[6]At the hyperfocal point $x = D/k$, Eq. (1a) yields $A = \infty$, Eq. (1b) yields $B = D/2k$. This represents the case when the depth of field extends from $x = D/2k$ to $x = \infty$ and forms the practical basis for fixed-focus cameras such as the well-known "Instamatic."

[7]*SPSE Handbook of Photographic Science and Engineering*, edited by Woodlief Thomas, Jr. (Wiley, New York, 1973).

[8]The two-point determination of Scheimpflug focusing and the two-point determination of depth of field are not unique to a view camera with calibrated monorail. At least one commercially available view camera has incorporated this approach through the use of conveniently arranged and calibrated adjustments that permit the photographer to read the desired f# and the Scheimpflug angle directly.

Pedagogy of spin in nonrelativistic quantum mechanics

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The quantum-mechanical properties of electron spin are developed systematically from a nonrelativistic Hamiltonian. The correct values of the magnetic moment of the electron, the spin-orbit coupling energy of atomic electrons, the Darwin correction, and the angular momentum of the electron are obtained directly without explicitly including any relativity beyond classical electromagnetism.

I. INTRODUCTION

Introducing the important concept of spin into courses on elementary quantum mechanics is one of the most difficult problems faced by university physics teachers. Textbooks generally follow one of two alternatives. Either Dirac's full relativistic treatment is used or spin is added on an *ad hoc* basis into the Schrödinger formalism. While the former is too complex for elementary courses, the latter is unsatisfactory in many ways since not only does spin fail to arise naturally but without cumbersome relativistic corrections the correct value for the spin-orbit energy cannot be obtained. Consequently, it is often stated that spin is an essentially relativistic phenomenon which cannot be fully incorporated into a nonrelativistic treatment. Of course magnetic fields are in a sense a relativistic correction to the electric field for moving particles and to this extent spin is also an intrinsicially relativistic phenomenon. However, the present authors feel that it is possible to present a systematic and convincing treatment of all the essential features of electron spin without recourse of Dirac's relativistic analysis or the addition of arbitrary terms to the Hamiltonian. The important results to be established in this paper are the magnetic moment of the electron, the spin-orbit coupling for atomic electrons, and the intrinsic spin of the electron.

II. FREE-PARTICLE HAMILTONIAN

In classical mechanics one can obtain the form of the free-particle Lagrangian and consequently the Hamiltonian by applying Hamilton's principle and the symmetry properties of space and time.[1] This leads to the usual Hamiltonian

$$H = \mathbf{p}^2/2m. \tag{1}$$

In quantum mechanics one may write a more general Hamiltonian of the form[2]

$$H = (\boldsymbol{\sigma} \cdot \mathbf{p})^2/2m, \tag{2}$$

which retains all the symmetry properties and reduces to Eq. (1) for free particles provided

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}, \tag{3}$$

from which the commutation relation

$$[\sigma_i,\sigma_j] = 2i\sigma_k \tag{4}$$

and the general identity

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}) \tag{5}$$

for any two operators **a** and **b**, can be derived. Equation (3) shows that the eigenvalues of each component of $\boldsymbol{\sigma}$ are ± 1. From these equations it is easy to show that the Pauli spin matrices provide a representation for the components of $\boldsymbol{\sigma}$ but from a pedagogical point of view it is important to stress the fact that the particular representation is not crucial. All the necessary information is contained in Eq. (3).

III. EXTERNAL MAGNETIC FIELDS

In order to determine the effect of an electromagnetic field on an otherwise free electron, one writes in the usual way[3]

$$H = [\boldsymbol{\sigma} \cdot (\mathbf{p} + e\mathbf{A})]^2/2m - e\phi, \tag{6}$$

where **p** is the canonical momentum operator, e is the absolute value of the charge on the electron, **A** is the vector potential that, if not explicitly time dependent, is purely magnetic, and ϕ is the scalar potential. Noting that the components of $\mathbf{p} + e\mathbf{A}$ do not in general commute, Eq. (6) can be expanded to give, using Eq. (5),

$$H + e\phi = \mathbf{p}^2/2m + e\mathbf{A} \cdot \mathbf{p}/m - ie\hbar\nabla \cdot \mathbf{A}/2m + e^2\mathbf{A}^2/2m + e\hbar\boldsymbol{\sigma} \cdot (\nabla \times \mathbf{A})/2m. \tag{7}$$

The last term in Eq. (7) can be written in a general magnetic field **B** as

$$e\hbar\boldsymbol{\sigma} \cdot B/2m = -\boldsymbol{\mu}_s \cdot \mathbf{B}, \tag{8}$$

while if the field is uniform the third term is zero and the

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second can be shown to be[3]

$$e\mathbf{L}\cdot\mathbf{B}/2m = -\boldsymbol{\mu}_l\cdot\mathbf{B}. \quad (9)$$

The fourth term can be neglected in weak fields but is the lowest-order term in photon scattering problems. Slater[4] shows that this term represents the energy from the induced current from the process of building up the magnetic field. Equation (8) shows clearly that the electron has an intrinsic magnetic moment $\boldsymbol{\mu}_s$ and an orbital magnetic moment $\boldsymbol{\mu}_l$ owing to the angular momentum **L**.

Comparing Eqs. (8) and (9) one is tempted to identify the electron spin with $\hbar\boldsymbol{\sigma}$. However, if **S** and **L** are to obey the same commutation relation then the correct expression for the electron spin must be

$$\mathbf{S} = \hbar\boldsymbol{\sigma}/2. \quad (10)$$

A more convincing argument will be given below.

IV. INTERACTION WITH AN ATOMIC NUCLEUS

The Hamiltonian for the electron–nucleus interaction can be written in terms of the retarded scalar and vector potentials owing to a point charge moving with velocity[5] **v**. In the zeroth-order approximation to the Hamiltonian, in powers of $(v/c)^2$, the scalar potential due to the electron at the nucleus is just the Coulomb potential

$$\phi = -e/4\pi\epsilon_0 r \quad (11)$$

and the vector potential is zero. In order to include the magnetic interaction it is necessary to consider the lowest-order term in **A** and the first-order correction to ϕ arising from the retardation. By means of a suitable gauge transformation the distinction between the electrostatic and the magnetic effects can be maintained, ϕ retains the form of the Coulomb potential, and the vector potential becomes[5]

$$\mathbf{A} = -e\mathbf{v}/8\pi\epsilon_0 rc^2 = \phi p/2mc^2 \quad (12)$$

provided the probability current density is perpendicular to **r** as it is in the case of a bound state of a hydrogen atom.[6]

The Hamiltonian for the interaction can now be written in the rest frame of the nucleus as

$$H = [\boldsymbol{\sigma}\cdot(\mathbf{p} + Ze\mathbf{A})]^2/2m + Ze\phi + 3p^4/8m^3c^2 \quad (13)$$

or

$$H - V = [\boldsymbol{\sigma}\cdot(\mathbf{p} + V\mathbf{p}/2mc^2)]^2/2m + 3p^4/8m^3c^2 \quad (14)$$

with

$$V = Ze\phi. \quad (15)$$

The last term in Eqs. (13) and (14) gives the first-order correction to the nonrelativistic expression for the mechanical energy. Expanding Eq. (14) (see Appendix) leads to

$$H = p^2/2m + V - (\hbar^2/4m^2c^2)(\partial V/\partial r)(\partial/\partial r) + (1/2m^2c^2)(1/r)(\partial V/\partial r)\mathbf{S}\cdot\mathbf{L} - p^4/8m^3c^2. \quad (16)$$

This can now be compared to the equation obtained by expanding Dirac's exact equation to this order.[3,7] The first two terms give the Hamiltonian in the absence of magnetic interactions, the third gives the Darwin term and the fourth the correct expression for the spin-orbit coupling. The last term agrees with the corresponding term in the full relativistic derivation. Both this and the Darwin term shift the energy levels by a small amount whereas the spin-orbit coupling term splits the energy levels and is of much greater importance.

V. ANGULAR MOMENTUM OF THE ELECTRON

If the spin orbit coupling term in Eq. (16) is written H', then

$$[H',\mathbf{L}] = f(r)(-i\hbar\boldsymbol{\sigma}\times\mathbf{L}), \quad (17)$$

$$[H',\boldsymbol{\sigma}] = f(r)(2i\hbar\boldsymbol{\sigma}\times\mathbf{L}), \quad (18)$$

so that neither **L** nor $\boldsymbol{\sigma}$ commutes with H' whereas

$$\mathbf{J} = \mathbf{L} + \hbar\boldsymbol{\sigma}/2 \quad (19)$$

clearly does commute with H' and the spin angular momentum is given correctly by Eq. (10). Since the eigenvalues of each component of $\boldsymbol{\sigma}$ are ± 1, the allowed values of the components of S are $\pm\hbar/2$ while that of S^2 is $3\hbar^2/2$.

VI. CONCLUSIONS

In this paper a new approach to the problem of teaching spin in quantum mechanics is presented. Advanced texts generally follow Dirac's fully relativistic treatment[3,6] that is beyond the reach of many undergraduates. Introductory texts generally introduce the various aspects of spin on an *ad hoc* basis. The anomolous magnetic moment of the electron is then simply quoted as a consequence of Dirac's theory. The spin-orbit coupling term is included by the somewhat indirect procedure of considering the magnetic field produced by the nucleus in the rest frame of the electron and then transforming back to the rest frame of the nucleus via the Thomas precession. However, the transformation alone requires a lengthy and involved calculation.[8]

The present authors feel that the approach outlined above is straightforward and enables students to appreciate the essential physics involved in the derivation of each term. The derivation of the potentials for the electron–nucleus interaction provides an instructive exercise in the manipulation of scalar and vector potentials by means of guage transformations.

The confusion in many textbooks concerning the "relativistic" or "nonrelativistic" nature of electron spin stems from the fact that the contribution to the energy of the electron–nucleus interaction arising from the vector potential is of the same order as the relativistic mass correction. However, it has been shown in this paper that the scalar and vector potentials can be included in this order on the basis of classical electromagnetic theory so that the Darwin term and the spin-orbit coupling term come out correctly. Fisher[9] has presented a very interesting analysis of the spin-orbit coupling in which he shows that a magnetic dipole moving with velocity βc has an electric dipole moment $\mathbf{p} = \boldsymbol{\beta}\times\boldsymbol{\mu}/c$. The coupling between this and the electric field from the rest of the atom then produces the spin-orbit coupling. Sakurai[10] explains the Darwin term in terms of zitterbewegung and negative energy components.

Perhaps the most important aspect of spin from a pedagogical point of view arises from the observation that spin cannot be developed in terms of differential operators and explicit wave functions as can be done for orbital angular momentum, whereas the commutation properties of σ suffice to determine the properties of electron spin completely. This enables one to stress the fundamental role of the operator formalism in quantum mechanics which students otherwise tend to regard as elegant but not essential.

The approach outlined in this paper has been used successfully in this department.

APPENDIX

Equation (14) can be written

$$H - V = (\boldsymbol{\sigma}\cdot\mathbf{p} + V\boldsymbol{\sigma}\cdot\mathbf{p}/2mc^2)^2/2m + (\tfrac{3}{8})p^4/m^3c^2. \quad \text{(A1)}$$

Expanding this equation and using Eq. (5) gives

$$H - V = p^2/2m + (\boldsymbol{\sigma}\cdot\mathbf{p}V\boldsymbol{\sigma}\cdot\mathbf{p})/(4m^2c^2) + (Vp^2/4m^2c^2) + (V^2p^2/8m^3c^4) + (\tfrac{3}{8})p^4/m^3c^2. \quad \text{(A2)}$$

The penultimate term is of order p^6/m^5c^4 and can be dropped and

$$\boldsymbol{\sigma}\cdot\mathbf{p}V\boldsymbol{\sigma}\cdot\mathbf{p} = \boldsymbol{\sigma}\cdot(\hat{\mathbf{p}}\hat{V})\boldsymbol{\sigma}\cdot\mathbf{p} + V(\boldsymbol{\sigma}\cdot\mathbf{p})^2, \quad \text{(A3)}$$

where $\hat{}$ indicates that $\mathbf{p}$ is to operate directly on the potential V. Equation (A2) then becomes

$$H - V = p^2/2m - i\hbar(\boldsymbol{\sigma}\cdot\nabla V)\boldsymbol{\sigma}\cdot\mathbf{p}/(4m^2c^2) + Vp^2/(2m^2c^2) + (\tfrac{3}{8})p^4/m^3c^2. \quad \text{(A4)}$$

Applying Eq. (5) to the second term gives

$$H - V = p^2/2m - i\hbar\nabla V\cdot\mathbf{p}/(4m^2c^2) + \hbar\boldsymbol{\sigma}\cdot(\nabla V\times\mathbf{p})/(4m^2c^2) + Vp^2/(2m^2c^2) + (\tfrac{3}{8})p^4/m^3c^2. \quad \text{(A5)}$$

For circular motion, or by taking average values and applying the virial theorem,

$$V = -2T = -p^2/m \quad \text{(A6)}$$

so that the last two terms in Eq. (A5) are of the order

$$-p^4/(2m^2c^2) + (\tfrac{3}{8})p^4/(m^3c^2) = -p^4/8m^3c^2, \quad \text{(A7)}$$

and Eq. (A5) reduces to Eq. (16) with the aid of Eq. (10) and the standard results

$$\nabla V = (1/r)(\partial V/\partial r)\mathbf{r}, \quad \text{(A8)}$$

$$\nabla V\cdot\mathbf{V} = (\partial V/\partial r)(\partial/\partial r). \quad \text{(A9)}$$

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PROBLEM

A mass m is held against a spring of spring constant k at the bottom of an incline which makes an angle θ with the horizontal. The spring is compressed a distance L. The coefficient of friction between the mass and the incline is μ. When the restraining force holding the mass is released, the mass will be projected up the include. How far up the incline will it travel? (The solution is on page 671.)