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Maxwell's equations from spacetime geometry and the role of Weyl curvature

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Abstract.

This research article demonstrates how the field equations of electrodynamics can be shown to be a special case of Einstein field equations of General Relativity. By establishing a special conjecture between the electromagnetic four-potential and the metric of the spacetime, it is first shown how the relativistic wave equation of electrodynamics is a condition for the metric to be Ricci-flat. Moreover, the four-current is identified with a certain four-gradient, which allows one to conjecture that electric charge is related to the covariant divergence of the electromagnetic four-potential. These considerations allow one to understand the Einstein field equations as a nonlinear generalization of Maxwell's equations. Finally, it is argued that the four-current induces Weyl curvature on the spacetime.

1. Introduction

Before the advent of Quantum Mechanics, and well in the 1950's, there was an eager strand of research in mathematical physics which tried to derive electromagnetism from purely geometric considerations. Probably one inspiration for such aspirations was the fact that Newtonian gravity and electrostatics share some key mathematical features in the sense that they are both described by Poisson's equation. The earliest attempts can be reasonably traced back to the German physicist Gustav Mie (1868-1957) and the Finnish physicist Gunnar Nordström (1881-1923). Fruitful efforts came, for example, from David Hilbert (1862-1943), Hermann Weyl (1885-1955), Theodor Kaluza (1885-1954), Arthur Eddington (1882-1944) and of course also from Albert Einstein (1879-1955). It is less well-known that, for example, Erwin Schrödinger (1887-1961) had such inclinations as well, see [1]. For a thorough historical review, see [2].

Theories by Rainich [3], Misner and Wheeler [4] are of course important in this respect. Indeed, John Wheeler and others continued such efforts within the research tradition of geometrodynamics. The inspirations for such research programmes came from the idea that perhaps the material world can be seen solely through the structure of the spacetime itself.

One approach to classical electrodynamics is through the Lagrangian approach, where it can be shown that a critical point for the action functional is achieved, if two of Maxwell's equations hold. The Lagrangian approach is the motivation in the present study as well. The present approach asks the question: under which conditions we can understand the classical electromagnetic Lagrangian through the curvature of spacetime? This article provides one possible



path for such understanding, whilst at the same time we establish a conjecture linking the very ontology of charge to some properties of the spacetime metric. The approach therefore resembles John Wheeler's "charge without charge". Einstein himself was of the view that "A theory in which the gravitational field and the electromagnetic field do not enter as logically different structures would be much preferable.", see [5]. The present study aims to do just that and also to endogenize the source currents by linking them to Weyl curvature.

2. Electromagnetic Lagrangian and the Einstein-Hilbert Action

The mathematical framework in the present study is that of pseudo-Riemannian geometry. Unlike in many approaches, where electromagnetism and spacetime geometry are considered together, we assume that only the canonical Levi-Civita connection is needed. Moreover, unlike in some other approaches, we assume the spacetime manifold to be four-dimensional, like in General Relativity. This guarantees us that the framework in General Relativity is preserved; the metric defines the curvature and the optimal metric defines the geodesics. In General Relativity the stress-energy tensor is the source of curvature in the spacetime. However, it is important to recall that to link the stress-energy content of the spacetime merely to the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ means that one only considers essentially the trace of the Riemann curvature tensor and thus Weyl curvature is ignored as such. In the present consideration Weyl curvature is essential to include, for in the final analysis it is needed to make the theory consistent with canonical electromagnetic theory.

2.1. Metric volume form and the Einstein-Hilbert functional

Suppose we consider a pseudo-Riemannian manifold (the 4-dimensional spacetime) (M, g) with a torsionless and metrical connection ∇ .

First, we show that the vacuum Einstein equation is the optimality equation for the Einstein-Hilbert functional, irrespective of the sign of the metric determinant¹. The object we are interested in is the total scalar curvature of the Lorentzian manifold. This can be defined through the Riemann-Christoffel curvature tensor $R_{\nu\sigma\mu}^{\lambda}$. This tensor can be contracted, R_{ν}^{λ} and the result is the Ricci curvature tensor $R_{\mu\nu}$. According to Einstein, Ricci curvature is essentially describing the local (mean-) curvature of spacetime, and the source of essentially this curvature is the mass and energy distribution (the symmetric stress-energy tensor). There is a natural invariant, called the scalar curvature R , which is the trace of the Ricci tensor. Minimizing this scalar curvature over invariant volume forms leads to the famous Einstein field equations of General Relativity. David Hilbert apparently discovered this already in 1915. The key point we want to demonstrate here is that the criticality condition is independent of the sign of the metric determinant g .

We want to make the following functional stationary with respect to the metric (without any source terms related to external stress-energy) :

$$S = \int R\sqrt{g}d^4x. \quad (1)$$

This functional is one of the simplest nontrivial curvature functionals. The inclusion of the metric determinant g is due to the requirement of coordinate-invariance, as the metric tensor is formally the square of the Jacobian. Therefore, the determinant of the metric tensor is the determinant of the Jacobian determinant squared: $g = (\det J)^2$. Taking the square root gives: $\sqrt{g} = \sqrt{(\det J)^2}$ giving $|\det J| = \sqrt{g}$, so that the invariant volume form is $dV = \sqrt{g}dx^4$.

¹ This idea was utilized partly when considering coordinate-invariance in Quantum Mechanics in [6]

Functional variation (vary with respect to the contravariant metric tensor $g^{\mu\nu}$) gives:

$$\delta S = \int (\delta R \sqrt{g} + R \delta \sqrt{g}) dx^4 = 0. \quad (2)$$

In the following we utilize Jacobi's formula: $\delta g = g g^{\mu\nu} \delta g_{\mu\nu}$. The second term under the integral is interesting so let us focus on it: $\int (\delta R \sqrt{g} + \frac{1}{2} R \frac{\delta g}{\sqrt{g}}) dx^4 = 0$. Substituting then the variation δg we have: $\int (\delta R \sqrt{g} + \frac{1}{2} R \sqrt{g} g^{\mu\nu} \delta g_{\mu\nu}) dx^4 = \int (\delta R + \frac{1}{2} R g^{\mu\nu} \delta g_{\mu\nu}) \sqrt{g} dx^4 = 0$. Using the well-known rules for manipulating metric tensors we have: $\int (\delta R - \frac{1}{2} R g_{\mu\nu} \delta g^{\mu\nu}) \sqrt{g} dx^4 = 0$. Remembering that $R = R_{\mu\nu} g^{\mu\nu}$, from which we can conclude that we have the Einstein equation in vacuum:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \quad (3)$$

The stationarity condition is the vacuum Einstein field equation, irrespective of the sign of the metric determinant. Therefore, whenever we optimize the scalar curvature over invariant volumes, the metric obeys the nonlinear Einstein field equation above. The complete technical arguments for the variation of the metric determinant and the Ricci scalar can be found in any good textbook on General Relativity, such as the classic [7].

2.2. The Lagrangian of electrodynamics and the Einstein-Hilbert Action

The key idea from which we can proceed to derive electromagnetism from the properties of the spacetime is the following: consider the electromagnetic four-potential A_μ and the metric tensor of the spacetime $g_{\mu\nu}$. We make the following key conjecture: the symmetric metric is given by the representation:

$$g_{\mu\nu} = A_\mu A_\nu \quad (4)$$

(tensor product), where A_μ is a general covector (the electromagnetic four-potential). We can then start to impose some desirable features for the tensor A_μ in order to impose some desirable features for the metric itself. For didactical reasons, it could be useful to think of the tensor A_μ as a "vector field" as in, say, continuum mechanics.

If one thinks of the tensor A_μ as a "vector field", we could look for a vector field, which would be optimal in some sense. We proceed in this way, and we want to minimize the "rotation" $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ of the vector field over the spacetime, whilst also we want the vector field to "travel along the level-sets of divergence", see Fig. 1. This latter requirement is inspired by the concept that the gradient of the divergence of A_ν in continuum mechanics would be related to variations in (mass) density. Therefore, the requirement thus intuitively warrants that the "vector field" should flow through volume elements with minimal variations in density.

Therefore, we look for a critical point for the following functional:

$$\int_M \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A^\mu \nabla_\mu (\nabla^\nu A_\nu) \right) \sqrt{g} d^4 x. \quad (5)$$

The invariant volume form $\sqrt{g} d^4 x$ ensures coordinate invariance, when integrating over the manifold. The rotating part of the above-defined cost functional is due to the tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The physical intuition for the latter integrand is the idea that the vector field should be orthogonal to the gradient of the divergence, i.e. parallel to the level-sets of divergence. Note that the above functional is exactly the negative of the classical Lagrangian of electrodynamics, if we identify the four-current with $J_\mu = \nabla_\mu \nabla^\nu A_\nu$.

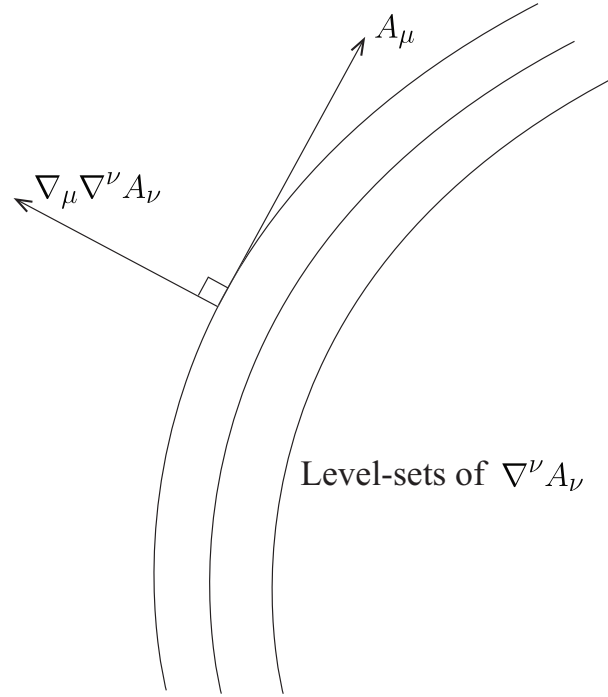


Figure 1. The four-potential A_μ should be orthogonal to the gradient of the divergence.

We now show how the above functional is actually equivalent with mild assumptions to the Einstein-Hilbert action. The first integrand is only the gradient energy of A_μ if the gradient tensor $\nabla_\nu A_\mu$ is antisymmetric:

$$F^{\mu\nu} F_{\mu\nu} = 4\nabla^\mu A^\nu \nabla_\mu A_\nu. \quad (6)$$

Consider now the covariant derivative of the tensor $F_{\mu\nu}$:

$$\nabla_\sigma F_{\mu\nu} = \nabla_\sigma \nabla_\mu A_\nu - \nabla_\sigma \nabla_\nu A_\mu. \quad (7)$$

Use the Ricci identity, which is essentially the definition for the Riemann-Christoffel curvature tensor:

$$\nabla_\sigma \nabla_\mu A_\nu = \nabla_\mu \nabla_\sigma A_\nu + R_{\nu\sigma\mu}^\lambda A_\lambda. \quad (8)$$

Substitute into the covariant derivative of the tensor $F_{\mu\nu}$:

$$\nabla_\sigma F_{\mu\nu} = \nabla_\mu \nabla_\sigma A_\nu + R_{\nu\sigma\mu}^\lambda A_\lambda - \nabla_\sigma \nabla_\nu A_\mu. \quad (9)$$

Next, we raise an index with the contravariant metric tensor $g^{\mu\nu}$:

$$\nabla_\sigma g^{\mu\nu} F_{\mu\nu} = \nabla^\nu \nabla_\sigma A_\nu + R_{\sigma}^\lambda A_\lambda - \nabla_\sigma \nabla^\mu A_\mu = 0. \quad (10)$$

The equation must be equal to zero, as the metric tensor is symmetric and the tensor $F_{\mu\nu}$ is antisymmetric. We also have made use of the metric compatibility of the Levi-Civita connection. Contract with $\nu = \sigma$ and we have:

$$\nabla^2 A_\sigma + R_{\sigma}^\lambda A_\lambda - \nabla_\sigma \nabla^\mu A_\mu = 0. \quad (11)$$

For convenience, raise an index by multiplying with $g^{\sigma\nu}$:

$$\nabla^2 A^\nu + R^{\nu\lambda} A_\lambda - \nabla^\nu \nabla^\mu A_\mu = 0. \quad (12)$$

Finally, multiply with A_ν (remember that $g_{\mu\nu} = A_\mu A_\nu$):

$$R = A_\nu \nabla^\nu \nabla^\mu A_\mu - A_\nu \nabla^2 A^\nu. \quad (13)$$

The above equation defines the scalar curvature in terms of the A_ν and its covariant derivatives. Using Green's First Identity (with vanishing boundary terms), we have $\int_M \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \sqrt{g} d^4x = \int_M -A_\mu \nabla^2 A^\mu \sqrt{g} d^4x$. The covariant d'Alembertian is defined as $\nabla^2 = \nabla^\mu \nabla_\mu$.

Comparing equation 5 we see that

$$\int_M \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A^\mu \nabla_\mu (\nabla^\nu A_\nu) \right) \sqrt{g} d^4x = \int_M R \sqrt{g} d^4x, \quad (14)$$

which is just the Einstein-Hilbert Action.

This equivalence provides us already a hint that the Einstein field equations and Maxwell's equations must be connected with this choice of metric identification.

3. Ricci-flat solutions and identification of the four-current with Weyl curvature

As we have shown now constructively that the classical electromagnetic field theory can be seen through finding a critical point for the Einstein-Hilbert action, it is clear that the electromagnetic field equation for this coupling of the metric with the four-potential is the Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0. \quad (15)$$

Consider now a simple candidate solution for the metric: Ricci-flatness. Ricci-flatness is important, as Ricci-flat solutions for the metric are indeed special solutions of the vacuum Einstein field equations. Ricci-flatness can be defined through the Christoffel symbols as follows:

$$R_{\mu\alpha\nu}^\alpha = R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\alpha,\nu}^\alpha + \Gamma_{\beta\alpha}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\alpha}^\beta = 0. \quad (16)$$

Examine what follows when $R^{\mu\nu} = 0$ and thus $R = 0$:

$$R = A_\nu \nabla^\nu \nabla^\mu A_\mu - A_\nu \nabla^2 A^\nu = 0 \quad (17)$$

and thus

$$\nabla^\nu \nabla^\mu A_\mu - \nabla^2 A^\nu = 0. \quad (18)$$

If we now make the identification $J^\nu = \nabla^\nu \nabla^\mu A_\mu$, we have:

$$\nabla^2 A^\nu = J^\nu, \quad (19)$$

which comprises two of Maxwell's equations.

Consider the covariant divergence of the Faraday tensor $F_{\mu\nu}$. Consider again equation 9:

$$\nabla_\sigma F_{\mu\nu} = \nabla_\mu \nabla_\sigma A_\nu + R_{\nu\sigma\mu}^\lambda A_\lambda - \nabla_\sigma \nabla_\nu A_\mu. \quad (20)$$

Raise an index by multiplying with the contravariant metric tensor $g^{\sigma\nu}$:

$$\nabla^\nu F_{\mu\nu} = \nabla_\mu \nabla^\nu A_\nu + R_{\mu}^\lambda A_\lambda - \nabla^\nu \nabla_\nu A_\mu. \quad (21)$$

With the metric being Ricci-flat, the covariant divergence of the Faraday tensor vanishes. If we want to require conformity with canonical electrodynamics, where the ordinary divergence of the Faraday tensor equals the four-current, we need to couple the four-current to the Weyl curvature of spacetime. Remember that the Weyl tensor is the traceless part of the Riemann curvature tensor. In other words, the four-current must induce Weyl curvature on the spacetime.

In terms operationalizing the coupling of the four-current to the Weyl curvature, consider the definition of the covariant derivative of the Faraday tensor:

$$\nabla_{\sigma} F_{\mu\nu} = \partial_{\sigma} F_{\mu\nu} - \Gamma_{\sigma\mu}^{\lambda} F_{\lambda\nu} - \Gamma_{\sigma\nu}^{\lambda} F_{\mu\lambda}. \quad (22)$$

Raise an index by multiplying with the contravariant metric tensor $g^{\sigma\nu}$:

$$\nabla^{\nu} F_{\mu\nu} = \partial^{\nu} F_{\mu\nu} - g^{\sigma\nu} \Gamma_{\sigma\mu}^{\lambda} F_{\lambda\nu} - g^{\sigma\nu} \Gamma_{\sigma\nu}^{\lambda} F_{\mu\lambda}. \quad (23)$$

As the covariant divergence of the Faraday tensor on a Ricci-flat spacetime vanishes, the requirement that the field equation conforms to classical electrodynamics, it is required to identify the four-current with the following:

$$J_{\mu} = g^{\sigma\nu} \Gamma_{\sigma\mu}^{\lambda} F_{\lambda\nu} + g^{\sigma\nu} \Gamma_{\sigma\nu}^{\lambda} F_{\mu\lambda}, \quad (24)$$

where the Christoffels symbols are not zero, in spite of the vanishing Ricci curvature, but they reflect the Weyl curvature of the spacetime.

The rest of the four Maxwell's equations are given by the algebraic Bianchi identity:

$$F_{[\lambda\mu;\nu]} = 0, \quad (25)$$

This cyclic permutation can be seen easily from the algebraic Bianchi identity which says that

$$R_{\sigma\mu\nu}^{\lambda} + R_{\mu\nu\sigma}^{\lambda} + R_{\nu\sigma\mu}^{\lambda} = 0 \quad (26)$$

where the semicolon refers to covariant differentiation. Substituting the definition of the Faraday tensor in the above and using the Ricci identity we end up with the algebraic Bianchi identity, which guarantees us Faraday's Law and the absence of magnetic monopoles. This is just due to the symmetry properties of the curvature tensor as we do not have torsion, ie. the Christoffel connections enjoy symmetry.

We also require that $\nabla^{\mu} J_{\mu} = 0$, which is the familiar conservation of charge statement. For us it means that the divergence of electromagnetic four-potential must obey the covariant wave equation

$$\nabla^2 \phi = 0, \quad (27)$$

where $\phi = \nabla^{\mu} A_{\mu}$ is the covariant divergence.

4. Discussion and conclusions

Electromagnetism is induced by the twisting geometry of the spacetime. As the metric tensor $g_{\mu\nu} = A_{\mu} A_{\nu}$ depends solely on the electromagnetic four-potential, Ricci-flatness requires that Maxwell's equations are satisfied. The classical electromagnetic action for electrodynamics is understood through minimizing rotation and preferring the level sets of divergence and thus through minimizing the total scalar curvature of the manifold. The classical action of electromagnetism is the Einstein-Hilbert Action and electromagnetism can be understood therefore

from the frameworks of General Relativity. For a concise set of conceptual analogies between geometry and electromagnetism, see Table. 1.

As the identification presented makes the hypothesis that the four-current is the four-gradient of the four-divergence of the electromagnetic potential, and as current is transport of charge, we make the claim that charge is directly related to the divergence of the electromagnetic four-potential. The four-current then is a re-balancing mechanism, which transports charge to make the scalar curvature to vanish. It is also concluded that the four-current is coupled to the Weyl curvature of spacetime, so that the vanishing of covariant divergence of the Faraday tensor conforms with the canonical formulation of electrodynamics. This could be in principle interesting also from an engineering point of view, as the traceless part of the Riemannian curvature tensor is not directly considered in classical Einstein field equations. Finally, as Maxwell's equations are the requirement that the spacetime manifold is Ricci-flat, we can understand the vacuum Einstein field equation as a nonlinear generalization of Maxwell's equations. In a way, Einstein's objective to unify the classical fields seems to be the correct approach, but the complete picture was missing the role of Weyl curvature. The strength of the present approach is simplicity, there is no need for higher dimensions, torsion tensors, asymmetric metrics or the like.

Table 1. The bridge between geometry and electromagnetism

Geometry	Electromagnetism
metric tensor	electromagnetic four-potential
Ricci scalar	classical Lagrangian in electrodynamics
Einstein field equation	generalized electrodynamic wave equation
Ricci-flatness	Maxwell's equations
Bianchi identity	Maxwell's equations
divergence	charge
Weyl curvature	four-current

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