new solutions can be realized in the physical world. Moreover, we emphasize that the new solutions correspond to phenomenon distinct to already observed wave motion with superluminal [5–8] (or even negative [9,10]) group velocities. In the case, e.g., of experiments [5–8] only the peaks of the waves travel (for a while) with superluminal velocity whereas their fronts always travel at the velocity of light.

We start by recalling how to write electromagnetic field configurations in terms of Hertz potentials [11,12]. Suppose we have a Hertz potential $\vec{\Pi}_m$ of magnetic type. In what follows we use units such that the velocity of light is $c = 1$. Then, the associated electromagnetic field is given by

$$\vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{\Pi}_m), \qquad \vec{B} = \nabla \times \nabla \times \vec{\Pi}_m. \tag{1}$$

Let us take $\vec{\Pi}_m = \Phi \hat{e}_z$. Then, since the Hertz potential (in vacuum) satisfies a homogeneous wave equation, we have that

$$\Box \Phi = 0. \tag{2}$$

The *Sommerfeld* problem (not to be confused with a *Cauchy* problem) to be considered here is the

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following. In a given inertial frame (the laboratory [2]) find a solution $\Phi_X : (t, \vec{x}) \mapsto C$ (where C is the field of complex numbers) for Eq. (2) satisfying the following boundary conditions [3] at the $z = 0$ plane,

$$\begin{cases} \Phi_X(t,\rho,0) = \mathbf{T}(t) \int\limits_{-\infty}^{\infty} d\omega\, B(\omega) J_0(\omega\rho\sin\eta) e^{-i\omega t}, \\ \partial\Phi_X(t,\rho,z)/\partial z\big|_{z=0} = i\mathbf{T}(t)\cos\eta \int\limits_{-\infty}^{\infty} d\omega \\ \qquad \times B(\omega) J_0(\omega\rho\sin\eta) \\ \qquad \times k(\omega) e^{-i\omega t}, \end{cases} \tag{3}$$

where $\mathbf{T}(t) = [\Theta(t+T) - \Theta(t-T)]$, Θ is the Heaviside function, $k(\omega) = \omega$, and η is a constant called the axicon angle [13–18] and $B(k)$ is an appropriate frequency distribution. As showed in [14] the solution of Eq. (2) (for $z > 0, t > T$) which satisfies the Sommerfeld conditions is

$$\Phi_X(t,\rho,z) = \begin{cases} \int\limits_{-\infty}^{\infty} d\omega\, B(\omega) J_0(\omega\rho\sin\eta) e^{-i\omega(t - z\cos\eta)} \\ \quad \text{for } |t - z\cos\eta| < T, \\ 0 \quad \text{for } |t - z\cos\eta| > T. \end{cases} \tag{4}$$

We call Φ_X a scalar superluminal X-pulse. Now, as is well known, the energy density for a complex field configuration, like the Φ_X, is

$$u = (\partial_t\Phi_X)(\partial_t\Phi_X^*) + (\partial_x\Phi_X)(\partial_x\Phi_X^*) + (\partial_y\Phi_X)(\partial_y\Phi_X^*) + (\partial_z\Phi_X)(\partial_z\Phi_X^*), \tag{5}$$

and the energy of the field configuration can be calculated by the volume integral of u on a constant time hyperplane, say $t = T' > T$. The calculation is easy when done in *cylindrical* coordinates. Recalling that from Eq. (4) it follows that the support of the pulse at $t = T'$ is $\Delta z = 2T/\cos\eta$, we have

$$\mathcal{E} = \frac{8\pi T}{\sin^2\eta\cos\eta} \int\limits_{-\infty}^{\infty} |B(k)|^2 k\, dk, \tag{6}$$

where the kinetic and potential energy terms give equal contributions. Eq. (6) gives *finite* energy for the scalar X-pulse for an infinity of frequency distribution functions $B(k)$, such that $|B(k)|^2$ be null for $k < 0$. A trivial example is $B(k) = [\Theta(k) - \Theta(k - k_0)]$, with k_0 a constant. Now, we study the electromagnetic case. The non null components of the electromagnetic field [4] corresponding to a magnetic Hertz potential $\vec{\Pi}_m = \Phi_X \hat{e}_z$ are (for $z > 0, t > T$)

$$\begin{cases} E_\theta = i\sin\eta \int\limits_{-\infty}^{\infty} dk\, B(k) k^2 J_1(k\rho\sin\eta) e^{-ik(t - z\cos\eta)}, \\ B_\rho = \frac{-i}{2}\sin 2\eta \int\limits_{-\infty}^{\infty} dk\, B(k) k^2 J_1(k\rho\sin\eta) \\ \qquad \times e^{-ik(t - z\cos\eta)}, \quad \text{for } |t - z\cos\eta| < T, \\ B_z = \sin^2\eta \int\limits_{-\infty}^{\infty} dk B(k) k^2 J_0(k\rho\sin\eta) e^{-ik(t - z\cos\eta)}, \\ E_\theta = B_\rho = B_z = 0, \quad \text{for } |t - z\cos\eta| > T. \end{cases} \tag{7}$$

Now, using the standard energy density of the electromagnetic field [11,12], the energy of the superluminal electromagnetic X pulse results,

$$\mathcal{E}_X = \frac{1}{2} \int\limits_0^{2\pi} \int\limits_{z_{\min}}^{z_{\max}} \int\limits_0^{\infty} \left[E_\theta E_\theta^* + B_\rho B_\rho^* + B_z B_z^*\right] \rho\, d\rho\, dz\, d\theta = \frac{4\pi T}{\cos\eta} \int\limits_{-\infty}^{\infty} |B(k)|^2 k^3\, dk. \tag{8}$$

Eq. (8) gives finite energy for superluminal solutions of Maxwell equations satisfying Sommerfeld boundary conditions (here expressed through conditions for the associated Hertz potential) for an infinity of possible frequency distributions $B(k)$, as in the scalar case.

We have four comments before ending this Letter:

(i) What does our finite energy solution (for the scalar wave equation) look like for an observer in a Lorentz frame $Z \in \sec TM$,

$$Z = \frac{1}{\sqrt{1 - V^2}} (\partial_t + V\partial_z), \tag{9}$$

which is moving with velocity $V = \cos\eta$ relative to the laboratory (the frame $L = \partial_t \in \sec TM$)

As can be easily verified the transformed solution is

$\Phi'_X(t', \rho, z')$

[2] The laboratory is modeled by time like vector field $L = \partial/\partial t \in \sec TM$.

[3] The necessity for these boundary conditions is proved in [19].

[4] Called a superluminal electromagnetic X pulse [19,20].

$$= \begin{cases} \int_{-\infty}^{\infty} d\omega\, B(\omega) J_0(\omega\rho \sin\eta) e^{-i\omega \sin\eta t'} \\ \quad \text{for } |t'| < T/\sin\eta, \\ 0 \quad \text{for } |t'| > T/\sin\eta. \end{cases} \tag{10}$$

The solution is independent of the spatial coordinate z and corresponds to a standing wave occupying all the rest space of the Z frame and that exists only for the time interval $\Delta t' = 2T/\sin\eta$. Is this result non physical? If not, what is the meaning of such a wave for the observers of the Z frame? As a Minkowski diagram can show, the wave stands for a finite period of time according to the time order of the Z frame because it is going to the *past* of the Z's observers. This must be a normal phenomenon if relativity theory is true and *genuine* superluminal motion exists. The observers at the Z frame will compute an infinite energy for that wave, but since they *know* relativity theory they will interpret the whole phenomena as follows: the wave that stands for a finite period of time at our frame is a superluminal finite energy wave produced in a laboratory (the L frame) that is moving with velocity $-1/\cos\eta$ relative to our frame (i.e., Z frame). Of course, the Z frames physicists cannot produce such a wave in their frame, due to two reasons. The first is that the wave according to them has infinite energy and the second, which is the crucial one, is simply because the device which produced it is at rest in another frame (the L frame). According to the Principle of Relativity the Z frame physicists can duplicate in their frame the device used in the L frame and launch a wave like the one given by Eq. (4) (with boundary conditions like in Eq. (3)) with the (t, ρ, z) substituted by (t', ρ, z'). Of course, if that would be possible, we would arrive at well known paradoxical situations, [5] that fortunately need not to be discussed here (see (iii) below).

Note also that the Z frame mathematicians aware of the interpretation given by their fellow physicists can obtain directly the solution given by Eq. (10) by solving a generalized *mixed* boundary value problem, where the boundary conditions are:

$$\begin{aligned} &\Phi'_X(t', \rho, z')\big|_{z'=-\cos\eta t'} \\ &\quad = \big[\Theta(\sin\eta t' + T) - \Theta(\sin\eta t' - T)\big] \\ &\qquad \times \int_{-\infty}^{\infty} d\omega\, B(\omega) J_0(\omega\rho\sin\eta) e^{-i\omega\sin\eta t'}, \\ &\left(\gamma \frac{\partial}{\partial z'} - \gamma V \frac{\partial}{\partial t'}\right) \Phi'_X(t', \rho, z')\bigg|_{z'=-\cos\eta t'} \\ &\quad = i\cos\eta \big[\Theta(\sin\eta t' + T) - \Theta(\sin\eta t' - T)\big] \\ &\qquad \times \int_{-\infty}^{\infty} d\omega\, B(\omega)\omega J_0(\omega\rho\sin\eta) e^{-i\omega\sin\eta t'}. \end{aligned} \tag{11}$$

(ii) Of course, an analogous analysis holds for the finite energy superluminal solutions of Maxwell equations that we have just found. It is worth saying here that the existence of such solutions does not conflict with the famous result on the Cauchy problem concerning the Maxwell equations. That result says: any electromagnetic field configuration with compact support at $t = 0$, let us say for $|\vec{x}| \leqslant R$, is such that the field is null for $t > 0$ for all $|\vec{x}| \geqslant R + t$. [6]

(iii) Is it possible to build a physical device to launch a finite energy superluminal electromagnetic X pulse? Our answer is *no*. Indeed, finite aperture approximations (FAA) to exact superluminal X-like solutions of Maxwell equations (which, of course have finite energy) have already been produced [7,8]. However, these FAA are such that their peaks move with velocity $v > 1$ but their front always moves with the speed of light. This result has been predicted in [16,18] and is endorsed by the experimental results of [7,8] as proved in [13]. Now, concerning the solutions we just found, in order for them to be produced (by an antenna) as real physical waves it is necessary to produce waves that extend in all the $z = 0$ plane where the antenna is located for the time interval $-T < t < T$. Of course, this is physically impossible because it would require that the antenna should be an infinite one.

(iv) Besides the superluminal solutions just found, there are also finite energy *subluminal* solutions (to be reported elsewhere). We must say that even if the new superluminal solutions cannot be produced by physical devices the only possible reason for their *non* existence in our universe is that of a possible violation of the principle of relativity. Eventually these new su-

[5] More details on this issue can be found in [18].

[6] A proof of an analogous theorem for the homogeneous wave equation can be found in [13]. For Maxwell equations, see [14].

perluminal solutions may also find applications in the understanding of some fundamental issues concerning the nonlocality problem in quantum mechanics [21].

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