Classical and quantum theories of spin

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Abstract

A great effort has been devoted to formulate a classical relativistic theory of spin compatible with quantum relativistic wave equations. The main difficulty in order to connect classical and quantum theories rests in finding a parameter which plays the role of proper time at a purely quantum level. We present a partial review of several proposals of classical and quantum spin theories from the pioneer works of Thomas and Frenkel, revisited in the classical BMT work, to the semiclassical model of Barut and Zanghi [Phys. Rev. Lett. 52, 2009 (1984)]. We show that the last model can be obtained from a semiclassical limit of the Feynman proper time parametrization of the Dirac equation. At the quantum level we derive spin precession equations in the Heisenberg picture. Analogies and differences with respect to classical theories are discussed in detail.

1 Introduction

The interest in formulating classical theories to study the precession of the spin lays in the fact that many problems, experimentally tested, could be more simply derived from a consistent set of covariant classical equations of motion in proper time. However at the quantum level the standard formulation of relativistic wave equations did not explicitly include the concept of proper time. Then some new ingredients are needed in order to retrieve the covariant classical equations of motion starting from quantum theory. In the semiclassical treatments of Rubinow and Keller [1] and Rafanelli and Schiller [2] the proper time was introduced through a WKB expansion of the Dirac equation. On the other hand many quantum proper time derivatives were proposed in the literature [3]. These are heuristic proposals based on classical analogies, which go back to the earlier works of Fock [4] and Beck [5]. Later on, in the sixties, there was a growing interest for parametrized theories in an invariant parameter (which looks like the proper time), known under several names such as "proper time" formalisms or parametrized relativistic quantum theories (PRQT). These formulations have their origin in the works of Stückelberg [6], Feynman [7, 8, 9], Nambu [10], and Schwinger [11]. From the point of view of PRQT, "proper time" derivatives can be thought of as the derivatives with respect to the evolution parameter in the Heisenberg picture.

In Section 2 we sketch the main lines of the classical theories of spin. For this case we put special emphasis in the traditional point of view, as the one developed by Frenkel [12] and Thomas [13] up to the revival of such pioneer works carried out by Bargmann, Michel, and Telegdi [14]. In Section 3 we revisit the most relevant quantum theories of spinning systems and show how to reobtain the BMT equation in the classical limit in different ways. In Section 4 we focus our attention on the theory associated with the proper time parametrization of the Dirac equation originally proposed by Feynman [7] and we discuss about the necessity to formulate an adequate framework out of the mass shell. We show the way to connect this formalism with the results of Section 3. Finally in Section 5, using the formalism of Section 4, we rederive in the semiclassical limit, via relativistic coherent states, the formulation of Barut and Zanghi [15] of the classical spinning system.

2 Classical theories

Classical theories of spinning particles were formulated as a generalization to any inertial frame of the classical spin precession in the presence of external electromagnetic fields (\mathbf{E}, \mathbf{B}) in the rest frame, namely (c = 1)

$$\frac{d\mathbf{S}}{dt} = \mu \times \mathbf{B},\tag{1}$$

where μ is the magnetic moment of the electron, which is assumed to be proportional to the spin \mathbf{S} , $\mu = \frac{ge}{2m} \mathbf{S}$, being *m* the mass, *e* the charge, and *g* the gyromagnetic factor of the electron. It is also assumed that the electric moment is zero in the rest frame.

There are two natural covariant generalizations to any frame of the spin **S** and consequently of Eq. (1). We can consider an axial four-vector S_{α} or an antisymmetric second-order tensor $S_{\alpha\beta}$, with only three independent components, such that $S_0 = 0$ and equivalently $S_{0i} = 0$ in the rest frame. In a covariant way these conditions read [13, 16, 17, 18]

$$S_{\alpha}u^{\alpha} = 0; \qquad \qquad S_{\alpha\beta}u^{\beta} = 0, \qquad (2)$$

where u^{α} is the four-velocity $u^{\alpha} = \frac{dx^{\alpha}}{ds}$ (a unit time-like vector $u^{\alpha}u_{\alpha} = 1$).

These objects are then related by¹

$$S^{\alpha} = \frac{1}{2} \varepsilon^{\alpha\beta\mu\nu} u_{\beta} S_{\mu\nu}; \qquad \qquad S_{\alpha\beta} = \varepsilon_{\alpha\beta\mu\nu} u^{\mu} S^{\nu}. \qquad (3)$$

Frenkel [12] and Thomas [13] considered the most general equation for the proper time variation of the spin vector or tensor –this includes non-minimal couplings between matter and external electromagnetic fields– compatible with Eq. (1), by taking into account all the relativistic invariants one can construct, assuming that the precession equation is linear in the spin (S_{α} or $S_{\alpha\beta}$) and in the electromagnetic fields ($F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$). Considering also the Lorentz force law²

$$\frac{d\pi_{\mu}}{ds} = eF_{\mu\nu}u^{\nu},\tag{4}$$

where $\pi_{\mu} = m u_{\mu}$, the spin precession equations result

$$\frac{dS_{\alpha}}{ds} = \frac{e}{m} \left[\frac{g}{2} F_{\alpha}^{\ \beta} S_{\beta} + \left(\frac{g}{2} - 1 \right) u_{\alpha} \left(S_{\lambda} F^{\lambda \nu} u_{\nu} \right) \right], \tag{5}$$

$$\frac{dS_{\alpha\beta}}{ds} = \frac{e}{2m} \left(1 - \frac{g}{2} \right) \left(S_{\beta\nu} u_{\alpha} - S_{\alpha\nu} u_{\beta} \right) F_{\lambda}^{\ \nu} u^{\lambda} + \frac{ge}{2m} \left(S_{\alpha\lambda} F_{\beta}^{\ \lambda} - S_{\beta\lambda} F_{\alpha}^{\ \lambda} \right). \tag{6}$$

Equation (5) is the celebrated BMT equation, which was rediscovered by Bargmann, Michel, and Telegdi [14].

3 Quantum theories

3.1 Semiclassical proper time derivative

Rubinow and Keller [1] continued and old attempt of Pauli [20] to solve the Dirac equation for a particle in an electromagnetic field minimally coupled, $\pi_{\mu} \equiv p_{\mu} - eA_{\mu}, p_{\mu} = i\partial_{\mu} (\hbar = 1),$

$$\left(\gamma^{\mu}\pi_{\mu} - m\right)\psi = 0, \tag{7}$$

using the WKB method, proposing a solution of the form

$$\psi_{\rm WKB} = \psi_0 e^{-iS},\tag{8}$$

¹The conventions are $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ for the Minkowski metric tensor, $\varepsilon^{0123} = 1$ for the Levi- Civita tensor, and $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$.

²At this level we are neglecting gradients in the fields, e.g. $\nabla (\mu \cdot \mathbf{B})$, but however retaining non-minimal couplings at the level of the spin precession equation. Neglecting consistently non-minimal terms Kramers [19] was able to demonstrate that the gyromagnetic factor is g = 2. This result is commonly believed to be explained only by the Dirac equation.

where S is an scalar function and ψ_0 a spinor. They were able to obtain the appropriate solution ψ_{WKB} also for the modified Dirac equation of a particle with an anomalous magnetic moment (by adding to Eq. (7) a Pauli term, $\left(\frac{g}{2}-1\right)\frac{e}{2m}F^{\mu\nu}S_{\mu\nu}$, where $\frac{g}{2}-1$ is the anomalous gyromagnetic factor, and $S_{\mu\nu} \equiv \frac{1}{2}\sigma_{\mu\nu} \equiv \frac{i}{4}\left[\gamma_{\mu},\gamma_{\nu}\right]$). For any variable $a = a(x^{\mu},\gamma^{\mu})$ it can be associated a density $\mathcal{A}(x^{\mu})$, such that

$$\mathcal{A} \equiv \overline{\psi}_{\rm WKB} a \psi_{\rm WKB},\tag{9}$$

where $\overline{\psi} \equiv \psi^{\dagger} \gamma^{0}$ is the Dirac adjoint. As particular cases of Eq. (9) we have [21]

$$\mathcal{U}_{\mu} = \frac{1}{m} \left(\partial_{\mu} S - e A_{\mu} \right) = \overline{\psi}_{\text{WKB}} \gamma_{\mu} \psi_{\text{WKB}} \tag{10}$$

and

$$S^{\mu} = \overline{\psi}_{\rm WKB} \frac{i}{2} \gamma^5 \gamma^{\mu} \psi_{\rm WKB}, \qquad \qquad S^{\mu\nu} = \overline{\psi}_{\rm WKB} S^{\mu\nu} \psi_{\rm WKB}, \qquad (11)$$

which are the semiclassical densities corresponding to the Michel and Wightman [22] polarization four-vector $t^{\mu} \equiv i\gamma^5\gamma^{\mu}$ and to the spin tensor $S^{\mu\nu}$, which satisfy relations analogous to those of Eq. (3). In order to obtain manifestly covariant equations of motion Rubinow and Keller [1] introduced the concept of proper time derivative in this semiclassical approximation, for which the notion of trajectory retrieves its meaning, through

$$\frac{d\mathcal{A}}{ds} = \frac{\partial \mathcal{A}}{\partial x^{\mu}} \frac{dx^{\mu}}{ds} = \mathcal{U}^{\mu} \frac{\partial \mathcal{A}}{\partial x^{\mu}},\tag{12}$$

and derived the BMT equation for the spin vector (11).

Rafanelli and Schiller [2], based on Fock's [4] work were also able to reobtain the BMT equation (5) in an easier way, applying the WKB method to the squared Dirac equation. Following these ideas we have shown [21] a deeper connection with Fock's work, *i.e.*

$$\frac{d\mathcal{A}}{ds} = \overline{\psi}_{\rm WKB} \left(\frac{da}{ds}\right)_{\rm Fock} \psi_{\rm WKB},\tag{13}$$

where the Fock proper time derivative will be defined in Section 3.2. As we will see the BMT equation will be obtained as a particular case of (13).

3.2 Quantum proper time derivatives

At a purely quantum level it is necessary to introduce the notion of proper time derivative in order to write explicitly covariant equations of motion in this parameter. The first in introducing the concept of proper time was Fock [4], through a parametrization which corresponds, in the Heisenberg picture, to a proper time derivative

$$\left(\frac{da}{ds}\right)_{\rm Fock} \equiv -\frac{i}{2m} \left[H^2, a\right],\tag{14}$$

for any dynamical variable a, where $H \equiv \gamma^{\mu} \pi_{\mu}$. For example, for $a = S_{\mu\nu}$ we have

$$\left(\frac{dS_{\alpha\beta}}{ds}\right)_{\rm Fock} = \frac{e}{m} \left(S_{\alpha\lambda}F_{\beta}^{\ \lambda} - S_{\beta\lambda}F_{\alpha}^{\ \lambda}\right),\tag{15}$$

which is the analogous of Eq. (6) for g = 2.

Some years later Beck [5] proposed a first-order proper time derivative

$$\left(\frac{da}{ds}\right)_{\text{Beck}} \equiv -i\left[H,a\right].$$
(16)

Applying (16) to $S_{\alpha\beta}$ we obtain [23]

$$\left(\frac{dS_{\alpha\beta}}{ds}\right)_{\text{Beck}} = \pi_{\alpha}\gamma_{\beta} - \pi_{\beta}\gamma_{\alpha},\tag{17}$$

where³

$$\gamma^{\mu} = \left(\frac{dx^{\mu}}{ds}\right)_{\text{Beck}}.$$
(18)

Equation (17) has not classical analogy yet. We will come back on this relation in Section 4, where we show that by eliminating the interference between positive and negative mass states (covariant *Zitterbewegung*) we retrieve the Frenkel-Thomas equation.⁴

The last derivations of quantum equations of spin motion come from heuristic proper time derivatives but not from first principles. They are formal relations assumed to be valid on the mass shell. However there is no reason for justifying Eq. (18) which uses that

$$[x_{\mu}, p_{\nu}] = -i\eta_{\mu\nu},\tag{19}$$

since at the classical level the covariant canonical Poisson brackets $\{x_{\mu}, p_{\nu}\} = \eta_{\mu\nu}$ are incompatible with the constraint $\pi^{\mu}\pi_{\mu} = m^{2.5}$ Then we need to extend our formalism beyond the mass shell condition. This is the goal of Section 4.

³This relation is the covariant generalization of $\frac{d\mathbf{x}}{dt} = \alpha$ in Dirac's theory, which originated the localization problem. Compare Eq. (18) with Eq. (10).

 $^{^{4}}$ We are considering only the minimal coupling case. The generalization of these results with the inclusion of a Pauli term into the covariant Hamiltonian H is straightforward.

⁵It is easy to see that $0 = \{x_{\mu}, m^2\} = \{x_{\mu}, \pi^{\nu}\pi_{\nu}\} = 2\pi_{\mu}$, which leads to an absurd.

4 Feynman parametrization

We are looking for a first quantized formalism which provides a consistent unification of relativity and quantum mechanics. The main requirements we need to take into account are [3]:

- Dirac's theory must be somehow included.
- The framework must not be restricted to the mass shell.
- The equations of motion must be analogous to those of the classical
- theory in which the evolution parameter is the proper time.

The first and third conditions are the Bohr correspondence principle, while the second one means that we have to enlarge the Poincaré algebra to be able to include a four-vector position operator x^{μ} (which localizes the charge) that satisfies the canonical commutation relations (19). We have developed [24, 25, 26, 27, 28] a consistent theory based on the Feynman parametrization of the Dirac equation

$$-i\frac{d}{ds}\left|\psi\right\rangle = H\left|\psi\right\rangle.$$
(20)

Here we only want to briefly discuss the three points raised above. Equation (20) is a Schördinger-like equation where the evolution parameter can be identified with the proper time. Dirac equation is recovered as an eigenvalue equation for the mass operator H,

$$H \left| \psi_m \right\rangle = m \left| \psi_m \right\rangle,\tag{21}$$

where $|\psi_m\rangle$ are stationary states in s. So, in general, an arbitrary state in this theory does not have a definite mass. These facts explain the first two points. Now let us focus our attention on the third one. The Beck derivative (16) becomes the equation of motion for the dynamical variables in the Heisenberg picture. We now consider semiclassical relativistic coherent states of the form $|\psi\rangle_c = u |\varphi\rangle_c$, where u is a constant spinor on the positive mass shell and $|\varphi\rangle_c$ is a relativistic coherent state for the orbital part, such that

$$\Lambda_{+} \left| \psi \right\rangle_{c} = \left[1 + O(\hbar) \right] \left| \psi \right\rangle_{c}, \qquad (22)$$

where $\Lambda_{+} \equiv \frac{1}{2} \left(I + \frac{\gamma^{\mu} p_{\mu}}{m} \right)$ is the projector on positive mass states. Taking mean values with these semiclassical states Eq. (16) becomes [25]

$$\left\langle \frac{da}{ds} \right\rangle_c = -\frac{i}{2m_c} \left\langle \left[\pi^\mu \pi_\mu - eF^{\mu\nu} S_{\mu\nu}, a \right] \right\rangle_c, \qquad (23)$$

where $m_c \equiv \left\langle (\pi^{\mu} \pi_{\mu})^{\frac{1}{2}} \right\rangle_c$ plays the role of a classical variable mass. Equation (23) resembles Fock's derivative. The important relation (23) allows us to retrieve the classical equations of motion in the limit $\hbar \to 0$, *e.g.* it allows us to

recover the BMT equation for g = 2 [using the result of (15) into Eq. (23)]. This formalism also solves the problems of relativistic wave equations –which were the cause of the reformulation given by quantum field theory– consistently incorporating into a one-charge theory the presence of particles and antiparticles.⁶

In Refs. [26, 27] we have integrated the Heisenberg equations of motion for π_{μ} and γ_{μ} , *i.e.*

$$\frac{d\gamma_{\mu}}{ds} = 4S_{\mu\nu}\pi^{\nu},\tag{24}$$

$$\frac{d\pi_{\mu}}{ds} = eF_{\mu}^{\ \nu}\gamma_{\nu},\tag{25}$$

in order to study the covariant *Zitterbewegung* in external electromagnetic fields (see also Ref. [29]). It has permitted us to describe particle creation in a constant external electric field. In such a case we have obtained a pictorial representation of the corresponding Feynman diagram as a *zig-zag* motion of the charge in space-time.

Summarizing, we have seen that it can be formulated a consistent (off-shell) framework from which to recover the classical covariant (on-shell) equations of motion in the semiclassical limit. In Section 5 we are going to study the semiclassical limit of the Feynman parametrization keeping the covariant *Zitterbewegung* without projecting on the positive mass shell.

5 Barut-Zanghi model for spin

The Feynman parametrization given in Eq. (20) can be derived from the action

$$I = \left[-i\overline{\psi}(x^{\mu}, s) \frac{\partial\psi(x^{\mu}, s)}{\partial s} - \overline{\psi}(x^{\mu}, s)\gamma^{\mu}\pi_{\mu}\psi(x^{\mu}, s) \right] d^{4}xds.$$
(26)

Performing a semiclassical limit as the one considered in Eq. (23), $\psi(x^{\mu}) = z\varphi_c(x^{\mu})$, where z is now an arbitrary constant spinor,⁷ we have [21]

$$I_{c} = \left[-i\overline{z}\frac{dz}{ds} + \epsilon \frac{d\langle x^{\mu}\rangle_{c}}{ds} \langle p_{\mu}\rangle_{c} - \overline{z}\gamma^{\mu}z \langle \pi_{\mu}\rangle_{c} \right] ds, \qquad (27)$$

where $\epsilon \equiv \overline{z}z = \pm 1$ takes into account particle and antiparticle states, according to the Stückelberg interpretation.⁸ From now on we will drop the averages $\langle \rangle_c$

⁶Feynman [30] has pointed out: "The various creation and annihilation operators in the conventional electron field view are required because the number of particles is not conserved, *i.e.*, pairs must be created or destroyed. On the other hand charge is conserved which suggests that if we follow the charge, not the particle, the results can be simplified."

⁷In this case the spinor z, in general, does not satisfy condition (22).

⁸A discussion of the importance of the factor ϵ in derivating the Dirac equation, starting from the Stückelberg interpretation, can be seen in Ref. [31].

interpreting all orbital variables as mean values in semiclassical states. The semiclassical action (27) is the action proposed by Proca [32] and Barut and Zanghi [15] as a classical theory for a spinning system that undergoes a real *Zitterbewegung.*⁹ Nevertheless in their formulation ϵ does not appear so that their action only describes particle motions.¹⁰

The Hamilton equations derived from the action (27) in the extended classical phase-space of coordinates $(z, \overline{z}, x^{\mu}, p_{\mu})$ are

$$\frac{dz}{ds} = i\gamma^{\mu}\pi_{\mu}z, \qquad \qquad \frac{d\overline{z}}{ds} = -i\overline{z}\gamma^{\mu}\pi_{\mu}, \qquad (28)$$

$$\frac{dx^{\mu}}{ds} = \epsilon \overline{z} \gamma^{\mu} z, \qquad \qquad \frac{d\pi_{\mu}}{ds} = \epsilon e F^{\mu\nu} \overline{z} \gamma^{\nu} z. \tag{29}$$

From these equations we have that $\mathcal{H} \equiv u^{\mu}\pi_{\mu}$ is a constant of motion (the analogue of Feynman's Hamiltonian $\gamma^{\mu}\pi_{\mu}$), where $u^{\mu} \equiv \epsilon \overline{z} \gamma^{\mu} z$ [cf. Eq. (18)]. Free particle solutions are

$$u^{\mu}(s) = p^{\mu} \frac{\mathcal{H}}{p^2} + \left[u^{\mu}(0) - p^{\mu} \frac{\mathcal{H}}{p^2} \right] \cos(2ps) + \frac{1}{2p} \frac{du^{\mu}(0)}{ds} \sin(2ps),$$
(30)

 $p_{\mu} = \text{const},$

where $p \equiv \sqrt{p^{\mu}p_{\mu}}$ can be identified with the positive mass of the particle (p enters here as a constant of the motion). Equations (30) show the classical analog of the phenomenon of *Zitterbewegung* of the charge [compare them with the operator solution obtained by Barut and Thacker [35] and in Refs. [26, 27]]. The four-velocity has the smooth term $p^{\mu}\frac{\mathcal{H}}{p^2}$ –as it is expected for the center of mass of particles– which is the term free of interferences between positive and negative masses, *i.e.* $p^{\mu}\frac{\mathcal{H}}{p^2} = \overline{z} (\Lambda_+ \gamma^{\mu} \Lambda_+ + \Lambda_- \gamma^{\mu} \Lambda_-) z$. The other contribution to the velocity is an oscillatory motion with the characteristic frequency $\omega = 2p$, which is a covariant classical analog of the *Zitterbewegung* [36, 37, 35].

As Barut and Zanghi [15] have pointed out their formulation has a more natural form in a five-dimensional space-time with metric signature (+, -, -, -, -). In Refs. [26, 27, 28] we have shown that the Feynman parametrization (20) naturally arises as a "massless" Dirac equation in this manifold.

Instead of the variables z and \overline{z} we can work in terms of the spin variables. Then we can obtain [15, 38] a new set of dynamical equations equivalent to (28) and (29):

 $^{^{9}\}mathrm{The}$ quantization of this theory was performed by Barut and Pavsic [33] and Barut and Unäl [34].

¹⁰Notice that for the temporal component of the first of Eqs. (29) we have $\frac{dx^0}{ds} = \epsilon z^{\dagger} z$. As $z^{\dagger} z$ is always positive, according to the Stückelberg [6] interpretation, it is impossible to have antiparticles unless $\epsilon = -1$.

$$\frac{dx^{\mu}}{ds} = u^{\mu},\tag{31}$$

$$\frac{du_{\mu}}{ds} = 4S_{\mu\nu}\pi^{\nu},\tag{32}$$

$$\frac{d\pi_{\mu}}{ds} = eF_{\mu\nu}u^{\nu},\tag{33}$$

$$\frac{dS_{\alpha\beta}}{ds} = \pi_{\alpha}u_{\beta} - \pi_{\beta}u_{\alpha}, \qquad (34)$$

for the new set of dynamical variables $(x^{\mu}, u^{\mu}, p_{\mu}, S_{\mu\nu})$, where now $S_{\mu\nu} \equiv \frac{1}{4}i\epsilon\overline{z}[\gamma_{\mu}, \gamma_{\nu}]z$. Equation (31) must be compared with (18) and (10), while Eqs (33) and (34) are respectively the Lorentz force law and the spin precession equation for the spin tensor in the case of minimal coupling [cf. Eq. (17)].¹¹ Equations (32) and (33) are the analogous of Eqs. (24) and (25) respectively.¹² In this semiclassical formulation conditions (2) and $u^{\alpha}u_{\alpha} = 1$ are only satisfied after taking the projection on the positive mass subspace. This eliminates the covariant *Zitterbewegung* and it is essentially equivalent to average in s.¹³ In this case, in the same way as it happened for Eq. (23), we reobtain the form of the classical equations of Frenkel and Thomas.

We have seen that the modified semiclassical theory of Barut and Zanghi is compatible with the Dirac equation and its proper time parametrization. This formulation corresponds to a one-charge theory that admits particle and antiparticle states, so it describes *Zitterbewegung* as well as particle creation processes. The classical theories such as those developed by Frenkel, Thomas, and Bargmann, Michel, and Telegdi, describe only a positive-mass system. In other words, to obtain in the classical limit the BMT equation it is necessary to project the generalized equations of motion on the positive mass subspace. This is the essential step in order to go from Beck's proper time derivative to Fock's one.

In summary we have established a deeper connection among classical and quantum theories of spinning systems, coming from a semiclassical treatment –which, however, inherits the old problems of the Dirac theory– to heuristic quantum proper time derivatives, which acquire meaning in the proper time parametrization of the Dirac equation. Using this parametrization we can retrieve, through a semiclassical limit, the classical covariant equations of motion

 $^{^{11}}$ The classical relativistic spinning particle with anomalous magnetic moment was considered by Barut and Cruz [38], who arrived to the BMT equation after averaging in s.

 $^{^{12}}$ The analogies between the equations of motion obtained from the Beck derivative and those obtained from the Barut and Zanghi classical formulation were also remarked by Barut and Unäl [34].

¹³See, for example, Eq. (30) in which, after averaging in s, only $p^{\mu} \frac{\mathcal{H}}{p^2}$ remains.

in proper time. It supports another evidence of the effectiveness of the Feynman parametrization to deal with spin systems. 14

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 $^{^{14}}$ The decisive argument supporting this parametrization is that Feynman [7, 9] himself derived QED from it (for an account of these ideas see Schweber's [39] review and Refs. [27, 28]).

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