Superluminal Solutions to the Klein-Gordon Equation and a Causality Problem

$\boxed{\text{A.A.Borghardt}}$, M.A.Belogolovskii* and D.Ya.Karpenko

Donetsk Physical and Technical Institute,

National Academy of Sciences of Ukraine,

Str. R.Luxemburg 72, 83114 Donetsk, Ukraine

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Abstract

We present a new axially symmetric monochromatic free-space solution to the Klein-Gordon equation propagating with a superluminal group velocity and show that it gives rise to an imaginary part of the causal propagator outside the light cone. We address the question about causality of the spacelike paths and argue that the signal with a well-defined wavefront formed by the superluminal modes would propagate in vacuum with the light speed.

*Electronic address: bel@kinetic.ac.donetsk.ua

A long-standing question of objects or waves traveling faster than the light speed in vacuum c has received a renewed interest in the past decade. In particular, it was shown experimentally that light pulses can propagate with abnormally large group velocities greater than c although an interpretation of the observations still remains a matter for debate and controversies (recent highlights on the problem as well as all important references on this subject can be found in a special issue of the IEEE Journal of Selected Topics in Quantum Electronics [1]). In this context, it is necessary to emphasize that superliminal group velocities at any case do not contradict the causality principle in the special theory of relativity because, in contrast to a widespread opinion, group velocity, like phase velocity, is not synonymous with a signal speed (see, for instance, overviews of the problem by Vainshtein [2] and recently by Chiao and Milonni [3] and Büttiker and Washburn [4]). A real signal should start from zero at some instant and hence should have a well-defined wavefront. Its arrival at a given point can occur only when the point is reached by the front. Because it is associated with infinite frequencies, the asymptotic behavior of a group velocity defines the front spreading. It means that group velocity at finite frequencies is no longer a meaningful concept and there are no restrictions on its value which can be even infinite [5, 6]. What is important is only the high-frequency asymptote that, of course, cannot propagate faster than c, an upper bound to the signal velocity.

Below we are concerned to cylindrically symmetrical free-space solutions of the Klein-Gordon (KG) equation with sub- and superluminal group velocities v. The former ones belong to a class of localized diffraction-free modes that have been found for a homogeneous wave equation [7] and studied in a number of papers (see Ref. 8 and references therein). The discussion of the latter solutions as well as their relation to the causal Green function, up to our knowledge, has not been done yet.

The KG equation follows from the well-known relation between energy E, momentum p, and particle rest mass m

$$E^2 = c^2(p^2 + m^2c^2) \tag{1}$$

through the operator transformation $E \to i\hbar\partial/\partial t$ and $\mathbf{p} \to -\mathbf{i}\boldsymbol{\hbar}\nabla$. The particle velocity is defined as a first derivative $\mathbf{v} = \partial\mathbf{E}/\partial\mathbf{p}$. Comparing with the basic formula of the relativistic mechanics $E = mc^2/\sqrt{1 - v^2/c^2}$, Eq. (1) contains well-known additional solutions for negative-energy states that were understood as antimatter (within the classical theory it should be interpreted as a negative rest mass). What we want to emphasize here is that this equation is also valid for a relativistic particle with $v > c$. This statement, of course, cannot be regarded as a proof of the existence of tachyons, the faster-than-light objects, but only as a hint that the corresponding wave equation may contain modes with a superluminal v. Another hint comes from the physics of a free relativistic particle. A causal propagator for the KG equation is known to be manifestly nonvanishing outside the light cone [9] and, as it was stated in Ref. 10, the path-integral formulation of relativistic quantum mechanics does include the contribution of spacelike trajectories. What we do not agree with the authors of the paper [10] is that this behavior is acausal, i.e., backward in time in some Lorentz frames. Below we shall find a dispersion law for superluminal solutions and show that in any case a signal velocity formed by the modes and defined as that of a wavefront cannot exceed the light speed in vacuum. Moreover, we shall demonstrate that an expression for a causal propagator outside the light cone does follow directly from the dispersion law with superluminal velocities.

Let us look for superluminal modes within a class of axially (φ invariant) symmetric monochromatic solutions of the KG equation propagating in the z direction $\psi(\mathbf{r},\mathbf{t}) = \phi(\rho)\exp(\mathbf{i k_z z} - \mathbf{i}\omega\mathbf{t})$, where $\rho = \sqrt{x^2+y^2}$, ω is the angular frequency, and k_z is the axial wavenumber. For $\phi(\rho)$ we obtain the following equation

$$[\Delta_\rho + ((\omega/c)^2 - k_z^2 - (mc/\hbar)^2)]\phi(\rho) = 0 \tag{2}$$

with a two-dimensional Laplace operator $\Delta_\rho = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in the plane perpendicular to the z-axis. The usually used solutions of Eq. (2) are plane waves proportional to $\exp(ik_x x + ik_y y)$, another ones are Bessel functions whose form depends on the sign of the expression in brackets. If it is positive, i.e.,

$$(\omega/c)^2 - k_z^2 - (mc/\hbar)^2 = Q^2 > 0, \tag{3}$$

then $J_0(Q\rho)$, the Bessel function of the first kind and zero order, is a solution of Eq. (2), and

$$\psi(\rho, z, t) = J_0(Q\rho)\exp(ik_z z - i\omega t). \tag{4}$$

This axially symmetric mode for a wave equation ($m = 0$) was found and realized experimentally by Durnin et al. [7] and is known now as a Bessel beam [8]. Its group velocity $v_z = \partial\omega/\partial k_z$ is always less than c but the relating wave packet would propagate with the light speed in vacuum (as it was shown by detailed calculations in Ref. 8 and follows from the asymptotic behavior of v_z).

Another possibility, i.e.,

$$(\omega/c)^2 - k_z^2 - (mc/\hbar)^2 = -q^2 < 0, \tag{5}$$

has not been discussed yet. The corresponding solution of Eq. (2) is a familiar modified Bessel function of the first kind and zero order $K_0(q\rho)$ and the whole solution to the KG equation looks as

$$\psi(\rho, z, t) = K_0(q\rho)\exp(ik_z z - i\omega t). \tag{6}$$

Together with the relation $k_z(\omega) = \pm\sqrt{(\omega/c)^2 + q^2 - (mc/\hbar)^2}$, Eq. (6) is the main result of the Letter. It is evident that the relating group velocity v_z exceeds the value of c for all available k_z. But for great ω it asymptotically goes to c and it means that the signal formed by these modes will propagate with the light speed in vacuum never destroying the causality principle.

Following the paper [11], we can now construct fundamental solutions of the KG equation (Green functions) for timelike and spacelike paths in the Minkowski 4D space. Eq. (4) with the dispersion law (3) yields [11] a real part of the causal Feynman propagator Δ^c [9] valid inside the light cone

$$\begin{aligned} &\frac{c}{2\pi}\int_0^\infty QdQ J_0(Q\rho)\int dk_z \frac{\sin\omega(k_z)|t|}{\omega(k_z)}\exp(ik_z z) \\ &= \frac{1}{2\pi}\int_0^\infty QdQ J_0(Q\rho) J_0(\tau\sqrt{Q^2 + (mc/\hbar)^2}) = \mathrm{Re}\Delta^c(\lambda), \end{aligned}$$

where $\tau = \sqrt{c^2t^2 - z^2}$ is the 2D timelike interval and $\lambda = \sqrt{c^2t^2 - r^2}$ is the 4D timelike interval. From the solution (6) with the dispersion relation (5) we obtain (see Ref. 11 for details) the Green function outside the light cone

$$\begin{aligned} &\frac{1}{2\pi c}\int_{mc/\hbar}^\infty qdq K_0(q\rho)\int d\omega \frac{\sin k_z(\omega)|z|}{k_z(\omega)}\exp(-i\omega t) \\ &= \frac{1}{2\pi}\int_{mc/\hbar}^\infty qdq K_0(q\rho) J_0(\widetilde{\tau}\sqrt{q^2 - (mc/\hbar)^2}) = \frac{mc}{2\pi\hbar\widetilde{\lambda}} K_1(mc\widetilde{\lambda}/\hbar), \end{aligned}$$

where $\widetilde{\tau} = \sqrt{z^2 - c^2t^2}$ is the 2D spacelike interval, $\widetilde{\lambda} = \sqrt{r^2 - c^2t^2}$ is the 4D spacelike interval, K_1 is the modified Bessel function of the first order. The latter result is just an imaginary part of the causal Feynman propagator Δ^c [9] for spacelike paths. In the limit $\hbar \to 0$ it vanishes for all classically forbidden trajectories.

Resuming, we worked out a novel solution to the free-space Klein- Gordon equation propagating with a superluminal group velocity outside the light cone. We have rejected any conclusions [10] about acausality of the spacelike paths and argued that a signal carried by a wave packet belonging to this class of modes propagates with the light speed in vacuum c. In our opinion, there are no objections to possible realizations of such solutions in nature. If they do exist, their thermodynamical properties would be very exotic due to the unusual spectrum (5).

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[1] IEEE J. Selected Topics in Quant. Electr. **9**, No. 1 (2003)

[2] L.A.Vainshtein, Sov. Phys. Usp. **19**, 189 (1976)

[3] R.Y.Chiao and P.W.Milonni, Opt. Photon. News **13**, 26 (2002).

[4] M.Büttiker and S.Washburn, Nature **422**, 271 (2003).

[5] A.M.Steinberg, P.G.Kwiat, and R.Y.Chiao, Phys. Rev. Lett. **71**, 708 (1993).

[6] E.L.Bolda, J.C.Garrison, and R.Y.Chiao, Phys. Rev. A **49**, 3938 (1994).

[7] J.Durnin, J.J.Miceli Jr., and J.H.Eberly, Phys. Rev. Lett. **58**, 1499 (1987).

[8] J.T.Lunardi, Phys. Lett. A **291**, 66 (2001).

[9] R.P.Feynman, Phys.Rev. **76**, 749 (1949).

[10] I.H.Redmount and W.-M.Suen, Int. J. Mod. Phys. A **8**, 1629 (1993).

[11] A.A.Borghardt and D.Ya.Karpenko, J. Nonlinear Math. Phys. **5**, 357 (1998).