BUILDING SPACETIME from SPIN

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Dirac's equation:

$$(i\not{\partial} - m)\,\psi = 0$$

DIRAC OPERATOR

SPINOR FIELD

$$\psi : \mathbb{R}^4 \to \mathbb{C}^4$$

MINKOWSKI SPACETIME

DIRAC SPINORS

HOW ARE THEY RELATED??

$$\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \overline{\mathbb{C}}^2$$

DIRAC SPINORS — LEFT-HANDED WEYL SPINORS — RIGHT-HANDED WEYL SPINORS

$\mathbb{C}^2$ & $\overline{\mathbb{C}}^2$ ARE IRREDUCIBLE REPRESENTATIONS ("IRREPS") OF $SL(2,\mathbb{C})$:

$$\begin{tikzcd}
SU(2) \arrow[r, hook] \arrow[d, two heads, "2-1"'] & SL(2,\mathbb{C}) \arrow[d, two heads, "2-1"] \\
SO(3) \arrow[r, hook] & SO_0(3,1)
\end{tikzcd}$$

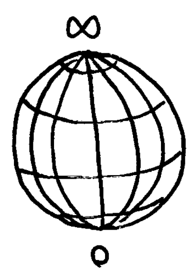
ROTATION GROUP (SPACE) — LORENTZ GROUP (SPACETIME)

THESE ARE ALL SECRETLY THE SAME :

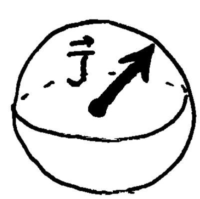
1) PURE STATES OF A WEYL SPINOR – unit vectors in $\mathbb{C}^2$ modulo phase

2) POINTS OF $\mathbb{C}P^1$ – 1-dimensional subspaces of $\mathbb{C}^2$

3) POINTS OF THE RIEMANN SPHERE



4) ANGULAR MOMENTUM VECTORS $\vec{J}$ with $\vec{J} \cdot \vec{J} = j(j+1)$, $j = \frac{1}{2}$



5) PROJECTION OPERATORS
$P : \mathbb{C}^2 \to \mathbb{C}^2$ WITH $\mathrm{tr}(P) = 1$

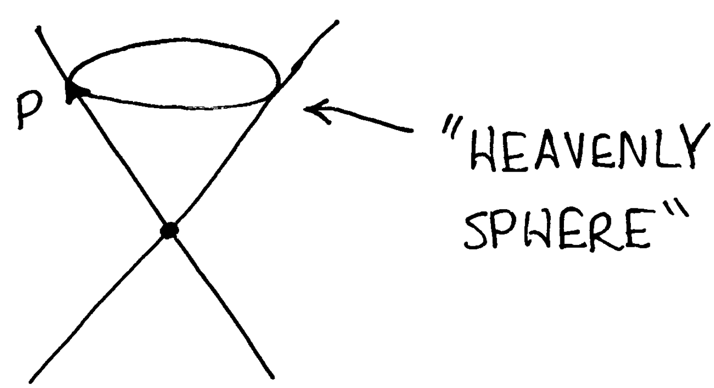
$(P^2 = P \,,\; P = P^*)$

BUT SPACE OF HERMITIAN 2×2 MATRICES IS MINKOWSKI SPACETIME:

$$\det \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

AND THESE PROJECTIONS HAVE $\det(P) = 0$, SO THEY ARE ALSO:

6) LIGHT RAYS THROUGH THE ORIGIN IN MINKOWSKI SPACETIME!



$SL(2,\mathbb{C})$ ACTS AS CONFORMAL TRANSFORMATIONS OF THIS SPHERE!

EVERY FINITE-DIMENSIONAL REP OF $SL(2,\mathbb{C})$, & THUS THE LORENTZ GROUP, IS BUILT FROM $\mathbb{C}^2$ AND $\overline{\mathbb{C}}^2$. HOW MUCH PHYSICS CAN WE DO JUST WITH THESE?

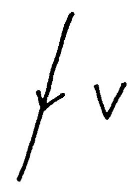
ANSWER: QUANTUM GRAVITY IN <u>3D</u> SPACETIME!

FEYNMAN'S DICTIONARY

as generalized to an arbitrary group by Penrose & others...

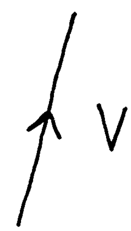
- IRREP V — "particle"

  e_i ← basis of V



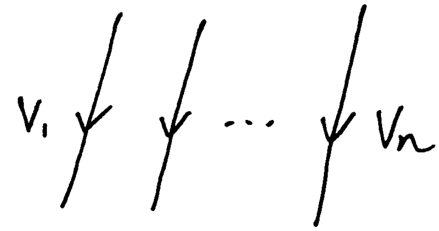
- DUAL REP V^* — "antiparticle"

  e^i ← dual basis of V^*



- TENSOR PRODUCT $V_1 \otimes \cdots \otimes V_n$ — "collection of particles"

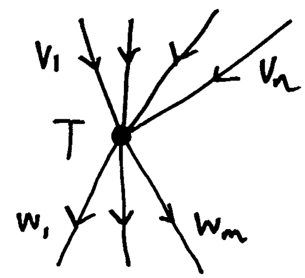
  $e_{i_1} \otimes \cdots \otimes e_{i_n}$ ← basis of $V_1 \otimes \cdots \otimes V_n$



- INTERTWINER "interaction"

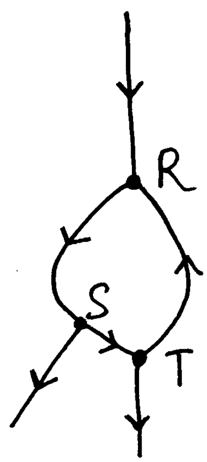
$$T: V_1 \otimes \cdots \otimes V_n \to W_1 \otimes \cdots \otimes W_n$$

$$T^{i_1 \cdots i_n}_{j_1 \cdots j_m}$$



WE CAN COMBINE INTERTWINERS TO GET NEW ONES:

$$R^{ij}_{k} S^{k}_{lm} T^{l}_{nj}$$



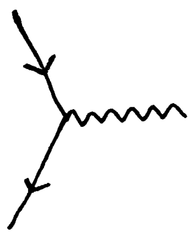
"Feynman diagram"

IF WE USE UNITARY REPS OF THE POINCARÉ GROUP THESE ARE THE "FEYNMAN RULES"!

QED:

= SOLUTIONS OF DIRAC EQUATION, ψ

= SOLUTIONS OF MAXWELL EQUATIONS, A



= INTERTWINER COMING FROM $\int \bar{\psi} \not{A} \psi$

ONLY PROBLEM:

INFINITIES,

COMING FROM THESE INFINITE-DIMENSIONAL REPS OF THE POINCARÉ GROUP!

PENROSE DECIDED TO TRY

FINITE-DIMENSIONAL,

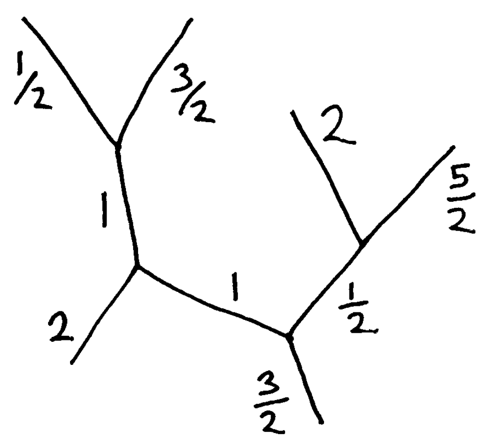
NON-UNITARY IRREPS OF

$SL(2,\mathbb{C})$. IF WE USE ONLY

"LEFT-HANDED" ONES, THERE IS

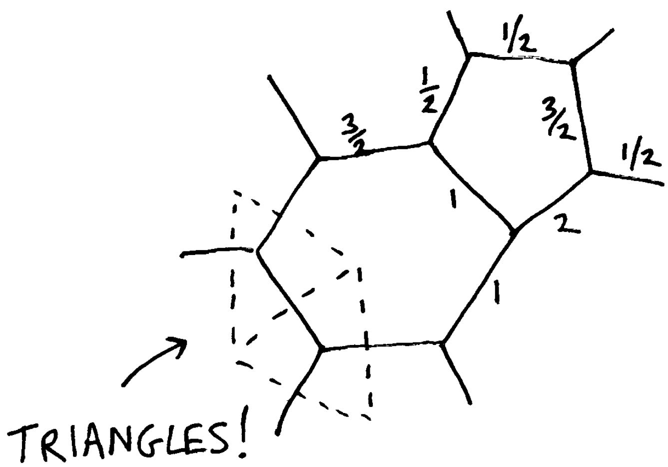
ONE FOR EACH <u>SPIN</u>:

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

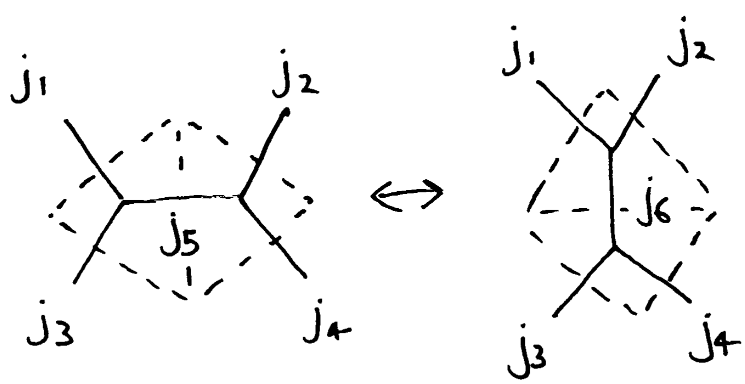


SPIN NETWORK

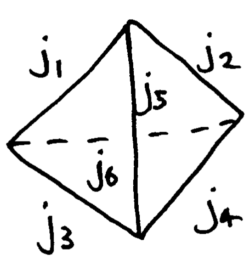
THE SPIN-j REPS ARE UNITARY REPS OF SU(2), SO SPIN NETWORKS TURN OUT TO BE BETTER SUITED FOR 3D RIEMANNIAN QUANTUM GRAVITY:



SPACE IS DESCRIBED USING TRIVALENT SPIN NETWORKS



THE BASIC "INTERACTION"



AMPLITUDE FOR INTERACTION COMPUTED BY EVALUATING THIS SPIN NETWORK!

"6J SYMBOL"

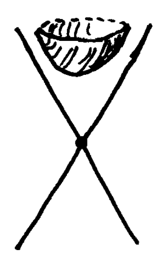
HOW ABOUT 4D LORENTZIAN QUANTUM GRAVITY?

Here it's natural to try unitary irreps of $SL(2,\mathbb{C})$ – "extensors". For any $j > 0$ get an irrep:

$$\mathcal{H}_j = \{ f : \nabla^2 f = -(j^2+1) f \}$$

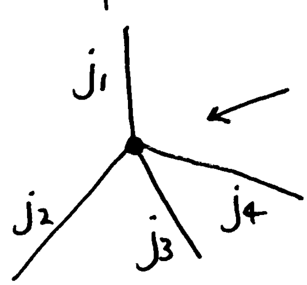
INFINITE-DIMENSIONAL!

function on "mass shell" –



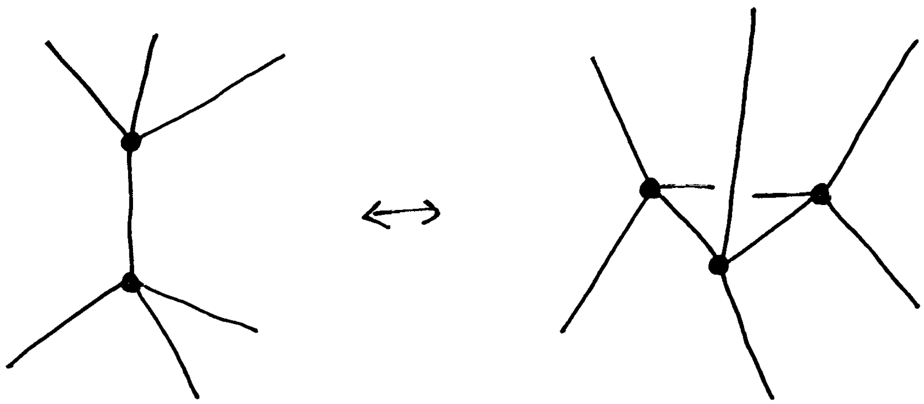
3d hyperbolic space

Space could be described by a 4-valent "Lorentzian spin network":

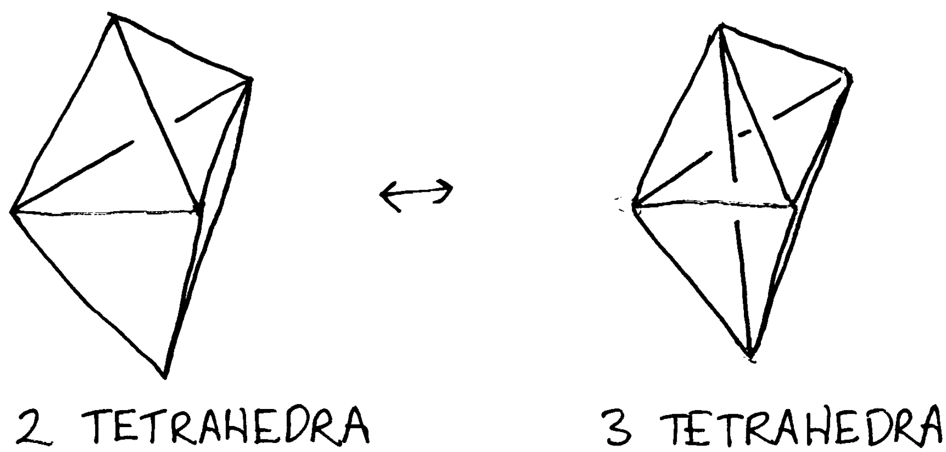


"Barrett-Crane intertwiner"

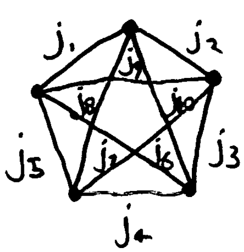
Basic interaction could be:



DUAL PICTURE:



... with interaction amplitude given by "Lorentzian 10j symbols":



Proved to converge even though reps are infinite-dimensional!

"BARRETT-CRANE MODEL"