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CLASSICAL SPINNING PARTICLES INTERACTING WITH EXTERNAL GRAVITATIONAL FIELDS

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In this paper we study the coupling between the pseudoclassical spinning particle and an arbitrary gravitational field. The gravitational field is treated as a gauge field in order to deal with possible contributions from the torsion of space-time. We find that the spinning particle cannot be coupled directly to the torsion. We study the classical equations of motion which turn out to be the same as derived by Papapetrou in order to describe the so called pole-dipole singularity in general relativity. We discuss also the structure of the energy-momentum tensor for the spinning particle.

1. Introduction

Recently there has been a certain amount of interest in a "classical" (pseudoclassical) description of a relativistic particle with spin [1–3]. Furthermore, the interactions with electromagnetic, Yang-Mills and weak gravitational fields have been studied [4,5].

It appears to be of some interest to study the interaction with an arbitrary gravitational field. Firstly, we can obtain information about the classical motion of spinning test objects. Secondly, we can try to understand eventual couplings between spin and torsion of the space-time [6,7]. In fact, it has been very often claimed in the literature that spin and torsion are strictly related.

In order to deal with these problems it appears to be very convenient to discuss the gravitational field as a gauge field [8]. In fact in this context it is easy to take into account the torsion. Furthermore, this treatment is convenient also from a technical point of view. For instance, one is led to introduce, in a natural way, the classical analogue of the γ-matrices as x_μ independent objects.

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The plan of the work is the following: in sect. 2 we build up a Lagrangian for the interaction of the pseudoclassical spinning particle with an external gravitational field. The general prescriptions to write down Lagrangians invariant under a general gauge group were discussed in ref. [5]. Correspondingly we introduce 16 "vierbein" fields to deal with translations and 24 "local connections" to deal with Lorentz transformations as explained by Kibble [6]. The Lagrangian we obtain in this way can obviously be translated in the ordinary language by introducing the metric tensor and the affine connection. However, starting from the "gauge-like" formulation, the affine connection does not need to be a symmetric one. The free Lagrangian of the spinning particle gives rise to first-class constraints [2]: one being the mass constraint, while the other one (Dirac constraint) gives rise to the Dirac equation after quantization. As explained in refs. [2,5], any interaction must preserve the first-class character of the constraints in order to have a smooth limit for the zero coupling constant. This requirement is not generally satisfied; for instance, in the electromagnetic case, this forces the particle to have a vanishing anomalous magnetic moment.

The consistency of the constraints is studied in sect. 3. Firstly we analyze the case of a symmetric affine connection and we find that the Lagrangian of sect. 2 satisfies the mentioned criterium. Secondly, we analyze the non-symmetric case. We find that it is necessary to modify our Lagrangian. The required modification consists in the substitution of the affine connection with its symmetric part, that is with the Christoffel symbol. But this means that *the pseudoclassical spinning particle cannot be coupled directly to the torsion.* This statement must be understood in the same sense as the other statement about the vanishing of the anomalous magnetic moment. Because if one considers second quantization effects, then as the electron can get an anomalous moment, it may well be possible that the spinning particle can have some interaction with the torsion. We also make some consideration about quantization, and in particular we find that the Dirac constraint gives rise to the usual form [9] of the Dirac equation for the interaction with a gravitational field.

In sect. 4 we study the equations of motion which follow from our Lagrangian. In particular we find that the spinning particle does not perform a geodesic motion. This is not a very surprising result, because this fact was known in general relativity already in 1951. At that time, Papapetrou [10] derived from general relativity the equations of motion for a classical test object, whose space-time singularity was a delta plus a delta-derivative function (pole-dipole singularity). This object can be pictorially seen as an infinitesimal dipole evolving in space-time. The interesting fact is that the *Papapetrou equations turn out to be coincident with our equations.* This cannot be an accident, and in order to clarify this point we study the energy-momentum tensor of the spinning particle. The singularities of this tensor are precisely the expected ones, that is a delta plus a delta-derivative. These facts open some possibilities to interpret these pseudoclassical theories, which up to this moment are essentially formal theories. In the weak field case we consider the operator corresponding to the energy-momentum tensor and we show that its matrix elements are those of the usual Dirac theory and coincide with the results found by Brink et al. [4]. About this point it is interesting to notice that the delta and the delta-derivative terms add together to form the Gordon decomposition of the Dirac current [4,5].

In the appendix we further develop the connection between the gauge-like and the metric approach to the gravitational field. In particular we give an explicit proof of a formula relating the affine connection with Christoffel symbols and the torsion tensor.

2. The interacting Lagrangian

We recall from ref. [2] that the Lagrangian describing a pseudoclassical spinning particle without any internal symmetry is

$$L_{\text{free}} = -\tfrac{1}{2}\, i\xi_\mu\dot{\xi}^\mu - \tfrac{1}{2}\, i\xi_5\dot{\xi}_5 - mc\sqrt{\left(\dot{x}_\mu - \frac{i}{mc}\,\xi_\mu\dot{\xi}_5\right)^2}\,, \tag{2.1}$$

where ξ_μ and ξ_5 are pseudovector and pseudoscalar Grassmann variables, respectively.

This Lagrangian in invariant under global Poincaré transformations

$$\begin{aligned}
\delta x^\mu &= x^\mu - \bar{x}^\mu = \epsilon^\mu + \epsilon^{\mu}_{.\nu}x^\nu \equiv \epsilon^\mu - \tfrac{1}{2}(\epsilon^{\alpha}_{.\beta}S^{\beta}_{.\alpha})^{\mu}_{.\nu}\, x^\nu\,,\\
\delta\xi^\mu &= \xi^\mu - \bar{\xi}^\mu = \epsilon^{\mu}_{.\nu}\xi^\nu = -\tfrac{1}{2}(\epsilon^{\alpha}_{.\beta}\, S^{\beta}_{.\alpha})^{\mu}_{.\nu}\xi^\nu\,,\\
\delta\xi_5 &= 0\,,
\end{aligned} \tag{2.2}$$

where

$$\epsilon^{\mu}_{.\nu} = -\eta_{\nu\rho}\epsilon^{\rho}_{.\sigma}\eta^{\sigma\mu}\,, \tag{2.3}$$

and $S^{\beta}_{.\alpha}$ are the generators of the Lorentz group in the four-vector representation, i.e.

$$(S^{\beta}_{.\alpha})^{\mu}_{.\nu} = \eta^{\beta\mu}\eta_{\alpha\nu} - \eta^{\beta}_{\nu}\eta^{\mu}_{\alpha}\,. \tag{2.4}$$

Here $\eta_{\alpha\beta}$ is the flat Minkowski metric

$$\eta_{\alpha\beta} = (1, -1, -1, -1)\,.$$

Now we want to extend these transformations to local ones [6,8]. In order to do that we observe that under local transformations, δx^μ and $\delta\xi^\mu$ are completely independent variations. Thus it will be convenient to use different indices according to the transformation properties. We will use Greek letters for quantities like x^μ, $\dot{x}^\mu$, which will be called "world type", whereas we will use capital latin letters for quantities like ξ^A ($A = 0, 1, 2, 3$), which will be called "local type". Thus under local transformations we have

$$\delta x^\mu = \epsilon^\mu(x) + \epsilon^\mu_{.\nu}(x)\, x^\nu \equiv \theta^\mu(x)\,, \tag{2.5}$$

and

$$\delta \xi^A = \epsilon^A_{.B}(x)\, \xi^B\,. \tag{2.6}$$

It follows for $\mathrm{d}x^\mu$

$$\delta \mathrm{d}x^\mu = \theta^\mu_{,\nu}(x)\, \mathrm{d}x^\nu\,. \tag{2.7}$$

Eqs. (2.6) and (2.7) define the transformation laws for a local four-vector and a world four-vector, respectively.

We want to emphasize again that $\epsilon^A_{.B}(x)$ (still satisfying the condition (2.3)) and $\theta^\mu_{,\nu}(x)$ are completely independent functions, due to the fact that we are making local translations in addition to local Lorentz transformations. In particular, the $\theta^\mu(x)$ can be considered as the parameters of a local translation, that is the parameters of a general coordinate transformation.

In order to make the argument of the square root in (2.1) invariant under (2.6) and (2.7) we need to introduce a "vierbein" field G^A_μ ★ such as to transform a world four-vector into a local one; i.e. we require

$$\delta(G^A_\mu\, \mathrm{d}x^\mu) = \epsilon^A_{.B}(x)\, G^B_\mu \mathrm{d}x^\mu\,. \tag{2.8}$$

It follows

$$\delta G^A_\mu = -\theta^\nu_{,\mu}(x)\, G^A_\nu + \epsilon^A_{.B}(x)\, G^B_\mu\,. \tag{2.9}$$

Now the combination

$$\left(G^A_\mu\, \dot{x}^\mu - \frac{i}{mc}\, \xi^A\, \dot{\xi}_5\right)^2$$

is clearly invariant under (2.6) and/or (2.7).

The next problem is that $\mathrm{d}\xi^A$ is not a local four-vector, but its transformation properties are

$$\delta \mathrm{d}\xi^A = \frac{\partial \epsilon^A_{.B}(x)}{\partial x^\mu}\, \mathrm{d}x^\mu \xi^B + \epsilon^A_{.B}(x)\, \mathrm{d}\xi^B\,. \tag{2.10}$$

In order to build up a local quantity we need to introduce a gauge field $(A_\mu)^A_{.B}$

★ We recall here that the G^A_μ can be introduced as any other kind of gauge fields. One has only to notice that the generators of the translation group are $\partial/\partial x^\mu$, see ref. [6], footnote 13.

such as to compensate for the inhomogeneous term in (2.10). As usual we form the quantity

$$\mathrm{d}\xi^A + (A_\mu)^A_{.B}\, \mathrm{d}x^\mu \xi^B ,$$

with the requirement that it be a local four-vector under (2.6) and (2.7),

$$\delta(\mathrm{d}\xi^A + (A_\mu)^A_{.B}\, \mathrm{d}x^\mu \xi^B)$$
$$= \epsilon^A_{.B}(x)(\mathrm{d}\xi^B + (A_\mu)^B_{.C}\, \mathrm{d}x^\mu \xi^C) .$$

It follows

$$\delta(A_\mu)^A_{.B} = -\theta^\nu_{,\mu}(A_\nu)^A_{.B} + \epsilon^A_{.A'}(A_\mu)^{A'}_{.B} - \epsilon^{B'}_{.B}(A_\mu)^A_{.B'} - \frac{\partial \epsilon^A_{.B}(x)}{\partial x^\mu} . \tag{2.11}$$

Furthermore, we have

$$(A_\mu)_{AB} = -(A_\mu)_{BA} .$$

Correspondingly we get the Lagrangian

$$L = -\tfrac{1}{2}\, i\eta_{AB}\xi^A(\dot\xi^B + (A_\mu)^B_{.C}\, \dot x^\mu \xi^C) - \tfrac{1}{2}\, i\xi_5\dot\xi_5 - mc\sqrt{\eta_{AB}\left(G^A_\mu \dot x^\mu - \frac{i}{mc}\,\xi^A\dot\xi_5\right)\left(G^B_\nu \dot x^\nu - \frac{i}{mc}\,\xi^B\dot\xi_5\right)} , \tag{2.12}$$

which is manifestly invariant under local Lorentz transformations. We can easily transform L into a form which contains world quantities only. To this end let us suppose that it is possible to invert the "vierbein" fields, i.e. to find fields such that

$$H^\lambda_A G^A_\mu = \delta^\lambda_\mu , \qquad H^\mu_A G^B_\mu = \delta^B_A . \tag{2.13}$$

In this situation we can define a world four-vector starting from ξ^A,

$$\zeta^\mu = H^\mu_A \xi^A , \qquad \zeta_5 \equiv \xi_5 . \tag{2.14}$$

Substituting in (2.12) we get

$$L = -\tfrac{1}{2}\, ig_{\mu\nu}\zeta^\mu(\dot\zeta^\nu + \Gamma^\nu_{\lambda\rho}\dot x^\rho \zeta^\lambda) - \tfrac{1}{2}\, i\zeta_5\dot\zeta_5 - mc\sqrt{g_{\mu\nu}\left(\dot x^\mu - \frac{i}{mc}\,\zeta^\mu\dot\zeta_5\right)\left(\dot x^\nu - \frac{i}{mc}\,\zeta^\nu\dot\zeta_5\right)} , \tag{2.15}$$

where we have defined the following quantities:

$$\Gamma^\nu_{\lambda\rho} = H^\nu_A(G^A_{\lambda,\rho} + (A\rho)^A_{.B}\, G^B_\lambda) , \tag{2.16}$$

$$g_{\mu\nu} = \eta_{AB} G^A_\mu G^B_\nu \ . \tag{2.17}$$

Furthermore, we have used the following relation:

$$H^\mu_A = \eta_{AB} g^{\mu\nu} G^B_\nu \ , \tag{2.18}$$

where $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$,

$$g^{\mu\nu} = \eta^{AB} H^\mu_A H^\nu_B \ . \tag{2.19}$$

The quantities $g_{\mu\nu}$ and $\Gamma^\nu_{\lambda\rho}$ can be identified as the matrix tensor and the affine connection, respectively (see the appendix).

Obviously we could have written directly the expression (2.15) for the interacting Lagrangian. However, as we said in the introduction, we prefer to stress the similarity of the non-Abelian gauge fields with the gravitational field. In this way we can easily compare the results we get here with the results obtained in ref. [5]; furthermore it is more natural, in this context, to consider the case of a space-time with torsion, that is, the case of a non-symmetric connection, as it will be done in the following section.

From the Lagrangian (2.12) ★ we get the following expressions for the conjugated momenta:

$$P_\mu = mc \frac{\eta_{AB} G^A_\mu \left(G^B_\nu \dot{x}^\nu - \dfrac{i}{mc} \xi^B \dot{\xi}_5\right)}{\sqrt{\eta_{AB}\left(G^A_\mu \dot{x}^\mu - \dfrac{i}{mc}\xi^A \dot{\xi}_5\right)\left(G^B_\nu \dot{x}^\nu - \dfrac{i}{mc}\xi^B\dot{\xi}_5\right)}}$$

$$+ \tfrac{1}{2} i \xi^A \eta_{AB} (A\mu)^B_{.C}\, \xi^C \ , \tag{2.20}$$

$$\Pi_A = \tfrac{1}{2} i \eta_{AB} \xi^B \ , \tag{2.21}$$

$$\Pi_5 = \tfrac{1}{2} i \xi_5 - \frac{i}{mc} H^\mu_A \xi^A (P_\mu - \tfrac{1}{2} i \xi^C \eta_{CD} (A_\mu)^D_{.E} \xi^E) \ , \tag{2.22}$$

which have the standard Poisson brackets [11]

$$\{x^\mu, P_\nu\} = -g^\mu_\nu \ ,$$

$$\{\xi^A, \Pi_B\} = -\delta^A_B \ , \qquad \{\xi_5, \Pi_5\} = -1 \ .$$

★ Here and in the following we will assume as Lagrangian variables x^μ and ξ^A. We could use ξ^μ as well, but due to the fact that their Dirac brackets (see eq. (2.30) in the following) is x^μ dependent, it is much more convenient to use the ξ^A variables. Then every time we will use the ξ^μ variables, they must be understood as functions of ξ^A and x^μ.

From (2.20) we get the following constraint

$$(P_\mu - \tfrac{1}{2}(A_\mu)^A_{.B}\, S^B_{.A})(P_\nu - \tfrac{1}{2}(A_\nu)^A_{.B}\, S^B_{.A})\, g^{\mu\nu} = m^2 c^2 \,, \tag{2.23}$$

where

$$S^A_{.B} = -\tfrac{1}{2}\, i\eta_{BC}[\xi^A, \xi^C] \,, \tag{2.24}$$

are the generators of the "spin part" of the Lorentz group. If we define the mechanical momentum

$$\mathcal{P}_\mu = P_\mu - \tfrac{1}{2}(A_\mu)^A_{.B}\, S^B_{.A} \,, \tag{2.25}$$

we get the mass-shell condition

$$\chi = \mathcal{P}_\mu \mathcal{P}_\nu g^{\mu\nu} - m^2 c^2 = 0 \,. \tag{2.26}$$

We recognize in (2.25) the typical combination for the covariant derivative.

From (2.21) and (2.22) we get two more constraints; moreover, we can require the further constraint [2,5]

$$\Pi_5 = -\tfrac{1}{2}\, i\xi_5 \,. \tag{2.27}$$

This constraint together with (2.21) forms a set of second-class constraints. By defining the corresponding Dirac brackets we get

$$\{\xi^A, \xi^B\}^* = i\eta^{AB} \,, \tag{2.28}$$

$$\{\xi_5, \xi_5\}^* = -i \,, \tag{2.29}$$

and for the variables ζ^μ

$$\{\zeta^\mu, \zeta^\nu\}^* = ig^{\mu\nu} \,. \tag{2.30}$$

The next problem is to determine the nature of the two remaining constraints (2.22) and (2.26). In fact, we know from our previous works [2,5] that an arbitrary interaction can change first-class constraints into second-class ones. In order to avoid this phenomenon (which would give a non-smooth zero interaction limit) we found that it is generally necessary to "renormalize" the mass appearing as a factor of the square root in eq. (2.12). The prescription is

$$m^2c^2 \Rightarrow m_{\mathrm{R}}^2 c^2 = m^2 c^2 - igF^a_{\mu\nu} I_a \xi^\mu \xi^\nu \,. \tag{2.31}$$

Here I_a is the generator of the gauge group and $F^a_{\mu\nu}$ is the covariant gauge tensor. In the present case we can establish the following correspondence

$$gI_a \Rightarrow \tfrac{1}{2}\, S^A_{.B} \,,$$

$$F^a_{\mu\nu} \Rightarrow -(R_{\mu\nu})^A_{.B} = (A_{\mu,\nu})^A_{.B} - (A_{\nu,\mu})^A_{.B} - (A_\mu)^A_{.C}(A_\nu)^C_{.B} + (A_\nu)^A_{.C}(A_\mu)^C_{.B} \,, \tag{2.32}$$

where $S^{.A}_{\cdot B}$ is defined in (2.24) and $(R_{\mu\nu})^{.A}_{\cdot B}$ is the Rieman curvature †.

It follows that the expected mass renormalization should be

$$m^2c^2 \Rightarrow m_R^2c^2 = m^2c^2 + \tfrac{1}{2}\, i(R_{\mu\nu})^{.A}_{\cdot B}\, S^{.B}_{\cdot A} H^{\mu}_{C} H^{\nu}_{D} \xi^C \xi^D \ ,$$

or in terms of the variables (2.14)

$$m_R^2c^2 = m^2c^2 + \tfrac{1}{2}\, g_{\tau\lambda} R^{\lambda}_{\rho\mu\nu} \zeta^{\rho}\zeta^{\tau}\zeta^{\mu}\zeta^{\nu} \ , \tag{2.33}$$

where we have introduced the Riemann tensor in world coordinates

$$R^{\lambda}_{\rho\mu\nu} = H^{\lambda}_{A} G^{B}_{\rho} (R_{\mu\nu})^{.A}_{\cdot B} = \Gamma^{\lambda}_{\rho\mu,\nu} - \Gamma^{\lambda}_{\rho\nu,\mu} + \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\sigma}_{\rho\mu} - \Gamma^{\lambda}_{\sigma\mu}\Gamma^{\sigma}_{\rho\nu} \ . \tag{2.34}$$

Furthermore we have

$$\zeta^{\rho}\zeta^{\tau}\zeta^{\mu}\zeta^{\nu} = -\epsilon^{\rho\tau\mu\nu}\zeta^0\zeta^1\zeta^2\zeta^3 \ .$$

It follows that for a symmetric affine connection there is no mass renormalization. In fact in this case $\Gamma^{\mu}_{\nu\lambda}$ coincides with the Christoffel symbol, consequently the Riemann tensor satisfies the cyclic identity [12]

$$R^{\lambda}_{\mu\nu\rho} + R^{\lambda}_{\nu\rho\mu} + R^{\lambda}_{\rho\mu\nu} = 0 \ .$$

It appears quite obvious from these considerations that, at least for symmetric connections, the constraints (2.22) and (2.26) are first class. This will be shown in the next section where we will examine the general case too. The result will be that the spinning particle is never coupled to the torsion tensor. To conclude this section we observe that after (2.27) the constraint (2.22) becomes

$$\chi_D = \mathcal{P}_{\mu}\zeta^{\mu} - mc\zeta_5 = 0 \ . \tag{2.36}$$

3. Constraints

Let us now study the conditions the interaction must verify in order to not change the character of the two constraints

$$\chi_D = \mathcal{P}_{\mu}\zeta^{\mu} - mc\zeta_5 \ , \tag{3.1}$$

$$\chi = \mathcal{P}_{\mu}\mathcal{P}_{\nu} g^{\mu\nu} - m^2c^2 \ . \tag{3.2}$$

† The $1/2g$ factor can be understood by comparison of (2.2) with the analogous transformation properties defined in ref. [5]. The minus sign is due to the different definition of the curl with respect to ref. [5].

By using the following brackets

$$\{\mathcal{P}_\mu, \mathcal{P}_\nu\}^* = \tfrac{1}{2}\,(S)^A_{.B}(R_{\mu\nu})^B_{.A}\ , \tag{3.3}$$

$$\{\mathcal{P}_\mu, \zeta^\nu\}^* = -\Gamma^\nu_{\rho\mu}\zeta^\rho\ , \tag{3.4}$$

we get

$$\{\chi_D, \chi_D\}^* = i(\mathcal{P}_\mu\mathcal{P}_\nu g^{\mu\nu} - m^2c^2 - \tfrac{1}{2}\,iS^A_{.B}(R_{\mu\nu})^B_{.A}\zeta^\mu\zeta^\nu) + 2\mathcal{P}_\mu\Gamma^\mu_{\rho\nu}\zeta^\rho\zeta^\nu\ , \tag{3.5}$$

and

$$\{\chi_D, \chi\}^* = S^A_{.B}(R_{\mu\rho})^B_{.A}\zeta^\mu g^{\rho\lambda}\mathcal{P}_\lambda + \mathcal{P}_\rho\mathcal{P}_\lambda\zeta^\mu(C^\rho_{\mu\nu}g^{\nu\lambda} + C^\lambda_{\mu\nu}g^{\rho\nu})\ , \tag{3.6}$$

where we have introduced the torsion tensor

$$C^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}\ . \tag{3.7}$$

From the analysis in the appendix we know that for $\Gamma^\rho_{\mu\nu}$ symmetric the following identity holds

$$\Gamma^\rho_{\mu\nu} = \left\{\begin{matrix}\rho\\ \mu\nu\end{matrix}\right\} = \tfrac{1}{2}\,g^{\rho\tau}(g_{\tau\mu,\nu} + g_{\tau\nu,\mu} - g_{\mu\nu,\tau})\ . \tag{3.8}$$

In this case the torsion vanishes; furthermore, by using the cyclic identity (2.35), we see that the two terms in (3.5) and (3.6) proportional to $(R_{\mu\nu})^A_{.B}$ are zero. Finally the last term on the right-hand side of (3.5) is zero due to the antisymmetry $\zeta^\rho\zeta^\nu$. Hence, in the symmetric case, χ and χ_D are first-class constraints and we do not need to modify the Lagrangian (2.12).

In the non-symmetric case, that is for a space-time with torsion, it is clear that we have to modify in some way our Lagrangian. The dangerous term is the last one on the right-hand side of eq. (3.5). If one looks at the derivation of this equation, one realizes that this term is present because in the Dirac bracket (3.4) the coefficient of ζ^ρ on the right-hand side is not a symmetric one. This suggests to look for a modification of $\mathcal{P}_\mu$ such as to obtain a symmetric coefficient in (3.4). By putting

$$\mathcal{P}'_\mu = \mathcal{P}_\mu + iL_{\mu\sigma\lambda}\zeta^\sigma\zeta^\lambda\ , \tag{3.9}$$

one gets

$$\{\mathcal{P}'_\mu, \zeta^\rho\}^* = -\ [\Gamma^\rho_{\sigma\mu} + (L_{\mu\sigma\lambda} - L_{\mu\lambda\sigma})\,g^{\lambda\rho}]\,\zeta^\sigma\ . \tag{3.10}$$

The symmetry condition determines $L_{\mu\sigma\lambda}$ uniquely,

$$L_{\mu\sigma\lambda} = \tfrac{1}{4}\ [C^\rho_{\sigma\mu}g_{\rho\lambda} + C^\rho_{\mu\lambda}g_{\rho\sigma} - C^\rho_{\lambda\sigma}g_{\rho\mu}]\ . \tag{3.11}$$

Now we can use eq. (A.15) of the appendix to get

$$L_{\mu\sigma\lambda} = -\tfrac{1}{2}\,g_{\lambda\alpha}\left(\Gamma^\alpha_{\sigma\mu} - \left\{\begin{matrix}\alpha\\ \sigma\mu\end{matrix}\right\}\right)\ . \tag{3.12}$$

By substituting into (3.10) and recalling that $L_{\mu\sigma\lambda} = -L_{\mu\lambda\sigma}$ we obtain

$$\{\mathcal{P}'_\mu, \zeta^\rho\}^* = -\left\{ \begin{matrix} \rho \\ \sigma\mu \end{matrix} \right\} \zeta^\sigma . \tag{3.13}$$

To obtain a mechanical momentum like (3.9) we need a new conjugated momentum

$$P'_\mu = P_\mu - iL_{\mu\sigma\lambda}\zeta^\sigma\zeta^\lambda ,$$

which can be derived by adding to our Lagrangian (2.15) the following term

$$-\tfrac{1}{2}\, ig_{\lambda\alpha}\left(\Gamma^\alpha_{\sigma\mu} - \left\{ \begin{matrix} \alpha \\ \sigma\mu \end{matrix} \right\}\right) \dot{x}^\mu \zeta^\sigma \zeta^\lambda .$$

By doing so we get the new Lagrangian

$$L = -\tfrac{1}{2}\, ig_{\mu\nu}\zeta^\mu\left(\dot{\zeta}^\nu + \left\{ \begin{matrix} \nu \\ \lambda\rho \end{matrix} \right\} \dot{x}^\rho \zeta^\lambda\right) - \tfrac{1}{2}\, i\zeta_5\dot{\zeta}_5$$
$$- mc\sqrt{g_{\mu\nu}\left(\dot{x}^\mu - \frac{i}{mc}\,\zeta^\mu\dot{\zeta}_5\right)\left(\dot{x}^\nu - \frac{i}{mc}\,\zeta^\nu\dot{\zeta}_5\right)} . \tag{3.14}$$

With this Lagrangian the same considerations as in the symmetric case hold, and in particular the constraints χ and χ_{D} are first class ones without any mass renormalization. Then we get the result that the spinning particle can be consistently coupled only to the symmetric part of the affine connection, i.e. to the Christoffel symbol. In other words *the spinning particle does not couple directly to the torsion, and correspondingly it is not a source for the torsion field* [6]. Then, in the following, we can forget eventual torsion properties of the space-time, because these are not relevant from our point of view.

Now let us spend some word about the quantization. From (2.28) and (2.29) we get the anticommutation rules for the quantum operators [2]

$$[\xi^A, \xi^B]_+ = -\hbar\eta^{AB} , \qquad [\xi_5, \xi_5]_+ = \hbar . \tag{3.15}$$

These relations can be satisfied by putting

$$\xi^A = \sqrt{\tfrac{1}{2}\,\hbar}\,\gamma_5\gamma^A , \qquad \xi_5 = \sqrt{\tfrac{1}{2}\,\hbar}\,\gamma_5 , \tag{3.16}$$

where γ_5 and γ^A are the usual Dirac matrices. We observe also that in this context the use of the variables ζ^μ is quite natural because they correspond to the use of the spin matrices conform to metric (see for instance ref. [9] and references therein). In fact we have

$$\zeta^\mu = G^\mu_A\,\xi^A = \sqrt{\tfrac{1}{2}\,\hbar}\,\gamma_5\gamma^\mu , \tag{3.17}$$

where

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu} . \tag{3.18}$$

In the x^μ representation, the momentum operator P_μ is

$$P_\mu = i\hbar\partial_\mu . \tag{3.19}$$

It follows for $\mathcal{P}_\mu$,

$$\mathcal{P}_\mu = i\hbar(\partial_\mu + \tfrac{1}{4}\, i(A_\mu)^{AB}\sigma_{AB}) = i\hbar\nabla_\mu , \tag{3.20}$$

where ∇_μ is just the covariant derivative upon a Dirac spinor [12], which by using (2.16) becomes

$$\nabla_\mu = \partial_\mu + \tfrac{1}{4}\, i(H^\lambda_A G^A_{\rho,\mu} - \Gamma^\lambda_{\rho\mu})\, g_{\tau\lambda}\sigma^{\tau\rho} . \tag{3.21}$$

In (3.20) and (3.21) we have

$$\sigma_{AB} = \tfrac{1}{2}\, i[\gamma_A, \gamma_B] ,$$

$$\sigma^{\mu\nu} = \tfrac{1}{2}\, i[\gamma^\mu, \gamma^\nu] , \tag{3.22}$$

respectively.

Now the Dirac constraint becomes the Dirac equation in a gravitational field [9],

$$(i\hbar\gamma^\mu\nabla_\mu - mc)\,\psi(x) = 0 . \tag{3.23}$$

4. Equations of motion

Starting from the Lagrangian (2.12) it is not difficult to obtain the equations of motion for the various quantities of interest. In particular we get

$$\dot{\mathcal{P}}_\mu - \left\{ \begin{matrix} \rho \\ \mu\nu \end{matrix} \right\} \dot{x}^\nu\, \mathcal{P}_\rho = -\tfrac{1}{2}\,(R_{\mu\nu})^A_{.B}\, \dot{x}^\nu\, S^B_{.A} , \tag{4.1}$$

$$\dot{\xi}_5 = \frac{1}{mc}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,(H^\mu_A\, \mathcal{P}_\mu \xi^A) , \tag{4.2}$$

$$\dot{\xi}^A + (A_\mu)^A_{.B}\, \dot{x}^\mu \xi^B = \frac{1}{mc}\,\eta^{AB} H^\mu_A\, \mathcal{P}_\mu \dot{\xi}_5 . \tag{4.3}$$

These equations can be also obtained very simply, by using the correspondence (2.32), from the analogous equations founded in ref. [5]. However, we must also take into account that $\mathcal{P}_\mu$ and ξ^A are not scalars under the gauge group; furthermore, in this case we do not have the mass renormalzation phenomenon.

We can get another interesting equation by considering the spin generators in world coordinates

$$S^{\mu}_{.\nu} = H^{\mu}_{A} G^{B}_{\nu} S^{A}_{.B} \,. \tag{4.4}$$

We find

$$\dot{S}^{\mu}_{.\nu} + \Gamma^{\mu}_{\lambda\rho} S^{\lambda}_{.\nu} \dot{x}^{\rho} - \Gamma^{\lambda}_{\nu\rho} S^{\mu}_{.\lambda} \dot{x}^{\rho} + \dot{x}^{\mu} \mathcal{P}_{\nu} - g^{\mu\lambda} g_{\nu\rho} \dot{x}^{\rho} \mathcal{P}_{\lambda} = 0 \,. \tag{4.5}$$

The physical meaning of this equation can be easily seen by going in the flat space limit, in which case we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau} (S^{\mu}_{.\nu} + x^{\mu} \mathcal{P}_{\nu} - x^{\nu} \mathcal{P}_{\mu}) = 0 \,, \tag{4.6}$$

that is the conservation of the total angular momentum. Then (4.5) is simply the balance of the angular momenta of particle and field.

These equations can be simplified by choosing a particular gauge [5], that is the gauge specified by the following Hamiltonian

$$H = -\frac{1}{2mc} (\mathcal{P}_{\mu} \mathcal{P}_{\nu} g^{\mu\nu} - m^2 c^2) \,. \tag{4.7}$$

The relevant Hamilton equations are

$$\dot{x}^{\mu} = \frac{1}{mc} g^{\mu\nu} \mathcal{P}_{\nu} \,, \tag{4.8}$$

$$\dot{\xi}_5 = 0 \,. \tag{4.9}$$

It follows from (4.8) that the constraint $\chi = 0$ is equivalent to the choice of the proper time gauge

$$\dot{x}^{\mu} \dot{x}^{\nu} g_{\mu\nu} = 1 \,. \tag{4.10}$$

Then in this gauge we have

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\rho\lambda} \dot{x}^{\rho} \dot{x}^{\lambda} = -\frac{1}{2mc} g^{\mu\rho} (R_{\rho\lambda})^{A}_{.B} \, \dot{x}^{\lambda} S^{B}_{.A} \,, \tag{4.11}$$

$$\dot{\xi}^{A} + (A_{\mu})^{A}_{.B} \, \dot{x}^{\mu} \xi^{B} = 0 \,, \tag{4.12}$$

and furthermore the spin generators are covariantly constant; in fact due to (4.8) the contribution of the orbital angular momentum is zero, and we have

$$\dot{S}^{\mu}_{.\nu} + \Gamma^{\mu}_{\lambda\rho} S^{\lambda}_{.\nu} \dot{x}^{\rho} - \Gamma^{\lambda}_{\nu\rho} S^{\mu}_{.\lambda} \dot{x}^{\rho} = 0 \,. \tag{4.13}$$

Some remark is in order with regard to eq. (4.11). The content of this equation is that *the spinning particle does not follow a geodesic in the space-time.* In fact, if we are in a curved space-time, it is impossible to perform a change of coordinates in such a way to gauge away the second member of eq. (4.11). *The spin is coupled to the curvature of the space-time.*

In order to understand this point we recall from ref. [5] that, studying the gauge current, we realized that the spinning particle is not a simple delta-like singularity in the space-time. The gauge current shows also a delta-derivative like singularity. In other words the spinning particle behaves like an infinitesimal pole-dipole (in the sense of the distribution theory) singularity. The behaviour of such a singularity in a gravitational field was studied by Papapetrou [10] long time ago. Generally speaking, if one is given a certain test singularity in the space-time, it is possible to derive the equations of motion of such a singularity starting from the Einstein field equations [13]. There are various ways to do that, the simplest possibility is to use the fact that in such a theory the energy-momentum tensor must have vanishing covariant divergence. Such a method has been used, for instance, by Fock [14] in the case of a pole singularity, that is of the scalar particle. In the case of a string-like singularity, Gürses and Gürsey [15] have used this procedure to derive the equations of motion for the Nambu string.

Finally, for a pole-dipole singularity, the equations of motion have been derived by Papapetrou, and they turn out to be the equations (4.1) and (4.5) ★. It follows that we do not have to worry if the spinning particle does not perform a geodesic motion, this is just what is required by general relativity.

It seems to us that the fact that the spinning particle obeys the Papapetrou equations cannot be considered as a mere coincidence, but that probably it is the key to better understand the physical meaning of the theories with Grassmann variables [17]

Now let us consider the energy-momentum tensor, which can be obtained by varying the action associated to the Lagrangian (2.15) [12]

$$T^{\mu\nu}(z) = -\frac{2c}{\sqrt{g}}\,\frac{\delta S}{\delta g_{\mu\nu}(z)}\,. \tag{4.14}$$

By working in the proper time gauge we get

$$T^{\mu\nu}(z) = \frac{mc^2}{\sqrt{g}}\int \mathrm{d}\tau \dot{x}^\mu \dot{x}^\nu \delta^4(x(\tau)-z)$$

$$+\frac{ic}{2\sqrt{g}}\int \mathrm{d}\tau\left[\zeta^\mu\dot{\zeta}^\nu + \zeta^\nu\dot{\zeta}^\mu + (\dot{x}^\mu\zeta^\nu + \dot{x}^\nu\zeta^\mu)\,\zeta^\rho\,\frac{\partial}{\partial x^\rho}\right]\delta^4(x(\tau)-z)\,. \tag{4.15}$$

Now we can use the equation of motion for ζ^μ,

$$\dot{\zeta}^\mu + \Gamma^\mu_{\rho\nu}\dot{x}^\nu\zeta^\rho = 0\,, \tag{4.16}$$

★ For some recent application of the Papapetrou equations and more references, see ref. [16].

to express $\dot{\zeta}^\mu$ in (4.15); we obtain

$$T^{\mu\nu}(z) = \frac{mc^2}{\sqrt{g}} \int \mathrm{d}\tau \dot{x}^\mu \dot{x}^\nu \delta^4(x(\tau) - z)$$

$$- \frac{ic}{2\sqrt{g}} \int \mathrm{d}\tau [\zeta^\mu \zeta^\rho \dot{x}^\sigma (\nabla^*_\rho)^\nu_\sigma + \zeta^\nu \zeta^\rho \dot{x}^\sigma (\nabla^*_\rho)^\mu_\sigma] \delta^4(x(\tau) - z) . \tag{4.17}$$

Here we have defined the differential operator

$$(\nabla^*_\rho)^\nu_\sigma = -g^\nu_\sigma \partial_\rho + \Gamma^\nu_{\rho\sigma} , \tag{4.18}$$

which has the following remarkable property

$$[\nabla^*_\rho, \nabla^*_\mu]^\nu_\sigma = (\nabla^*_\rho)^\alpha_\sigma (\nabla^*_\mu)^\nu_\alpha - (\nabla^*_\mu)^\alpha_\sigma (\nabla^*_\rho)^\nu_\alpha = R^\nu_{\sigma\rho\mu} . \tag{4.19}$$

The differential operator ∇^*_ρ is the adjoint of the operator necessary to define the covariant divergence of $T^{\mu\nu}$. In fact we recall the wellknown formula [12]

$$\sqrt{g}\, T^{\mu\nu}{}_{;\nu} = \partial_\nu(\sqrt{g}\, T^{\mu\nu}) + \Gamma^\mu_{\nu\alpha} \sqrt{g}\, T^{\nu\alpha} ,$$

which can be rewritten as

$$\sqrt{g}\, T^{\mu\nu}{}_{;\nu} = (g^\mu_\alpha \partial_\nu + \Gamma^\mu_{\nu\alpha})(\sqrt{g}\, T^{\nu\alpha}) \equiv (\nabla_\nu)^\mu_\alpha (\sqrt{g}\, T^{\nu\alpha}) . \tag{4.20}$$

By using (4.20) we can easily verify that $T^{\mu\nu}$ has zero covariant divergence, in fact we find

$$\sqrt{g}\, T^{\mu\nu}{}_{;\nu} = mc^2 \int \mathrm{d}\tau \left[\ddot{x}^\mu + \Gamma^\mu_{\alpha\nu} \dot{x}^\alpha \dot{x}^\nu - \frac{i}{2mc} R^\mu_{\sigma\rho\nu} \dot{x}^\sigma \zeta^\nu \zeta^\rho\right] \delta^4(x(\tau) - z)$$

$$+ \tfrac{1}{2} ic \int \mathrm{d}\tau \left[\frac{\mathrm{d}}{\mathrm{d}\tau} (\zeta^\mu \zeta^\rho) + \Gamma^\mu_{\alpha\sigma} \dot{x}^\sigma \zeta^\alpha \zeta^\rho + \Gamma^\rho_{\alpha\sigma} \dot{x}^\sigma \zeta^\mu \zeta^\alpha\right] \partial_\rho \delta^4(x(\tau) - z) , \tag{4.21}$$

and the coefficients of the delta function and its derivative are the equations of motion (4.11) and (4.13), respectively.

Now we see easily the reason why Papapetrou gets two equations of motion. In fact in order to get a vanishing divergence for $T^{\mu\nu}$ we have to put to zero independently the pole and the dipole terms.

Let us comment about the form of $T^{\mu\nu}$; we see that the first term is the usual contribution for the scalar particle. The second term is the spin term, and it is very reminiscent of an analogous contribution we found in ref. [5] for the local isospin current. Furthermore, this term plays an important role in the quantized theory. In fact let us consider the operator $T^{\mu\nu}$ at the zero order in the gravitational field ★

★ Here we use natural units $\hbar = c = 1$.

$$T^{\mu\nu}(z) = \frac{1}{4m}\int \mathrm{d}\tau \{[p^\mu p^\nu, \delta^4(x(\tau) - z)]_+$$

$$+ p^\mu \delta^4(x(\tau) - z)\, p^\nu + p^\nu \delta^4(x(\tau) - z)\, p^\mu\}$$

$$- \frac{1}{8m}\int \mathrm{d}\tau [p^\mu \sigma^{\nu\rho} + p^\nu \sigma^{\mu\rho}, \partial_\rho \delta^4(x(\tau) - z)]_+ \,.$$

By taking the Fourier transform and the matrix element between states of given momentum, we find

$$\bar{u}(p')\, T^{\mu\nu}(p' - p)\, u(p) = \tfrac{1}{4}\, \bar{u}(p')[\gamma^\mu (p' + p)^\nu + \gamma^\nu (p' + p)^\mu]\, u(p) \,. \tag{4.22}$$

This is just the usual result, which is obtained by matching the first and the second term in (4.17) *via* the Gordon decomposition of the Dirac current ★.

To conclude this section let us consider the motion of the spin four-vector

$$\Sigma^A = \tfrac{1}{2}\, \epsilon^{ABCD} H^\mu_B\, \mathcal{P}_\mu S_{CD} \,, \tag{4.23}$$

which in world coordinates is

$$\Sigma^\mu = H^\mu_A\, \Sigma^A = \frac{1}{\sqrt{g}}\, \epsilon^{\mu\nu\rho\sigma} \mathcal{P}_\nu S_{\rho\sigma} \,, \tag{4.24}$$

where

$$g = -\det \|g_{\mu\nu}\| = \det{}^{-2} \|H^\mu_A\| \,. \tag{4.25}$$

By using the previous equations of motion we find

$$\dot{\Sigma}^\mu + \Gamma^\mu_{\lambda\nu} \dot{x}^\nu\, \Sigma^\lambda = 2 g^{\mu\rho} R_{\rho\nu} \dot{x}^\nu\, \Gamma_5 \,, \tag{4.26}$$

where

$$R_{\rho\nu} = R^\mu_{\rho\mu\nu} \,, \tag{4.27}$$

$$\Gamma_5 = \xi_0 \xi_1 \xi_2 \xi_3 \,. \tag{4.28}$$

This equation is the analogue of the Bargmann-Michel-Telegdi, which was founded in ref. [5], for the spinning particle in a gravitational field.

Appendix

The relation between the gauge and the metric theory of gravitation has been considered in great detail in refs. [6,8]. For completeness we will report here some of the relevant results.

★ The same thing happens for the interaction with an arbitrary gauge field, see refs. [4,5].

Let us start by recalling eqs. (2.16) and (2.17),

$$\Gamma^{\mu}_{\nu\rho} = H^{\mu}_{A}(G^{A}_{\nu,\rho} + (A_{\rho})^{A}_{.B}G^{B}_{\nu}) , \tag{A.1}$$

$$g_{\mu\nu} = \eta_{AB}G^{A}_{\mu}G^{B}_{\nu} . \tag{A.2}$$

We want to show that $\Gamma^{\mu}_{\nu\rho}$ and $g_{\mu\nu}$ can be identified with the affine connection and the metric tensor, respectively. From (2.9) and (2.11) it follows

$$\delta\Gamma^{\mu}_{\nu\rho} = \theta^{\mu}_{,\lambda}\Gamma^{\lambda}_{\nu\rho} - \theta^{\lambda}_{,\nu}\Gamma^{\mu}_{\lambda\rho} - \theta^{\lambda}_{,\rho}\Gamma^{\mu}_{\nu\lambda} - \theta^{\mu}_{,\nu\rho} , \tag{A.3}$$

$$\delta g_{\mu\nu} = -\theta^{\lambda}_{,\mu}g_{\lambda\nu} - \theta^{\lambda}_{,\nu}g_{\mu\lambda} . \tag{A.4}$$

We see that these are the right transformation properties for an affine connection [12] and for a symmetric tensor of rank two.

Now let us consider a general quantity which is simultaneosuly a local and a world tensor; i.e.

$$\begin{aligned}\delta V^{(\mu_1\ldots\mu_l)(A_1\ldots A_n)}_{(\nu_1\ldots\nu_m)(B_1\ldots B_p)} &= \sum_{i=1}^{l} \theta^{\mu_i}_{,\mu'_i} V^{(\mu_1\ldots\mu'_i\ldots\mu_l)(A_1\ldots A_n)}_{(\nu_1\ldots\ldots\nu_m)(B_1\ldots B_p)} \\ &- \sum_{i=1}^{m} \theta^{\nu'_i}_{,\nu_i} V^{(\mu_1\ldots\mu_l)(A_1\ldots A_n)}_{(\nu_1\ldots\nu'_i\ldots\nu_m)(B_1\ldots B_p)} \\ &+ \sum_{i=1}^{n} \epsilon^{A_i}_{.A'_i} V^{(\mu_1\ldots\mu_l)(A_1\ldots A'_i\ldots A_n)}_{(\nu_1\ldots\nu_m)(B_1\ldots B_p)} \\ &- \sum_{i=1}^{p} \epsilon^{B'_i}_{.B_i} V^{(\mu_1\ldots\mu_l)(A_1\ldots A_n)}_{(\nu_1\ldots\nu_m)(B_1\ldots B'_i\ldots B_p)} .\end{aligned} \tag{A.5}$$

Then we can define a derivative which is covariant with respect to the two different types of transformations

$$\begin{aligned}V^{(\mu_1\ldots\mu_l)(A_1\ldots A_n)}_{(\nu_1\ldots\nu_m)(B_1\ldots B_p);\,\mu} &= V^{(\mu_1\ldots\mu_l)(A_1\ldots A_n)}_{(\nu_1\ldots\nu_m)(B_1\ldots B_p),\,\mu} \\ &+ \sum_{i=1}^{l} \Gamma^{\mu_i}_{\mu'_i\mu} V^{(\mu_1\ldots\mu'_i\ldots\mu_l)(A_1\ldots A_n)}_{(\nu_1\ldots\nu_m)(B_1\ldots B_p)} \\ &- \sum_{i=1}^{m} \Gamma^{\nu'_i}_{\nu_i\mu} V^{(\mu_1\ldots\mu_l)(A_1\ldots A_n)}_{(\nu_1\ldots\nu'_i\ldots\nu_m)(B_1\ldots B_p)} \\ &+ \sum_{i=1}^{n} (A_\mu)^{A_i}_{.A'_i} V^{(\mu_1\ldots\mu_l)(A_1\ldots A'_i\ldots A_n)}_{(\nu_1\ldots\nu_m)(B_1\ldots B_p)}\end{aligned}$$

$$- \sum_{i=1}^{p} (A_\mu)^{B_i'}_{\cdot B_i} V^{(\mu_1 \ldots \mu_l)(A_1 \ldots A_n)}_{(\nu_1 \ldots \nu_m)(B_1 \ldots B_i' \ldots B_p)} \ . \tag{A.6}$$

It follows from this definition and (A.1) that the covariant derivative of the "vierbein" fields and of $g_{\mu\nu}$ is zero; correspondingly $g_{\mu\nu}$ can be interpreted as a metric tensor.

Up to this moment we have not required that the affine connection is a symmetric one, thus in general it is not expressible as a function of the metric tensor only; however, we can show that it is uniquely determined by the metric and the torsion tensors. In fact from the definition (3.7) of the torsion tensor and eq. (A.1) we get

$$(A_\mu)^A_{\cdot B} G^B_\nu - (A_\nu)^A_{\cdot B} G^B_\mu = G^A_{\mu,\nu} - G^A_{\nu,\mu} + G^A_\rho C^\rho_{\mu\nu} \ . \tag{A.7}$$

By introducing the quantity

$$A^A_{BC} = H^\mu_B (A_\mu)^A_{\cdot C} \ , \tag{A.8}$$

that is the gauge field in local coordinates, we can solve eq. (A.7) for the antisymmetric part of A^A_{BC}

$$A^A_{CB} - A^A_{BC} = t^A_{CB} + C^A_{CB} \ , \tag{A.9}$$

where

$$t^A_{CB} = H^\mu_C H^\nu_B (G^A_{\mu,\nu} - G^A_{\nu,\mu}) \ , \tag{A.10}$$

and

$$C^A_{CB} = G^A_\rho H^\mu_C H^\nu_B C^\rho_{\mu\nu} \ . \tag{A.11}$$

It is convenient to "lower" the upper index in (A.9) by defining

$$A_{ABC} = \eta_{AA'} A^{A'}_{BC} \ , \tag{A.12}$$

and analogously for the other quantities. Now we are able to solve eq. (A.9) for A_{ABC},

$$A_{CAB} = \tfrac{1}{2}(t_{ACB} - t_{BAC} - t_{CBA}) + \tfrac{1}{2}(C_{ACB} - C_{BAC} - C_{CBA}) \ . \tag{A.13}$$

Coming back to $(A_\mu)^A_{\cdot B}$ we get

$$\begin{aligned}(A_\mu)^A_{\cdot B} &= \tfrac{1}{2}\eta^{AA'} G^C_\mu (t_{CA'B} - t_{BCA'} - t_{A'BC}) \\ &\quad + \tfrac{1}{2}\eta^{AA'} G^C_\mu (C_{CA'B} - C_{BCA'} - C_{A'BC}) \ .\end{aligned} \tag{A.14}$$

We see that the local connection $(A_\mu)^A_{\cdot B}$ is completely fixed by the "vierbein" fields and the torsion.

Now we can insert the expression (A.14) into (A.1); by doing so, one can see that the first term in (A.14) contributes to building up the Christoffel symbol (3.8), and the final result is

$$\Gamma^{\alpha}_{\beta\mu} = \left\{ \begin{matrix} \alpha \\ \beta\mu \end{matrix} \right\} + \tfrac{1}{2} g^{\alpha\nu}(g_{\mu\rho}C^{\rho}_{\nu\beta} - g_{\nu\rho}C^{\rho}_{\beta\mu} - g_{\beta\rho}C^{\rho}_{\mu\nu}) \,. \tag{A.15}$$

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