If the wavelength is large compared to r, then $\psi = 0$ and we get an attractive force,

$$F_{1-2} = -\frac{\rho\beta^2}{8\pi}(\omega P_a)^2\left(1-\frac{\beta_1}{\beta}\right)\left(1-\frac{\beta_2}{\beta}\right)\frac{V_1V_2}{r^2}. \quad \text{(A11)}$$

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The apparent shape of a rapidly moving sphere

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Several articles have considered the apparent shape of a relativistic sphere. One interesting finding showed that, under certain conditions, the surface of the sphere may appear concave. This article concentrates on the case where the observer is at the origin and the sphere moves in the positive x direction with its center on the x axis. The sphere actually hits the observer, and one of the findings presented here is that for any speed v, $0 < v \leqslant c$, part of the surface appears concave for some part of the motion along the x axis.

I. INTRODUCTION

There is now an extensive literature on the apparent shape of objects moving at relativistic speeds. By apparent shape we mean the perceived shape that a stationary observer sees when photons, emitted from different parts of an object at different times, arrive simultaneously at his position. Mathews and Lakshmanan[1] corrected earlier misconceptions in the literature, by Terrell,[2] for example, that objects which subtend small solid angles at the observer appear simply rotated, and retain the same apparent shape to all observers. They showed that the apparent shape is related to the rest shape by a combination of nonuniform shear and extension or contraction, all parallel to the direction of motion, and that this does not reduce to a rotation, even in the case where the angle subtended is small.

Because of its simple rest shape and relevance to astronomy, several articles[1–6] have considered the apparent shape of a relativistic sphere. In particular, Scott and van Driel[5] presented three-dimensional views of a sphere that show the apparent distortion that occurs at various speeds and distances from the observer. One of their most interesting findings was that, under certain conditions, the surface of the sphere may appear concave. In spite of all the earlier work performed on a moving sphere, the sphere is still an interesting and worthwhile object to study, and this article concentrates on the case where the observer is at the origin and the sphere moves in the positive x direction with its center on the x axis. This is a special case of the situation considered by Scott and van Driel[5] and a complete analytic treatment of the apparent shape can be given for all positions and speeds. Since the apparent shape is always rotationally symmetric about the x axis in this situation, the three-dimensional shape can be unambiguously inferred by plotting the outline in the (x,y) plane. A difference between this article and previous studies of spheres is that the sphere actually hits the observer, and one of the findings presented here is that for any speed v, $0 < v \leqslant c$, part of the surface appears concave for some part of the motion along the x axis.

II. GENERAL SHAPE FOR MOTION PARALLEL TO THE x AXIS

The observer is at rest at the origin O of a coordinate system S. The sphere is at rest in the system S' which is moving in the x direction with speed v, and has center (O,b,O) and radius R and S'. The x axes are parallel and the two origins are coincident at $t = t' = 0$. The objective shape of the sphere, as measured in S, is

$$\gamma^2(x-\beta ct)^2 + (y-b)^2 + z^2 = R^2, \quad (1)$$

where, as usual, $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$. The objective shape (1), as many articles have noted, is an oblate spheroid. To calculate the apparent shape, let the photons from all parts of the surface of the sphere arrive at the observer at time $t = T$. If a photon appears to have been emitted from the point $P = (x,y,z)$ in S, it must have been emitted at time

$$t = T - (x^2 + y^2 + y^2)^{1/2}/c. \quad (2)$$

Substituting (2) into (1) gives the apparent shape as

$$\gamma^2[x - \beta cT + \beta(x^2+y^2+z^2)^{1/2}]^2 + (y-b)^2 + z^2 = R^2. \quad (3)$$

It is convenient to use dimensionless coordinates $\bar{x} = x/R$, $\bar{y} = y/R$, $\bar{z} = z/R$, time parameter $\alpha = cT/R$, and impact parameter $\rho = b/R$. Rewriting Eq. (3) in terms of these and dropping the bars gives

$$\gamma^2[x - \beta\alpha + \beta(x^2 + y^2 + z^2)^{1/2}]^2 + (y - \rho)^2 + z^2 = 1. \tag{4}$$

The apparent shape is most easily analyzed by solving Eq. (4) for x as a function of y and z, which gives

$$x_\pm = \gamma^2\{E_\pm - \beta[E_\pm^2 + (1 - \beta^2)(y^2 + z^2)]^{1/2}\}. \tag{5}$$

Here,

$$E_\pm = \beta\alpha \pm \gamma^{-1}[1 - (y - \rho)^2 - z^2]^{1/2}, \tag{6}$$

where the ($\pm$) signs refer to the front and back surfaces.

Asymptotic expressions as $\alpha \to \pm\infty$ follow readily from Eqs. (5) and (6). When the sphere is moving toward the observer and $\alpha \to -\infty$, the sphere appears to have the shape

$$x_\pm \sim -[\beta/(1+\beta)]\alpha + [(1+\beta)/(1-\beta)]^{1/2} \times [1 - (y-\rho)^2 - z^2]^{1/2},$$

which is the prolate spheroid

$$[(1-\beta)/(1+\beta)][x - \beta\alpha/(1-\beta)]^2 + (y-\rho)^2 - z^2 = 1, \tag{7}$$

with the long axis pointing toward the observer. Similarly, when the sphere is receding from the observer and $\alpha \to +\infty$, the asymptotic shape is the oblate spheroid

$$[(1+\beta)/(1-\beta)][x - \beta\alpha/(1+\beta)]^2 + (y-\rho)^2 + z^2 = 1, \tag{8}$$

foreshortened in the x direction.

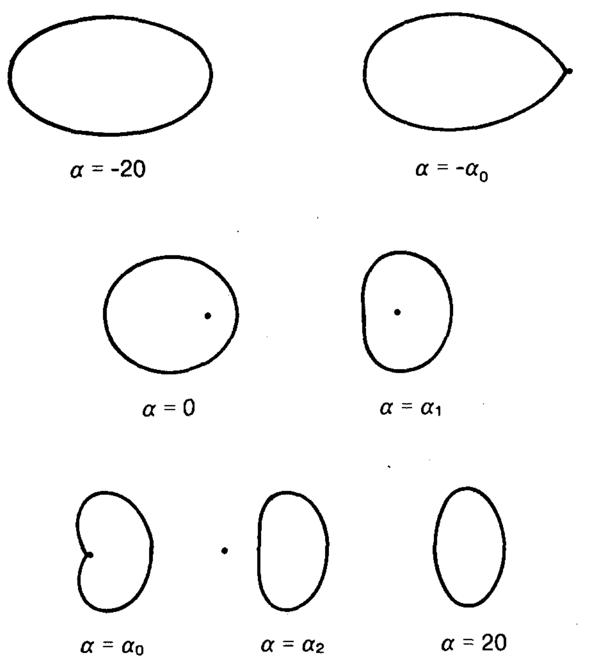

Fig. 1. Apparent shape of a sphere moving along the x axis with speed $\beta = 0.5$. The dots mark the position of the observer for each time except $\alpha = \pm 20$, when the observer is far from the sphere. The observer is inside the sphere for $-\alpha_0 < \alpha < \alpha_0$, where $\alpha_0 = 1.732$ for $\beta = 0.5$. In addition, the back surface of the sphere appears concave for $\alpha_1 < \alpha < \alpha_2$ where, for $\beta = 0.5$, $\alpha_1 = 1.1547$ and $\alpha_2 = 3.4641$. Expressions in terms of β are derived for α_0, α_1, and α_2 in the text.

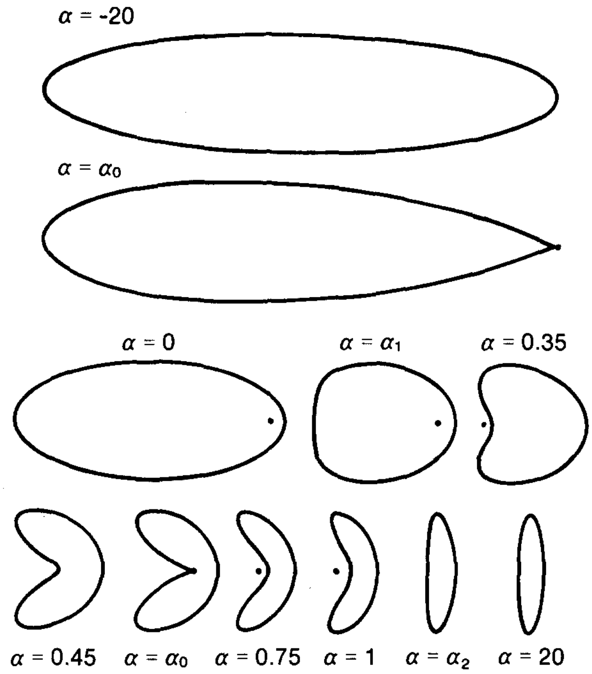

Fig. 2. Same as Fig. 1 but for $\beta = 0.9$. Here, $\alpha_0 = 0.4843$, $\alpha_1 = 0.2549$, and $\alpha_2 = 4.843$.

III. SPHERE TRAVELING DIRECTLY ALONG THE x AXIS

In this case, $\rho = 0$ and the rotational symmetry of the apparent shape about the x axis means no details are lost if we restrict attention to the intersection of the shape with the (x,y) plane. This leaves

$$x_\pm = \gamma^2\{E_\pm - \beta[E_\pm^2 + (1 - \beta^2)y^2]^{1/2}\}, \tag{9}$$

where

$$E_\pm = \beta\alpha \pm \gamma^{-1}(1 - y^2)^{1/2}. \tag{10}$$

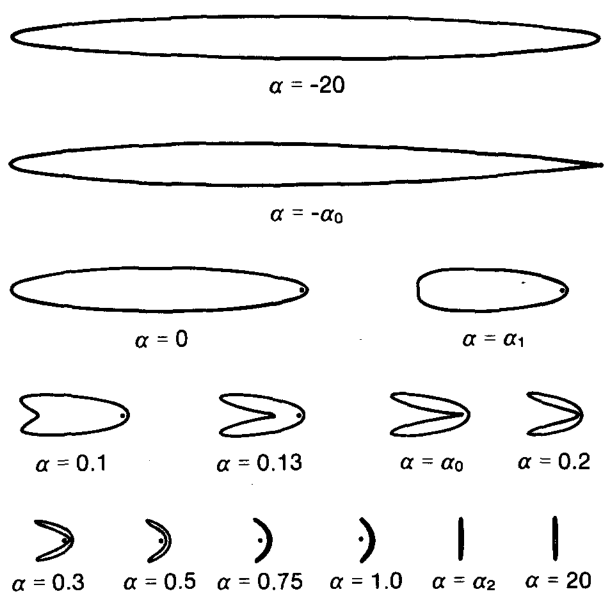

Fig. 3. Same as Figs. 1 and 2, but for $\beta = 0.99$. In this case $\alpha_0 = 0.1425$, $\alpha_1 = 0.0716$, and $\alpha_2 = 14.249$.

The evolution of the apparent shape with time can now be analyzed by studying a series of outlines $x_{\pm} = x_{\pm}(y)$ for various values of β and α. Figures 1–3 show the outline at different values of α for speeds $\beta = 0.5$, 0.9, and 0.99. All the essential features of the apparent shape appear here, from the elongated prolate ellipsoids at large negative times, through a variety of concave intermediate shapes, to the oblate spheroids at large positive times.

We first calculate when the observer is inside the sphere. The surface touches the observer when $x_{\pm}(0) = 0$, that is, $\alpha = \mp\alpha_0$ where $\alpha_0 = (1-\beta^2)^{1/2}/\beta$. The observer is inside the sphere for $-\alpha_0 \leqslant \alpha \leqslant \alpha_0$, which is just the time interval for a sphere with the objective shape (1) to pass by. This is because there is no time delay involved in observing when the surfaces touch the observer.

Figures 1–3 show that the front surface develops a sharp convex point at the observer when $\alpha = -\alpha_0$ and that a concave point develops on the back surface when $\alpha = \alpha_0$. This behavior can be explained by examining the slope of $x_{\pm}(y)$ at $y = 0$. Differentiating (9) with respect to y and denoting the derivative by $\dot{x}_{\pm}$ gives

$$\dot{x}_{\pm} = \gamma\{\mp 1 - \beta[E_{\pm}^2 + (1-\rho^2)y^2]^{-1/2} \times[\mp E + (1-\beta)^{1/2}(1-y^2)^{-1/2}]\} \times(1-y^2)^{-1/2}y. \quad (11)$$

Here, $\dot{x}_{\pm} \to 0$ as $y \to 0$ for all times other than $\alpha = \mp\alpha_0$. When $\alpha = \mp\alpha_0$

$$\lim_{y\to 0} \dot{x}_{\pm}(y)|_{\alpha=\mp\alpha_0} = -\operatorname{sgn}(y)/(1-\beta^2)^{1/2},$$

which is nonzero, and produces a discontinuity in the gradient at $y = 0$. This gives rise to the points, which become sharper as β increases and a cusp develops in the limit $\beta = 1$. There is nothing unusual about the behavior of the sphere in this regard, because it is likely that any smooth surface will develop a discontinuity in slope as it touches the observer. For example, the apparent shape of lines and planes oriented perpendicular to the motion has been studied several times.[1,7–10] A plane appears as a hyperboloid of revolution except when it touches the observer and degenerates to a cone.[10]

Another feature of the shape apparent from Figs. 1–3 is the existence of a positive interval of time during which part of the back surface appears concave. The limits α_1 and α_2 of this interval can be calculated by again considering the derivative (11) of $x_-(y)$. When the back surface is convex, the only place where the derivative can be zero is $y = 0$, but when it is concave, there will be two nonzero values of y (symmetric about $y = 0$) where the derivative will also be zero. These values are

$$y = \pm[4\beta^2\alpha^2 - (1-\beta^2)(1+\beta^2\alpha^2)^2]^{1/2}/(2\beta\alpha), \quad (12)$$

and the ends of the interval are

$$\alpha_1 = [(1-\beta)/(1+\beta)]^{1/2}/\beta$$

and

$$\alpha_2 = [(1+\beta)/(1-\beta)]^{1/2}/\beta.$$

Figures 1–3 show the appearance of the sphere at $\alpha = \alpha_1$ and $\alpha = \alpha_2$, and the limits are plotted in Fig. 4. As $\beta \to 0$, $\alpha_1 \to \infty$ and $\alpha_2 \to \infty$, and as $\beta \to 1$, $\alpha_1 \to 0$ and $\alpha_2 \to \infty$. Consequently, for every value of β in the range $0 < \beta < 1$, the back surface appears concave over a finite time interval.

The time $\alpha = \alpha_0$ when the sphere leaves the observer is also plotted in Fig. 4 and occurs inside the interval $\alpha_1 < \alpha < \alpha_2$. The apparent shape thus becomes concave when the observer is inside, and becomes convex again when the observer is outside.

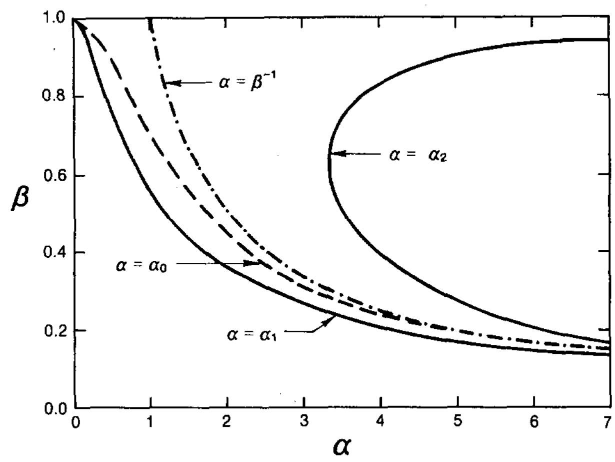

Fig. 4. The values $\alpha = \alpha_0$, $\alpha = \alpha_1$, and $\alpha = \alpha_2$ plotted as functions of β. Also plotted is $\alpha = \beta^{-1}$, at which time the maximum fraction of the back surface appears concave, and the back "rim" appears to pass the observer.

The reason for the concave appearance is as follows. For large negative α, the photons travel nearly parallel to the x axis and points on the back surface are physically closer to the observer as their distance from the x axis increases. The photons are thus emitted at increasingly later times as this distance increases and the back surface appears elongated and convex. However, for certain positions of the observer inside the sphere, points on the back surface can be physically further away from the observer with increasing distance from the x axis. Photons emitted from these points are emitted at earlier times, and to such an extent that the back surface appears to curve back in the negative x direction. This happens for some part of the back surface for all speeds $\beta > 0$ because the surface approximates a plane arbitrarily accurately as $|y| \to 0$, and planes always appear curved backwards. The concave appearance disappears as the sphere moves away from the observer because for large positive α the photons travel nearly parallel to the x axis again. Although photons emitted from points on the back surface away from the x axis must still be emitted at earlier times, the curvature of the surface toward positive x predominates and the sphere appears convex again.

The maximum extent in the x direction of the concave section occurs at $\alpha = \alpha_0$, when the back surface touches the observer, and is given by

$$\Delta x_{max} = \beta^2\gamma^2/2.$$

We see that $\Delta x_{max} \to \infty$ as $\beta \to 1$. Figures 1–3 show the apparent shape at $\alpha = \alpha_0$, and the shape at these times for the higher speeds of $\beta = 0.999$ and $\beta = 0.9999$ appear in Fig. 5. Another way to quantify the concave shape is to calculate how much of the back surface becomes concave for a given value of β. The maximum value of $|y|$ for which the surface is concave is $y_{max} = \pm\beta$, and occurs when $\alpha = \beta^{-1}$. Since $y_{max} \to \pm 1$ as $\beta \to 1$, the entire back surface of the sphere appears concave when $\alpha = 1$ and $\beta = 1$. The expression $\alpha = \beta^{-1}$ is also plotted in Fig. 4, which shows the observer to be outside the sphere again when the maximum fraction of the back surface is concave. This occurs just as the back "rim" appears to pass the observer.

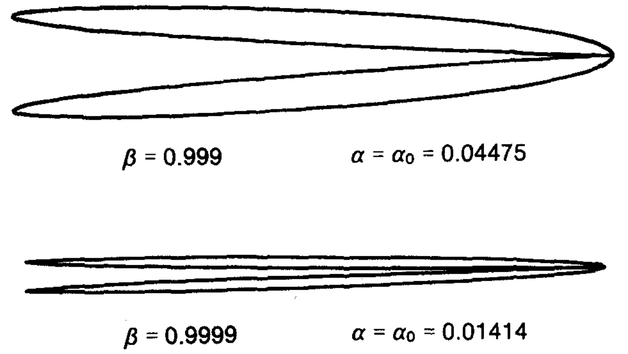

Fig. 5. Apparent shape at $\alpha = \alpha_0$ for $\beta = 0.999$ and $\beta = 0.9999$ showing the extremely pinched shape that develops in the back surface at this time, as $\beta \rightarrow 1$. Note that these two outlines are drawn to different scales.

There is one further point that can be noted about the apparent shape. The minimum value of α_2 (the time when the sphere appears convex again) occurs for $\beta = (\sqrt{5} - 1)/2 = 0.618$, and is

$$\alpha_{2,\min} = \frac{2}{\sqrt{5} - 1}\left(\frac{1 + \sqrt{5}}{1 - \sqrt{5}}\right)^{1/2} = 3.236\,.$$

The physical significance of these values, if any, is not clear.

The above results are, of course, consistent with the result proved by Penrose[3] and Boas[4] that the *outline* of a sphere always appears circular. In this case, the outline (which exists whenever the observer is outside the sphere) has to be circular because of the rotational symmetry about the x axis. More generally, if the motion was not along the x axis, the outline would still appear circular, no matter how distorted the three-dimensional shape was. When the observer is inside the sphere, there is no outline as no light rays can leave the surface tangentially and reach the observer.

IV. VOLUME

It is of interest to calculate the volume of the apparent shape as a function of α and β when the sphere is moving along the x axis. Since the shape is a volume of revolution, the volume is

$$V = \int_0^1 2\pi y[x_+(y) - x_-(y)]dy$$
$$= \frac{4\pi}{3}\gamma - \frac{2\pi\beta}{1-\beta^2}(I_+ - I_-)\,,$$

where

$$I_\pm = \int_0^1 y[\beta^2\alpha^2 + 1 - \beta^2$$
$$\pm \beta(1-\beta^2)^{1/2}\alpha(1-y^2)^{1/2}]^{1/2}\,dy\,.$$

This evaluates to

$$V = (4\pi/3)\gamma - (\pi\gamma^4/15\beta\alpha^2)$$
$$\times\{[3\beta\alpha\gamma^{-1} - \beta^2(\alpha^2-1) - 1]$$
$$\times[\beta^2(\alpha^2-1) + 1 + 2\beta\alpha\gamma^{-1}]^{3/2}$$
$$+ [3\beta\alpha\gamma^{-1} + \beta^2(\alpha^2-1) + 1]$$
$$\times[\beta^2(\alpha^2-1) + 1 - 2\beta\alpha\gamma^{-1}]^{3/2}\}\,. \qquad (13)$$

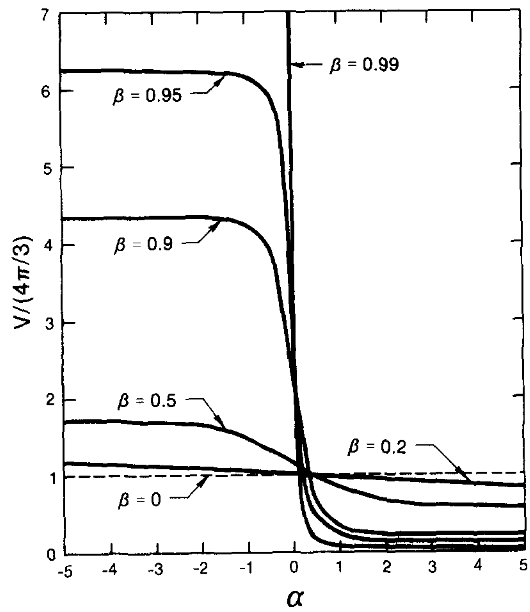

Fig. 6. The behavior as α varies of the apparent volume V from Eq. (13) normalized by the rest volume $4\pi/3$. The volume is shown for six values of β including $\beta = 0$.

Asymptotic expressions for (13) are

$$V \rightarrow (4\pi/3)[(1+\beta)/(1-\beta)]^{1/2}, \quad \text{as } \alpha \rightarrow -\infty$$

and

$$V \rightarrow (4\pi/3)[(1-\beta)/(1+\beta)]^{1/2}, \quad \text{as } \alpha \rightarrow +\infty,$$

and, as expected, these are the volumes of the prolate and oblate spheroids of Eqs. (7) and (8).

The volume (13) is plotted against α for several values of β in Fig. 6.

V. THE ULTRARELATIVISTIC LIMIT $\beta \rightarrow 1$

We now return to the full three-dimensional equations (5) and (6) and examine the apparent shape in the ultrarelativistic limit $\beta \rightarrow 1$. Expanding these equations in

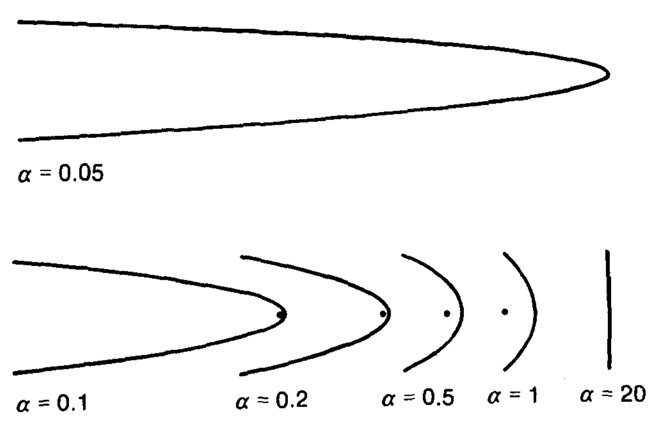

Fig. 7. The outline in the (x,y) plane in the ultrarelativistic limit $\beta \rightarrow 1$. For $\alpha > 0$, the apparent shape is a single curved surface that is part of a paraboloid of revolution.

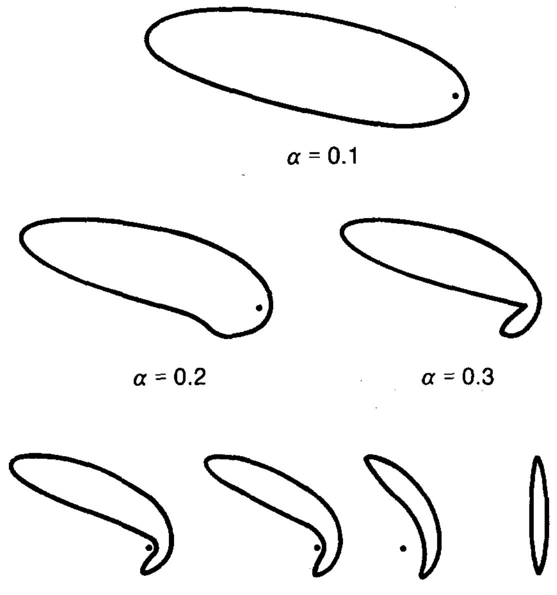

Fig. 8. The outline in the (x,y) plane at various times of a sphere that is not moving with its center on the x axis. Here, $\beta = 0.95$ and the impact parameter is $\rho = 0.5$, so the sphere still hits the observer.

powers of γ gives, for $\alpha < 0$,

$$x_\pm \sim \frac{\beta\alpha}{1-\beta} \pm \left(\frac{1+\beta}{1-\beta}\right)^{1/2} [1-(y-\rho)^2 - z^2]^{1/2} - \frac{1}{2\alpha}(y^2+z^2) + O(\gamma^{-1}), \quad (14)$$

and for $\alpha > 0$,

$$x_\pm \sim \frac{\beta\alpha}{1+\beta} \pm \left(\frac{1-\beta}{1+\beta}\right)^{1/2} [1-(y-\rho)^2 - z^2]^{1/2} - \frac{1}{2\alpha}(y^2+z^2) + O(\gamma^{-1}), \quad (15)$$

Equation (14) is not particularly interesting because it only tells us what we already know: The sphere appears stretched out in the x direction and the amount of stretching becomes infinite as $\beta \to 1$. Equation (15) is more interesting because the separate terms for the front and back surfaces vanish in the limit $\beta \to 1$, as do all the terms that have been discarded. This leaves

$$x_\pm = \alpha/2 - (1/2\alpha)(y^2+z^2), \quad (16)$$

which shows that the front and back surfaces have merged and the sphere appears as a one-dimensional curved sheet. If we consider the case $\rho = 0$ again, Eq. (16) indicates the shape is part of a paraboloid of revolution about the x axis. [Although (16) does not involve ρ explicitly, ρ does affect the limits of y and z.] When $\rho = 0$, the surface cuts the x axis at $x = \alpha/2$, and extends back to

$$x_{\min} = (\alpha^2 - 1)/(2\alpha).$$

Consequently, $x_{\min} \to -\infty$ as $\alpha \to 0$, and Fig. 7 shows a sequence of outlines in the (x,y) plane for α between $\alpha = 0.05$ and $\alpha = 20$. As $\alpha \to +\infty$, Eq. (16) reduces to $x_\pm = \alpha/2$, as expected, a flat disk of unit radius.

VI. MOTION WITH $\rho \neq 0$

A detailed analysis of the apparent shape of the type presented in Secs. III and IV for $\rho = 0$ is more difficult when $\rho \neq 0$ and this article makes no attempt at it. Figure 8 does present some outlines in the (x,y) plane for a sphere moving with speed $\beta = 0.95$ and $\rho = 0.5$. Since $\rho < 1$, the sphere still hits the observer and similar behavior to the $\rho = 0$ case is observed. However, the three-dimensional shape can no longer be inferred from these outlines because the shapes are no longer rotationally symmetric about the x axis. Here, the outlines give the apparent shapes of a moving ring in the (x,y) plane. Scott and van Driel[5] show some outlines of a sphere in the (x,y) plane when $\rho > 1$, and the outlines in Fig. 8 are very similar to their figures.

VII. CONCLUDING REMARKS

Although the apparent shape of a sphere is simple to analyze, particularly when it is moving along the x axis, it should be emphasized that none of the general characteristics of the shape are peculiar to the sphere. Any finite body (for example, a cube) moving in the same manner will exhibit a similar shape: elongated when approaching the observer and foreshortened when receding. In addition, provided the speed is high enough, the body will appear concave over a certain time interval and in the limit $\beta \to 1$ it will be squashed to a curved sheet when receding from the observer.

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