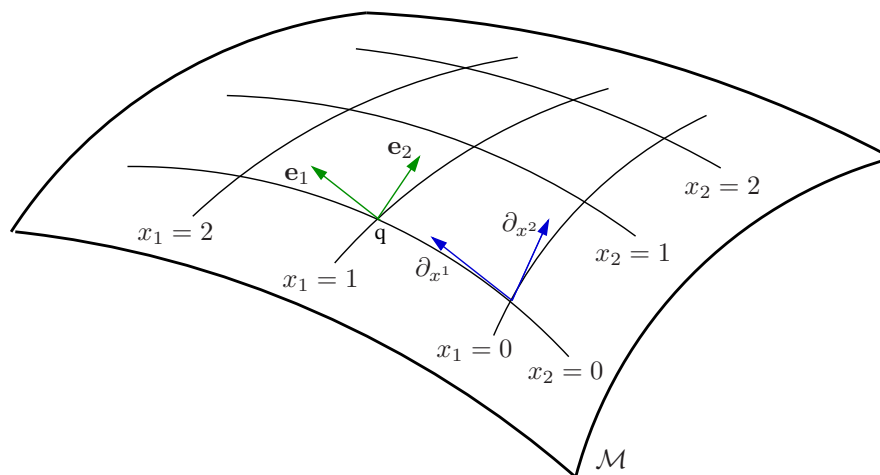


Catalogue of Spacetimes



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Date: 04. Nov 2010

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Chapter 1

Introduction and Notation

The *Catalogue of Spacetimes* is a collection of four-dimensional Lorentzian spacetimes in the context of the General Theory of Relativity (GR). The aim of the catalogue is to give a quick reference for students who need some basic facts of the most well-known spacetimes in GR. For a detailed discussion of a metric, the reader is referred to the standard literature or the original articles. Important resources for exact solutions are the book by Stephani et al[SKM⁺03] and the book by Griffiths and Podolský[GP09].

Most of the metrics in this catalogue are implemented in the Motion4D-library[MG09] and can be visualized using the GeodesicViewer[MG10]. Except for the Minkowski and Schwarzschild spacetimes, the metrics are sorted by their names.

1.1 Notation

The notation we use in this catalogue is as follows:

Indices: Coordinate indices are represented either by Greek letters or by coordinate names. Tetrad indices are indicated by Latin letters or coordinate names in brackets.

Einstein sum convention: When an index appears twice in a single term, once as lower index and once as upper index, we build the sum over all indices:

$$\zeta_\mu \zeta^\mu \equiv \sum_{\mu=0}^3 \zeta_\mu \zeta^\mu. \quad (1.1.1)$$

Vectors: A coordinate vector in x^μ direction is represented as $\partial_{x^\mu} \equiv \partial_\mu$. For arbitrary vectors, we use boldface symbols. Hence, a vector \mathbf{a} in coordinate representation reads $\mathbf{a} = a^\mu \partial_\mu$.

Derivatives: Partial derivatives are indicated by a comma, $\partial\psi/\partial x^\mu \equiv \partial_\mu \psi \equiv \psi_{,\mu}$, whereas covariant derivatives are indicated by a semicolon, $\nabla\psi = \psi_{;\mu}$.

Symmetrization and Antisymmetrization brackets:

$$a_{(\mu} b_{\nu)} = \frac{1}{2} (a_\mu b_\nu + a_\nu b_\mu), \quad a_{[\mu} b_{\nu]} = \frac{1}{2} (a_\mu b_\nu - a_\nu b_\mu) \quad (1.1.2)$$

1.2 General remarks

The Einstein field equation in the most general form reads[MTW73]

$$G_{\mu\nu} = \varkappa T_{\mu\nu} - \Lambda g_{\mu\nu}, \quad \varkappa = \frac{8\pi G}{c^4}, \quad (1.2.1)$$

with the symmetric and divergence-free Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R , the metric tensor $g_{\mu\nu}$, the energy-momentum tensor $T_{\mu\nu}$, the cosmological constant Λ , Newton's gravitational constant G , and the speed of light c . Because the Einstein tensor is divergence-free, the conservation equation $T^{\mu\nu}{}_{;\nu} = 0$ is automatically fulfilled.

A solution to the field equation is given by the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1.2.2)$$

with the symmetric, covariant metric tensor $g_{\mu\nu}$. The contravariant metric tensor $g^{\mu\nu}$ is related to the covariant tensor via $g_{\mu\nu} g^{\nu\lambda} = \delta_\mu^\lambda$ with the Kronecker- δ . Even though $g_{\mu\nu}$ is only a component of the metric tensor $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$, we will also call $g_{\mu\nu}$ the metric tensor.

Note that, in this catalogue, we mostly use the convention that the signature of the metric is $+2$. In general, we will also keep the physical constants c and G within the metrics.

1.3 Basic objects of a metric

The basic objects of a metric are the Christoffel symbols, the Riemann and Ricci tensors as well as the Ricci and Kretschmann scalars which are defined as follows:

Christoffel symbols of the first kind:¹

$$\Gamma_{\nu\lambda\mu} = \frac{1}{2} (g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}) \quad (1.3.1)$$

with the relation

$$g_{\nu\lambda,\mu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\mu\lambda\nu} \quad (1.3.2)$$

Christoffel symbols of the second kind:

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2} g^{\mu\rho} (g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}) \quad (1.3.3)$$

which are related to the Christoffel symbols of the first kind via

$$\Gamma_{\nu\lambda}^\mu = g^{\mu\rho} \Gamma_{\nu\lambda\rho} \quad (1.3.4)$$

Riemann tensor:

$$R_{\nu\rho\sigma}^\mu = \Gamma_{\nu\sigma,\rho}^\mu - \Gamma_{\nu\rho,\sigma}^\mu + \Gamma_{\rho\lambda}^\mu \Gamma_{\nu\sigma}^\lambda - \Gamma_{\sigma\lambda}^\mu \Gamma_{\nu\rho}^\lambda \quad (1.3.5)$$

or

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R_{\nu\rho\sigma}^\lambda = \Gamma_{\nu\sigma\mu,\rho} - \Gamma_{\nu\rho\mu,\sigma} + \Gamma_{\nu\rho}^\lambda \Gamma_{\mu\sigma\lambda} - \Gamma_{\nu\sigma}^\lambda \Gamma_{\mu\rho\lambda} \quad (1.3.6)$$

with symmetries

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}, \quad R_{\mu\nu\rho\sigma} = -R_{\nu\rho\sigma\mu}, \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \quad (1.3.7)$$

and

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 \quad (1.3.8)$$

Ricci tensor:

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} = R_{\mu\rho\nu}^\rho \quad (1.3.9)$$

Ricci and Kretschmann scalar:

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu} = R^\mu{}_\mu, \quad \mathcal{K} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = R^{\gamma\delta}{}_{\alpha\beta} R^{\alpha\beta}{}_{\gamma\delta} \quad (1.3.10)$$

¹The notation of the Christoffel symbols of the first kind differs from the one used by Rindler[Rin01], $\Gamma_{\mu\nu\lambda}^{\text{Rindler}} = \Gamma_{\nu\lambda\mu}^{\text{CoS}}$.

Weyl tensor:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2} (g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu}) + \frac{1}{3} R g_{\mu[\rho} g_{\sigma]\nu} \quad (1.3.11)$$

If we change the signature of a metric, these basic objects transform as follows:

$$\Gamma_{\nu\lambda}^{\mu} \mapsto \Gamma_{\nu\lambda}^{\mu}, \quad R_{\mu\nu\rho\sigma} \mapsto -R_{\mu\nu\rho\sigma}, \quad C_{\mu\nu\rho\sigma} \mapsto -C_{\mu\nu\rho\sigma}, \quad (1.3.12a)$$

$$R_{\mu\nu} \mapsto R_{\mu\nu}, \quad \mathcal{R} \mapsto -\mathcal{R}, \quad \mathcal{H} \mapsto \mathcal{H}. \quad (1.3.12b)$$

Covariant derivative

$$\nabla_{\lambda} g_{\mu\nu} = g_{\mu\nu;\lambda} = 0. \quad (1.3.13)$$

Covariant derivative of the vector field ψ^{μ} :

$$\nabla_{\nu} \psi^{\mu} = \psi^{\mu}_{;\nu} = \partial_{\nu} \psi^{\mu} + \Gamma_{\nu\lambda}^{\mu} \psi^{\lambda} \quad (1.3.14)$$

Covariant derivative of a r-s-tensor field:

$$\begin{aligned} \nabla_c T^{a_1 \dots a_r}_{b_1 \dots b_s} &= \partial_c T^{a_1 \dots a_r}_{b_1 \dots b_s} + \Gamma_{dc}^{a_1} T^{d \dots a_r}_{b_1 \dots b_s} + \dots + \Gamma_{dc}^{a_r} T^{a_1 \dots a_{r-1} d}_{b_1 \dots b_s} \\ &\quad - \Gamma_{b_1 c}^d T^{a_1 \dots a_r}_{d \dots b_s} - \dots - \Gamma_{b_s c}^d T^{a_1 \dots a_r}_{b_1 \dots b_{s-1} d} \end{aligned} \quad (1.3.15)$$

Killing equation:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \quad (1.3.16)$$

1.4 Natural local tetrad and initial conditions for geodesics

We will call a local tetrad natural if it is adapted to the symmetries or the coordinates of the spacetime. The four base vectors $\mathbf{e}_{(i)} = e_{(i)}^{\mu} \partial_{\mu}$ are given with respect to coordinate directions $\partial/\partial x^{\mu} = \partial_{\mu}$, compare Nakahara[Nak90] or Chandrasekhar[Cha06] for an introduction to the tetrad formalism. The inverse or dual tetrad is given by $\theta^{(i)} = \theta_{\mu}^{(i)} dx^{\mu}$ with

$$\theta_{\mu}^{(i)} e_{(j)}^{\mu} = \delta_{(j)}^{(i)} \quad \text{and} \quad \theta_{\mu}^{(i)} e_{(i)}^{\nu} = \delta_{\mu}^{\nu}. \quad (1.4.1)$$

Note that we use Latin indices in brackets for tetrads and Greek indices for coordinates.

1.4.1 Orthonormality condition

To be applicable as a local reference frame (Minkowski frame), a local tetrad $\mathbf{e}_{(i)}$ has to fulfill the orthonormality condition

$$\langle \mathbf{e}_{(i)}, \mathbf{e}_{(j)} \rangle_{\mathbf{g}} = \mathbf{g}(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}) = g_{\mu\nu} e_{(i)}^{\mu} e_{(j)}^{\nu} \stackrel{!}{=} \eta_{(i)(j)}, \quad (1.4.2)$$

where $\eta_{(i)(j)} = \text{diag}(\mp 1, \pm 1, \pm 1, \pm 1)$ depending on the signature $\text{sign}(\mathbf{g}) = \pm 2$ of the metric. Thus, the line element of a metric can be written as

$$ds^2 = \eta_{(i)(j)} \theta^{(i)} \theta^{(j)} = \eta_{(i)(j)} \theta_{\mu}^{(i)} \theta_{\nu}^{(j)} dx^{\mu} dx^{\nu}. \quad (1.4.3)$$

To obtain a local tetrad $\mathbf{e}_{(i)}$, we could first determine the dual tetrad $\theta^{(i)}$ via Eq. (1.4.3). If we combine all four dual tetrad vectors into one matrix Θ , we only have to determine its inverse Θ^{-1} to find the tetrad vectors,

$$\Theta = \begin{pmatrix} \theta_0^{(0)} & \theta_1^{(0)} & \theta_2^{(0)} & \theta_3^{(0)} \\ \theta_0^{(1)} & \theta_1^{(1)} & \theta_2^{(1)} & \theta_3^{(1)} \\ \theta_0^{(2)} & \theta_1^{(2)} & \theta_2^{(2)} & \theta_3^{(2)} \\ \theta_0^{(3)} & \theta_1^{(3)} & \theta_2^{(3)} & \theta_3^{(3)} \end{pmatrix} \Rightarrow \Theta^{-1} = \begin{pmatrix} e_{(0)}^0 & e_{(1)}^0 & e_{(2)}^0 & e_{(3)}^0 \\ e_{(0)}^1 & e_{(1)}^1 & e_{(2)}^1 & e_{(3)}^1 \\ e_{(0)}^2 & e_{(1)}^2 & e_{(2)}^2 & e_{(3)}^2 \\ e_{(0)}^3 & e_{(1)}^3 & e_{(2)}^3 & e_{(3)}^3 \end{pmatrix}. \quad (1.4.4)$$

There are also several useful relations:

$$e_{(a)\mu} = g_{\mu\nu} e_{(a)}^\nu, \quad \eta_{(a)(b)} = e_{(a)}^\mu e_{(b)\mu}, \quad e_{(b)\mu} = \eta_{(a)(b)} \theta_\mu^{(a)}, \quad (1.4.5a)$$

$$\theta_\mu^{(b)} = \eta^{(a)(b)} e_{(a)\mu}, \quad g_{\mu\nu} = e_{(a)\mu} \theta_\nu^{(a)}, \quad \eta^{(a)(b)} = \theta_\mu^{(a)} \theta_\nu^{(b)} g^{\mu\nu}. \quad (1.4.5b)$$

1.4.2 Tetrad transformations

Instead of the above found local tetrad that was directly constructed from the spacetime metric, we can also use any other local tetrad

$$\hat{\mathbf{e}}_{(i)} = \mathbf{A}_i^k \mathbf{e}_{(k)}, \quad (1.4.6)$$

where \mathbf{A} is an element of the Lorentz group $O(1,3)$. Hence $\mathbf{A}^T \boldsymbol{\eta} \mathbf{A} = \boldsymbol{\eta}$ and $(\det \mathbf{A})^2 = 1$.

Lorentz-transformation in the direction $n^a = (\sin \chi \cos \xi, \sin \chi \sin \xi, \cos \xi)^T = n_a$ with $\gamma = 1/\sqrt{1-\beta^2}$,

$$\Lambda_0^0 = \gamma, \quad \Lambda_a^0 = -\beta \gamma n_a, \quad \Lambda_0^a = -\beta \gamma n^a, \quad \Lambda_b^a = (\gamma - 1) n^a n_b + \delta_b^a. \quad (1.4.7)$$

1.4.3 Ricci rotation-, connection-, and structure coefficients

The Ricci rotation coefficients $\gamma_{(i)(j)(k)}$ with respect to the local tetrad $\mathbf{e}_{(i)}$ are defined by

$$\gamma_{(i)(j)(k)} := g_{\mu\lambda} e_{(i)}^\mu \nabla_{\mathbf{e}_{(k)}} e_{(j)}^\lambda = g_{\mu\lambda} e_{(i)}^\mu e_{(k)}^\nu \nabla_{\mathbf{e}_{(j)}} e_{(k)}^\lambda = g_{\mu\lambda} e_{(i)}^\mu e_{(k)}^\nu \left(\partial_\nu e_{(j)}^\lambda + \Gamma_{\nu\beta}^\lambda e_{(j)}^\beta \right). \quad (1.4.8)$$

They are antisymmetric in the first two indices, $\gamma_{(i)(j)(k)} = -\gamma_{(j)(i)(k)}$, which follows from the definition, Eq. (1.4.8), and the relation

$$0 = \partial_\mu \eta_{(i)(j)} = \nabla_\mu \left(g_{\beta\nu} e_{(i)}^\beta e_{(j)}^\nu \right), \quad (1.4.9)$$

where $\nabla_\mu g_{\beta\nu} = 0$, compare [Cha06]. Otherwise, we have

$$\gamma_{(j)(k)}^{(i)} = \theta_\lambda^{(i)} e_{(k)}^\nu \nabla_{\mathbf{e}_{(j)}} e_{(k)}^\lambda = -e_{(j)}^\lambda e_{(k)}^\nu \nabla_{\mathbf{e}_{(j)}} \theta_\lambda^{(i)}. \quad (1.4.10)$$

The contraction of the first and the last index is given by

$$\gamma_{(j)} = \gamma_{(j)(k)}^{(k)} = \eta^{(k)(i)} \gamma_{(i)(j)(k)} = -\gamma_{(0)(j)(0)} + \gamma_{(1)(j)(1)} + \gamma_{(2)(j)(2)} + \gamma_{(3)(j)(3)} = \nabla_{\mathbf{e}_{(j)}}. \quad (1.4.11)$$

The connection coefficients $\omega_{(j)(n)}^{(m)}$ with respect to the local tetrad $\mathbf{e}_{(i)}$ are defined by

$$\omega_{(j)(n)}^{(m)} := \theta_\mu^{(m)} \nabla_{\mathbf{e}_{(j)}} e_{(n)}^\mu = \theta_\mu^{(m)} e_{(j)}^\alpha \nabla_{\mathbf{e}_{(n)}} e_{(n)}^\mu = \theta_\mu^{(m)} e_{(j)}^\alpha \left(\partial_\alpha e_{(n)}^\mu + \Gamma_{\alpha\beta}^\mu e_{(n)}^\beta \right), \quad (1.4.12)$$

compare Nakahara[Nak90]. They are related to the Ricci rotation coefficients via

$$\gamma_{(i)(j)(k)} = \eta_{(i)(m)} \omega_{(k)(j)}^{(m)}. \quad (1.4.13)$$

Furthermore, the local tetrad has a non-vanishing Lie-bracket $[X, Y]^\nu = X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu$. Thus,

$$[\mathbf{e}_{(i)}, \mathbf{e}_{(j)}] = c_{(i)(j)}^{(k)} \mathbf{e}_{(k)} \quad \text{or} \quad c_{(i)(j)}^{(k)} = \theta^{(k)} [\mathbf{e}_{(i)}, \mathbf{e}_{(j)}]. \quad (1.4.14)$$

The structure coefficients $c_{(i)(j)}^{(k)}$ are related to the connection coefficients or the Ricci rotation coefficients via

$$c_{(i)(j)}^{(k)} = \omega_{(i)(j)}^{(k)} - \omega_{(j)(i)}^{(k)} = \eta^{(k)(m)} (\gamma_{(m)(j)(i)} - \gamma_{(m)(i)(j)}) = \gamma_{(j)(i)}^{(k)} - \gamma_{(i)(j)}^{(k)}. \quad (1.4.15)$$

1.4.4 Riemann-, Ricci-, and Weyl-tensor with respect to a local tetrad

The transformations between the coordinate representations of the Riemann-, Ricci-, and Weyl-tensors and their representation with respect to a local tetrad $\mathbf{e}_{(i)}$ are given by

$$R_{(a)(b)(c)(d)} = R_{\mu\nu\rho\sigma} e_{(a)}^\mu e_{(b)}^\nu e_{(c)}^\rho e_{(d)}^\sigma, \quad (1.4.16a)$$

$$R_{(a)(b)} = R_{\mu\nu} e_{(a)}^\mu e_{(b)}^\nu, \quad (1.4.16b)$$

$$\begin{aligned} C_{(a)(b)(c)(d)} &= C_{\mu\nu\rho\sigma} e_{(a)}^\mu e_{(b)}^\nu e_{(c)}^\rho e_{(d)}^\sigma \\ &= R_{(a)(b)(c)(d)} - \frac{1}{2} (\eta_{(a)[(c} R_{d)](b)} - \eta_{(b)[(c} R_{d)](a)}) + \frac{R}{3} \eta_{(a)[(c} \eta_{d)](b)}. \end{aligned} \quad (1.4.16c)$$

1.4.5 Null or timelike directions

A null or timelike direction $v = v^{(i)} \mathbf{e}_{(i)}$ with respect to a local tetrad $\mathbf{e}_{(i)}$ can be written as

$$v = v^{(0)} \mathbf{e}_{(0)} + \psi (\sin \chi \cos \xi \mathbf{e}_{(1)} + \sin \chi \sin \xi \mathbf{e}_{(2)} + \cos \chi \mathbf{e}_{(3)}) = v^{(0)} \mathbf{e}_{(0)} + \psi \mathbf{n}. \quad (1.4.17)$$

In the case of a null direction we have $\psi = 1$ and $v^{(0)} = \pm 1$. A timelike direction can be identified with an initial four-velocity $\mathbf{u} = c\gamma(\mathbf{e}_{(0)} + \beta \mathbf{n})$, where

$$\mathbf{u}^2 = \langle \mathbf{u}, \mathbf{u} \rangle_{\mathbf{g}} = c^2 \gamma^2 \langle \mathbf{e}_{(0)} + \beta \mathbf{n}, \mathbf{e}_{(0)} + \beta \mathbf{n} \rangle = c^2 \gamma^2 (-1 + \beta^2) = \mp c^2, \quad \text{sign}(\mathbf{g}) = \pm 2. \quad (1.4.18)$$

Thus, $\psi = c\beta\gamma$ and $v^{(0)} = \pm c\gamma$. The sign of $v^{(0)}$ determines the time direction.

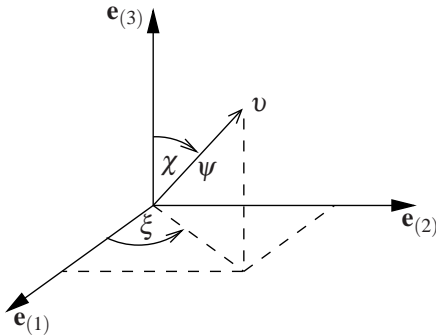


Figure 1.1: Null or timelike direction v with respect to the local tetrad $\mathbf{e}_{(i)}$.

The transformations between a local direction $v^{(i)}$ and its coordinate representation v^μ read

$$v^\mu = v^{(i)} e_{(i)}^\mu \quad \text{and} \quad v^{(i)} = \theta_\mu^{(i)} v^\mu. \quad (1.4.19)$$

1.4.6 Local tetrad for diagonal metrics

If a spacetime is represented by a diagonal metric

$$ds^2 = g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2, \quad (1.4.20)$$

the natural local tetrad reads

$$\mathbf{e}_{(0)} = \frac{1}{\sqrt{g_{00}}} \partial_0, \quad \mathbf{e}_{(1)} = \frac{1}{\sqrt{g_{11}}} \partial_1, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{g_{22}}} \partial_2, \quad \mathbf{e}_{(3)} = \frac{1}{\sqrt{g_{33}}} \partial_3, \quad (1.4.21)$$

given that the metric coefficients are well behaved. Analogously, the dual tetrad reads

$$\theta^{(0)} = \sqrt{g_{00}} dx^0, \quad \theta^{(1)} = \sqrt{g_{11}} dx^1, \quad \theta^{(2)} = \sqrt{g_{22}} dx^2, \quad \theta^{(3)} = \sqrt{g_{33}} dx^3. \quad (1.4.22)$$

1.4.7 Local tetrad for stationary axisymmetric spacetimes

The line element of a stationary axisymmetric spacetime is given by

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dt d\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\vartheta\vartheta}d\vartheta^2, \quad (1.4.23)$$

where the metric components are functions of r and ϑ only.

The local tetrad for an observer on a stationary circular orbit, ($r = \text{const}$, $\vartheta = \text{const}$), with four velocity $\mathbf{u} = c\Gamma(\partial_t + \zeta\partial_\varphi)$ can be defined as, compare Bini[BJ00],

$$\mathbf{e}_{(0)} = \Gamma(\partial_t + \zeta\partial_\varphi), \quad \mathbf{e}_{(1)} = \frac{1}{\sqrt{g_{rr}}}\partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{g_{\vartheta\vartheta}}}\partial_\vartheta, \quad (1.4.24a)$$

$$\mathbf{e}_{(3)} = \Delta\Gamma[\pm(g_{t\varphi} + \zeta g_{\varphi\varphi})\partial_t \mp (g_{tt} + \zeta g_{t\varphi})\partial_\varphi], \quad (1.4.24b)$$

where

$$\Gamma = \frac{1}{\sqrt{-(g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi})}} \quad \text{and} \quad \Delta = \frac{1}{\sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}. \quad (1.4.25)$$

The angular velocity ζ is limited due to $g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi} < 0$

$$\zeta_{\min} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \quad \text{and} \quad \zeta_{\max} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \quad (1.4.26)$$

with $\omega = -g_{t\varphi}/g_{\varphi\varphi}$.

For $\zeta = 0$, the observer is static with respect to spatial infinity. The locally non-rotating frame (LNRF) has angular velocity $\zeta = \omega$, see also MTW[MTW73], exercise 33.3.

Static limit: $\zeta_{\min} = 0 \Rightarrow g_{tt} = 0$.

The transformation between the local direction $v^{(i)}$ and the coordinate direction v^μ reads

$$v^0 = \Gamma(v^{(0)} \pm v^{(3)}\Delta w_1), \quad v^1 = \frac{v^{(1)}}{\sqrt{g_{rr}}}, \quad v^2 = \frac{v^{(2)}}{\sqrt{g_{\vartheta\vartheta}}}, \quad v^3 = \Gamma(v^{(0)}\zeta \mp v^{(3)}\Delta w_2), \quad (1.4.27)$$

with

$$w_1 = g_{t\varphi} + \zeta g_{\varphi\varphi} \quad \text{and} \quad w_2 = g_{tt} + \zeta g_{t\varphi}. \quad (1.4.28)$$

The back transformation reads

$$v^{(0)} = \frac{1}{\Gamma} \frac{v^0 w_2 + v^3 w_1}{\zeta w_1 + w_2}, \quad v^{(1)} = \sqrt{g_{rr}} v^1, \quad v^{(2)} = \sqrt{g_{\vartheta\vartheta}} v^2, \quad v^{(3)} = \pm \frac{1}{\Delta\Gamma} \frac{\zeta v^0 - v^3}{\zeta w_1 + w_2}. \quad (1.4.29)$$

Note, to obtain a right-handed local tetrad, $\det(e_{(i)}^\mu) > 0$, the upper sign has to be used.

1.5 Newman-Penrose tetrad and spin-coefficients

The Newman-Penrose tetrad consists of four null vectors $\mathbf{e}_{(i)}^* = \{\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}\}$, where \mathbf{l} and \mathbf{n} are real and \mathbf{m} and $\bar{\mathbf{m}}$ are complex conjugates; see Penrose and Rindler[PR84] or Chandrasekhar[Cha06] for a thorough discussion. The Newman-Penrose (NP) tetrad has to fulfill the orthonormality relation

$$\langle \mathbf{e}_{(i)}^*, \mathbf{e}_{(j)}^* \rangle = \eta_{(i)(j)}^* \quad \text{with} \quad \eta_{(i)(j)}^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (1.5.1)$$

A straightforward relation between the NP tetrad and the natural local tetrad, as discussed in Sec. 1.4, is given by

$$\mathbf{l} = \mp \frac{1}{\sqrt{2}} (\mathbf{e}_{(0)} + \mathbf{e}_{(1)}), \quad \mathbf{n} = \mp \frac{1}{\sqrt{2}} (\mathbf{e}_{(0)} - \mathbf{e}_{(1)}), \quad \mathbf{m} = \mp \frac{1}{\sqrt{2}} (\mathbf{e}_{(2)} + i\mathbf{e}_{(3)}), \quad (1.5.2)$$

where the upper/lower sign has to be used for metrics with positive/negative signature. The Ricci rotation-coefficients of a NP tetrad are now called *spin coefficients* and are designated by specific symbols:

$$\kappa = \gamma_{(2)(1)(1)}, \quad \rho = \gamma_{(2)(0)(3)}, \quad \varepsilon = \frac{1}{2} (\gamma_{(1)(0)(0)} + \gamma_{(2)(3)(0)}), \quad (1.5.3a)$$

$$\sigma = \gamma_{(2)(0)(2)}, \quad \mu = \gamma_{(1)(3)(2)}, \quad \gamma = \frac{1}{2} (\gamma_{(1)(0)(1)} + \gamma_{(2)(3)(1)}), \quad (1.5.3b)$$

$$\lambda = \gamma_{(1)(3)(3)}, \quad \tau = \gamma_{(2)(0)(1)}, \quad \alpha = \frac{1}{2} (\gamma_{(1)(0)(3)} + \gamma_{(2)(3)(3)}), \quad (1.5.3c)$$

$$\nu = \gamma_{(1)(3)(1)}, \quad \pi = \gamma_{(1)(3)(0)}, \quad \beta = \frac{1}{2} (\gamma_{(1)(0)(2)} + \gamma_{(2)(3)(2)}). \quad (1.5.3d)$$

1.6 Coordinate relations

1.6.1 Spherical and Cartesian coordinates

The well-known relation between the spherical coordinates (r, ϑ, φ) and the Cartesian coordinates (x, y, z) , compare Fig. 1.2, are

$$x = r \sin \vartheta \cos \varphi, \quad y = r \sin \vartheta \sin \varphi, \quad z = r \cos \vartheta, \quad (1.6.1)$$

and

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \vartheta = \arctan 2(\sqrt{x^2 + y^2}, z), \quad \varphi = \arctan 2(y, x), \quad (1.6.2)$$

where $\arctan 2()$ ensures that $\varphi \in [0, 2\pi)$ and $\vartheta \in (0, \pi)$.

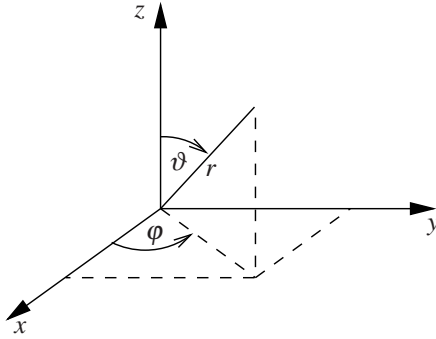


Figure 1.2: Relation between spherical and Cartesian coordinates.

The total differentials of the spherical coordinates read

$$dr = \frac{x dx + y dy + z dz}{r}, \quad d\vartheta = \frac{xz dx + yz dy - (x^2 + y^2) dz}{r^2 \sqrt{x^2 + y^2}}, \quad d\varphi = \frac{-y dx + x dy}{x^2 + y^2}, \quad (1.6.3)$$

whereas the coordinate derivatives read

$$\partial_r = \frac{\partial x}{\partial r} \partial_x + \frac{\partial y}{\partial r} \partial_y + \frac{\partial z}{\partial r} \partial_z = \sin \vartheta \cos \varphi \partial_x + \sin \vartheta \sin \varphi \partial_y + \cos \vartheta \partial_z, \quad (1.6.4a)$$

$$\partial_\vartheta = \frac{\partial x}{\partial \vartheta} \partial_x + \frac{\partial y}{\partial \vartheta} \partial_y + \frac{\partial z}{\partial \vartheta} \partial_z = r \cos \vartheta \cos \varphi \partial_x + r \cos \vartheta \sin \varphi \partial_y - r \sin \vartheta \partial_z, \quad (1.6.4b)$$

$$\partial_\varphi = \frac{\partial x}{\partial \varphi} \partial_x + \frac{\partial y}{\partial \varphi} \partial_y + \frac{\partial z}{\partial \varphi} \partial_z = -r \sin \vartheta \sin \varphi \partial_x + r \sin \vartheta \cos \varphi \partial_y, \quad (1.6.4c)$$

and

$$\partial_x = \frac{\partial r}{\partial x} \partial_r + \frac{\partial \vartheta}{\partial x} \partial_\vartheta + \frac{\partial \varphi}{\partial x} \partial_\varphi = \sin \vartheta \cos \varphi \partial_r + \frac{\cos \vartheta \cos \varphi}{r} \partial_\vartheta - \frac{\sin \varphi}{r \sin \vartheta} \partial_\varphi, \quad (1.6.5a)$$

$$\partial_y = \frac{\partial r}{\partial y} \partial_r + \frac{\partial \vartheta}{\partial y} \partial_\vartheta + \frac{\partial \varphi}{\partial y} \partial_\varphi = \sin \vartheta \sin \varphi \partial_r + \frac{\cos \vartheta \sin \varphi}{r} \partial_\vartheta + \frac{\cos \varphi}{r \sin \vartheta} \partial_\varphi, \quad (1.6.5b)$$

$$\partial_z = \frac{\partial r}{\partial z} \partial_r + \frac{\partial \vartheta}{\partial z} \partial_\vartheta + \frac{\partial \varphi}{\partial z} \partial_\varphi = \cos \vartheta \partial_r - \frac{\sin \vartheta}{r} \partial_\vartheta. \quad (1.6.5c)$$

1.6.2 Cylindrical and Cartesian coordinates

The relation between cylindrical coordinates (r, φ, z) and Cartesian coordinates (x, y, z) is given by

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad \text{and} \quad r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan 2(y, x), \quad (1.6.6)$$

where $\arctan 2()$ again ensures that the angle $\varphi \in [0, 2\pi)$.

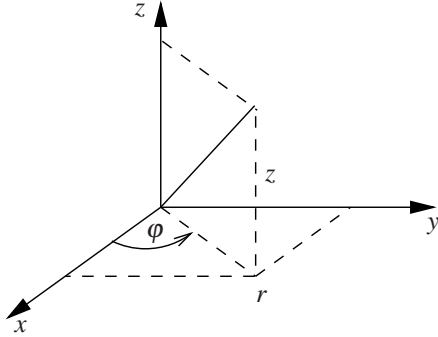


Figure 1.3: Relation between cylindrical and Cartesian coordinates.

The total differentials of the spherical coordinates are given by

$$dr = \frac{xdx + ydy}{r}, \quad d\varphi = \frac{-ydx + xdy}{r^2}, \quad (1.6.7)$$

and

$$dx = \cos \varphi dr - r \sin \varphi d\varphi, \quad dy = \sin \varphi dr + r \cos \varphi d\varphi. \quad (1.6.8)$$

The coordinate derivatives are

$$\partial_r = \frac{\partial x}{\partial r} \partial_x + \frac{\partial y}{\partial r} \partial_y = \cos \varphi \partial_x + \sin \varphi \partial_y, \quad (1.6.9a)$$

$$\partial_\varphi = \frac{\partial x}{\partial \varphi} \partial_x + \frac{\partial y}{\partial \varphi} \partial_y = -r \sin \varphi \partial_x + r \cos \varphi \partial_y, \quad (1.6.9b)$$

and

$$\partial_x = \frac{\partial r}{\partial x} \partial_r + \frac{\partial \varphi}{\partial x} \partial_\varphi = \cos \varphi \partial_r - \frac{\sin \varphi}{r} \partial_\varphi, \quad (1.6.10a)$$

$$\partial_y = \frac{\partial r}{\partial y} \partial_r + \frac{\partial \varphi}{\partial y} \partial_\varphi = \sin \varphi \partial_r + \frac{\cos \varphi}{r} \partial_\varphi. \quad (1.6.10b)$$

1.7 Embedding diagram

A two-dimensional hypersurface with line segment

$$d\sigma^2 = g_{rr}(r)dr^2 + g_{\varphi\varphi}(r)d\varphi^2 \quad (1.7.1)$$

can be embedded in a three-dimensional Euclidean space with cylindrical coordinates,

$$d\sigma^2 = \left[1 + \left(\frac{dz}{d\rho} \right)^2 \right] d\rho^2 + \rho^2 d\varphi^2. \quad (1.7.2)$$

With $\rho(r)^2 = g_{\varphi\varphi}(r)$ and $dr = (dr/d\rho)d\rho$, we obtain for the embedding function $z = z(r)$,

$$\frac{dz}{dr} = \pm \sqrt{g_{rr} - \left(\frac{d\sqrt{g_{\varphi\varphi}}}{dr} \right)^2}. \quad (1.7.3)$$

If $g_{\varphi\varphi}(r) = r^2$, then $d\sqrt{g_{\varphi\varphi}}/dr = 1$.

1.8 Equations of motion and transport equations

1.8.1 Geodesic equation

The geodesic equation reads

$$\frac{D^2 x^\mu}{d\lambda^2} = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \quad (1.8.1)$$

with the affine parameter λ . For timelike geodesics, however, we replace the affine parameter by the proper time τ .

The geodesic equation (1.8.1) is a system of ordinary differential equations of second order. Hence, to solve these differential equations, we need an initial position $x^\mu(\lambda = 0)$ as well as an initial direction $(dx^\mu/d\lambda)(\lambda = 0)$. This initial direction has to fulfill the constraint equation

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \kappa c^2, \quad (1.8.2)$$

where $\kappa = 0$ for lightlike and $\kappa = \mp 1$, ($\text{sign}(\mathbf{g}) = \pm 2$), for timelike geodesics.

The initial direction can also be determined by means of a local reference frame, compare sec. 1.4.5, that automatically fulfills the constraint equation (1.8.2). If we use the natural local tetrad as local reference frame, we have

$$\left. \frac{dx^\mu}{d\lambda} \right|_{\lambda=0} = v^\mu = v^{(i)} e_{(i)}^\mu. \quad (1.8.3)$$

1.8.2 Fermi-Walker transport

The Fermi-Walker transport, see e.g. Stephani[SS90], of a vector $\mathbf{X} = X^\mu \partial_\mu$ along the worldline $x^\mu(\tau)$ with four-velocity $\mathbf{u} = u^\mu(\tau) \partial_\mu$ is given by $\mathbb{F}_{\mathbf{u}} X^\mu = 0$ with

$$\mathbb{F}_{\mathbf{u}} X^\mu := \frac{dX^\mu}{d\tau} + \Gamma_{\rho\sigma}^\mu u^\rho X^\sigma + \frac{1}{c^2} (u^\sigma a^\mu - a^\sigma u^\mu) g_{\rho\sigma} X^\rho. \quad (1.8.4)$$

The four-acceleration follows from the four-velocity via

$$a^\mu = \frac{D^2 x^\mu}{d\tau^2} = \frac{Du^\mu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma_{\rho\sigma}^\mu u^\rho u^\sigma. \quad (1.8.5)$$

1.8.3 Parallel transport

If the four-acceleration vanishes, the Fermi-Walker transport simplifies to the parallel transport $\mathbb{P}_{\mathbf{u}} X^\mu = 0$ with

$$\mathbb{P}_{\mathbf{u}} X^\mu := \frac{DX^\mu}{d\tau} = \frac{dX^\mu}{d\tau} + \Gamma_{\rho\sigma}^\mu u^\rho X^\sigma. \quad (1.8.6)$$

1.8.4 Euler-Lagrange formalism

A detailed discussion of the Euler-Lagrange formalism can be found, e.g., in Rindler[Rin01]. The Lagrangian \mathcal{L} is defined as

$$\mathcal{L} := g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \quad \mathcal{L} \stackrel{!}{=} \kappa c^2, \quad (1.8.7)$$

where x^μ are the coordinates of the metric, and the dot means differentiation with respect to the affine parameter λ . For timelike geodesics, $\kappa = \mp 1$ depending on the signature of the metric, $\text{sign}(\mathbf{g}) = \pm 2$. For lightlike geodesics, $\kappa = 0$.

The Euler-Lagrange equations read

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0. \quad (1.8.8)$$

If \mathcal{L} is independent of x^ρ , then x^ρ is a cyclic variable and

$$p_\rho = g_{\rho\nu} \dot{x}^\nu = \text{const}. \quad (1.8.9)$$

Note that $[\mathcal{L}]_{\text{U}} = \frac{\text{length}^2}{\text{time}^2}$ for timelike and $[\mathcal{L}]_{\text{U}} = 1$ for lightlike geodesics, see Sec. 1.9.

1.8.5 Hamilton formalism

The super-Hamiltonian \mathcal{H} is defined as

$$\mathcal{H} := \frac{1}{2} g^{\mu\nu} p_\mu p_\nu, \quad \mathcal{H} \stackrel{!}{=} \frac{1}{2} \kappa c^2, \quad (1.8.10)$$

where $p_\mu = g_{\mu\nu} \dot{x}^\nu$ are the canonical momenta, see e.g. MTW[MTW73], para. 21.1. As in classical mechanics, we have

$$\frac{dx^\mu}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_\mu} \quad \text{and} \quad \frac{dp_\mu}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^\mu}. \quad (1.8.11)$$

1.9 Units

A first test in analyzing whether an equation is correct is to check the units. Newton's gravitational constant G , for example, has the following units

$$[G]_{\text{U}} = \frac{\text{length}^3}{\text{mass} \cdot \text{time}^2}, \quad (1.9.1)$$

where $[\cdot]_{\text{U}}$ indicates that we evaluate the units of the enclosed expression. Further examples are

$$[ds]_{\text{U}} = \text{length}, \quad [\mathbf{u}]_{\text{U}} = \frac{\text{length}}{\text{time}}, \quad [R_{\text{rrr}}^{\text{Schwarzschild}}]_{\text{U}} = \frac{1}{\text{time}^2}, \quad [R_{\vartheta\varphi\vartheta\varphi}^{\text{Schwarzschild}}]_{\text{U}} = \text{length}^2. \quad (1.9.2)$$

1.10 Tools

1.10.1 Maple/GRTensorII

The Christoffel symbols, the Riemann- and Ricci-tensors as well as the Ricci and Kretschmann scalars in this catalogue were determined by means of the software Maple together with the GRTensorII package by Musgrave, Pollney, and Lake.²

A typical worksheet to enter a new metric may look like this:

²The commercial software Maple can be found here: <http://www.maplesoft.com>. The GRTensorII-package is free: <http://grtensor.phy.queensu.ca>.

```

> grtw();
> makeg(Schwarzschild);

Makeg 2.0: GRTensor metric/basis entry utility
To quit makeg, type 'exit' at any prompt.
Do you wish to enter a 1) metric [g(dn,dn)],
                        2) line element [ds],
                        3) non-holonomic basis [e(1)...e(n)], or
                        4) NP tetrad [l,n,m,mbar]?

> 2:

Enter coordinates as a LIST (eg. [t,r,theta,phi]):
> [t,r,theta,phi]:

Enter the line element using d[coord] to indicate differentials.
(for example, r^2*(d[theta]^2 + sin(theta)^2*d[phi]^2)
[Type 'exit' to quit makeg]
ds^2 =

If there are any complex valued coordinates, constants or functions
for this spacetime, please enter them as a SET ( eg. { z, psi } ).

Complex quantities [default={}]:
> {}:

You may choose to 0) Use the metric WITHOUT saving it,
                  1) Save the metric as it is,
                  2) Correct an element of the metric,
                  3) Re-enter the metric,
                  4) Add/change constraint equations,
                  5) Add a text description, or
                  6) Abandon this metric and return to Maple.

> 0:

```

The worksheets for some of the metrics in this catalogue can be found on the authors homepage. To determine the objects that are defined with respect to a local tetrad, the metric must be given as non-holonomic basis.

The various basic objects can be determined via

Christoffel symbols $\Gamma_{\nu\rho}^{\mu}$	<code>gcalc(Chr2);</code>	<code>gcalc(Chr(dn,dn,up));</code>
partial derivatives $\Gamma_{\nu\rho,\sigma}^{\mu}$		<code>gcalc(Chr(dn,dn,up,pdn));</code>
Riemann tensor $R_{\mu\nu\rho\sigma}$	<code>gcalc(Riemman);</code>	<code>gcalc(R(dn,dn,dn,dn));</code>
Ricci tensor $R_{\mu\nu}$	<code>gcalc(Ricci);</code>	<code>gcalc(R(dn,dn));</code>
Ricci scalar \mathcal{R}	<code>gcalc(Ricciscalar);</code>	
Kretschmann scalar \mathcal{K}	<code>gcalc(RiemSq);</code>	

1.10.2 Mathematica

The calculation of the Christoffel symbols, the Riemann- or Ricci-tensor within *Mathematica* could read like this:

```

Clearing the values of symbols:
In[1]:= Clear[coord, metric, inversemetric, affine,
           t, r, Theta, Phi]

Setting the dimension:
In[2]:= n := 4

Defining a list of coordinates:
In[3]:= coord := {t, r, Theta, Phi}

Defining the metric:
In[4]:= metric := {{-(1 - rs/r) c^2, 0, 0, 0},
                   {0, 1/(1 - rs/r), 0, 0},
                   {0, 0, r^2, 0},
                   {0, 0, 0, r ^2 Sin[Theta]^2}}

In[5]:= metric // MatrixForm

```

Calculating the inverse metric:

```
In[6]:= inversemetric := Simplify[Inverse[metric]]

In[7]:= inversemetric // MatrixForm
```

Calculating the Christoffel symbols of the second kind:

```
In[8]:= affine := affine = Simplify[
  Table[(1/2) Sum[inversemetric[[Mu, Rho]] (
    D[metric[[Rho, Nu]], coord[[Lambda]]] +
    D[metric[[Rho, Lambda]], coord[[Nu]]] -
    D[metric[[Nu, Lambda]], coord[[Rho]]]),
  {Rho, 1, n}], {Nu, 1, n}, {Lambda, 1, n}, {Mu, 1, n}]]
```

Displaying the Christoffel symbols of the second kind:

```
In[9]:= listaffine :=
  Table[If[UnsameQ[affine[[Nu, Lambda, Mu]], 0],
  {Style[Subsuperscript[\[CapitalGamma],
  Row[{coord[[Nu]], coord[[Lambda]]}], coord[[Mu]], 18],
  "=",
  Style[affine[[Nu, Lambda, Mu]], 14]}],
  {Lambda, 1, n}, {Nu, 1, n}, {Mu, 1, n}]]

In[10]:= TableForm[Partition[DeleteCases[Flatten[listaffine],
  Null], 3],
  TableSpacing -> {1, 2}]
```

Defining the Riemann tensor:

```
In[11]:= riemann := riemann =
  Table[D[affine[[Nu, Sigma, Mu]], coord[[Rho]]] -
  D[affine[[Nu, Rho, Mu]], coord[[Sigma]]] +
  Sum[affine[[Rho, Lambda, Mu]]
  affine[[Nu, Sigma, Lambda]] -
  affine[[Sigma, Lambda, Mu]]
  affine[[Nu, Rho, Lambda]],
  {Lambda, 1, n},
  {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]]
```

Defining the Riemann tensor with lower indices:

```
In[12]:= riemannDn := riemannDn =
  Table[Simplify[
  Sum[metric[[Mu, Kappa]] riemann[[Kappa, Nu, Rho, Sigma]],
  {Kappa, 1, n}],
  {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]]

In[13]:= listRiemann :=
  Table[If[UnsameQ[riemannDn[[Mu, Nu, Rho, Sigma]], 0],
  {Style[Subscript[R, Row[{coord[[Mu]], coord[[Nu]], coord[[Rho]],
  coord[[Sigma]]}], 16], "=",
  riemannDn[[Mu, Nu, Rho, Sigma]]}],
  {Nu, 1, n}, {Mu, 1, n}, {Sigma, 1, n}, {Rho, 1, n}]]

In[14]:= TableForm[Partition[DeleteCases[Flatten[listRiemann],
  Null], 3],
  TableSpacing -> {2, 2}]
```

Defining the Ricci tensor:

```
In[15]:= ricci := ricci =
  Table[Simplify[
  Sum[riemann[[Rho, Mu, Rho, Nu]], {Rho, 1, n}],
  {Mu, 1, n}, {Nu, 1, n}]]

In[16]:= listRicci :=
  Table[If[UnsameQ[ricci[[Mu, Nu]], 0],
  {Style[Subscript[R, Row[{coord[[Mu]], coord[[Nu]]}], 16],
  "=",
  Style[ricci[[Mu, Nu]], 16]}], {Nu, 1, n}, {Mu, 1, n}]]

In[17]:= TableForm[Partition[DeleteCases[Flatten[listRicci],
  Null], 3],
  TableSpacing -> {1, 2}]
```

Defining the Ricci scalar:

```
In[18]:= ricciscalar := ricciscalar =
  Simplify[Sum[
```



```

Sum[inversemetric[[Mu, Nu]] ricci[[Nu, Mu]],
{Mu, 1, n}], {Nu, 1, n}]]

Defining the Kretschmann scalar:
In[19]:= riemannUp := riemannUp =
Table[Simplify[
Sum[inversemetric[[Nu, Kappa]]
riemann[[Mu, Kappa, Rho, Sigma]], {Kappa, 1, n}]],
{Mu, 1, n}], {Nu, 1, n}], {Rho, 1, n}], {Sigma, 1, n}]

In[20]:= kretschmann := kretschmann =
Simplify[Sum[ Sum[Sum[Sum[
riemannUp[[Mu, Nu, Rho, Sigma]]
riemannUp[[Rho, Sigma, Mu, Nu]],
{Mu, 1, n}]], {Nu, 1, n}], {Rho, 1, n}], {Sigma, 1, n}]]

```

Some example notebooks can be found on the authors homepage.

1.10.3 Maxima

Instead of using commercial software like *Maple* or *Mathematica*, Maxima also offers a tensor package that helps to calculate the Christoffel symbols etc. The above example for the Schwarzschild metric can be written as a maxima worksheet as follows:

```

/* load ctensor package */
load(ctensor);

/* define coordinates to use */
ct_coords:[t,r,theta,phi];

/* start with the identity metric */
lg:ident(4);
lg[1,1]:c^2*(1-rs/r);
lg[2,2]:-1/(1-rs/r);
lg[3,3]:-r^2;
lg[4,4]:-r^2*sin(theta)^2;
cmetric();

/* calculate the christoffel symbols of the second kind */
christof(mcs);

/* calculate the riemann tensor */
lriemann(mcs);

/* calculate the ricci tensor */
ricci(mcs);

/* calculate the ricci scalar */
scurvature();

/* calculate the Kretschmann scalar */
uriemann(mcs);
rinvariant();
ratsimp(%);

```

As you may have noticed, the Schwarzschild metric must be given with negative signature.

Chapter 2

Spacetimes

2.1 Minkowski

2.1.1 Cartesian coordinates

The Minkowski metric in Cartesian coordinates $\{t, x, y, z \in \mathbb{R}\}$ reads

$$\boxed{ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2.} \quad (2.1.1)$$

All Christoffel symbols as well as the Riemann- and Ricci-tensor vanish identically. The natural local tetrad is trivial,

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(x)} = \partial_x, \quad \mathbf{e}_{(y)} = \partial_y, \quad \mathbf{e}_{(z)} = \partial_z, \quad (2.1.2)$$

with dual

$$\theta^{(t)} = c dt, \quad \theta^{(x)} = dx, \quad \theta^{(y)} = dy, \quad \theta^{(z)} = dz. \quad (2.1.3)$$

2.1.2 Cylindrical coordinates

The Minkowski metric in cylindrical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \varphi \in [0, 2\pi), z \in \mathbb{R}\}$,

$$\boxed{ds^2 = -c^2 dt^2 + dr^2 + r^2 d\varphi^2 + dz^2,} \quad (2.1.4)$$

has the natural local tetrad

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r} \partial_\varphi, \quad \mathbf{e}_{(z)} = \partial_z. \quad (2.1.5)$$

Christoffel symbols:

$$\Gamma_{\varphi\varphi}^r = -r, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}. \quad (2.1.6)$$

Partial derivatives

$$\Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\varphi\varphi,r}^r = -1. \quad (2.1.7)$$

Ricci rotation coefficients:

$$\gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r} \quad \text{and} \quad \gamma_{(r)} = \frac{1}{r}. \quad (2.1.8)$$

2.1.3 Spherical coordinates

In spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$, the Minkowski metric reads

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (2.1.9)$$

Christoffel symbols:

$$\Gamma_{\vartheta\vartheta}^r = -r, \quad \Gamma_{\varphi\varphi}^r = -r \sin^2 \vartheta, \quad \Gamma_{r\vartheta}^{\vartheta} = \frac{1}{r}, \quad (2.1.10a)$$

$$\Gamma_{\varphi\varphi}^{\vartheta} = -\sin \vartheta \cos \vartheta, \quad \Gamma_{r\varphi}^{\varphi} = \frac{1}{r}, \quad \Gamma_{\vartheta\varphi}^{\varphi} = \cot \vartheta. \quad (2.1.10b)$$

Partial derivatives

$$\Gamma_{r\vartheta,r}^{\vartheta} = -\frac{1}{r^2}, \quad \Gamma_{\varphi\varphi,r}^{\varphi} = -\frac{1}{r^2}, \quad \Gamma_{\vartheta\vartheta,r}^r = -1, \quad (2.1.11a)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^{\varphi} = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,r}^r = -\sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^{\vartheta} = -\cos(2\vartheta), \quad (2.1.11b)$$

$$\Gamma_{\varphi\varphi,\vartheta}^r = -\sin(2\vartheta). \quad (2.1.11c)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_{\varphi}. \quad (2.1.12)$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{r}. \quad (2.1.13)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2}{r}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{r}. \quad (2.1.14)$$

2.1.4 Conform-compactified coordinates

The Minkowski metric in conform-compactified coordinates $\{\psi \in [-\pi, \pi], \xi \in (0, \pi), \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ reads[HE99]

$$ds^2 = -d\psi^2 + d\xi^2 + \sin^2 \xi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (2.1.15)$$

This form follows from the spherical Minkowski metric (2.1.9) by means of the coordinate transformation

$$ct + r = \tan \frac{\psi + \xi}{2}, \quad ct - r = \tan \frac{\psi - \xi}{2}, \quad (2.1.16)$$

resulting in the metric

$$d\tilde{s}^2 = \frac{-d\psi^2 + d\xi^2}{4 \cos^2 \frac{\psi + \xi}{2} \cos^2 \frac{\psi - \xi}{2}} + \frac{\sin^2 \xi}{4 \cos^2 \frac{\psi + \xi}{2} \cos^2 \frac{\psi - \xi}{2}} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.1.17)$$

and by the conformal transformation $ds^2 = \Omega^2 d\tilde{s}^2$ with $\Omega^2 = 4 \cos^2 \frac{\psi + \xi}{2} \cos^2 \frac{\psi - \xi}{2}$.

Christoffel symbols:

$$\Gamma_{\xi\vartheta}^{\vartheta} = \cot \xi, \quad \Gamma_{\xi\varphi}^{\varphi} = \cot \xi, \quad \Gamma_{\vartheta\vartheta}^{\xi} = -\sin \xi \cos \xi, \quad (2.1.18a)$$

$$\Gamma_{\vartheta\varphi}^{\varphi} = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^{\xi} = -\sin \xi \cos \xi \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^{\vartheta} = -\sin \vartheta \cos \vartheta. \quad (2.1.18b)$$

Partial derivatives

$$\Gamma_{\xi\vartheta,\xi}^{\vartheta} = -\frac{1}{\sin^2\xi}, \quad \Gamma_{\xi\varphi,\xi}^{\varphi} = -\frac{1}{\sin^2\xi}, \quad \Gamma_{\vartheta\vartheta,\xi}^{\xi} = -\cos(2\xi), \quad (2.1.19a)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^{\varphi} = -\frac{1}{\sin^2\vartheta}, \quad \Gamma_{\varphi\varphi,\xi}^{\xi} = -\cos(2\xi)\sin^2\vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^{\vartheta} = -\cos(2\vartheta), \quad (2.1.19b)$$

$$\Gamma_{\varphi\varphi,\vartheta}^{\xi} = -\frac{1}{2}\sin(2\xi)\sin(2\vartheta). \quad (2.1.19c)$$

Riemann-Tensor:

$$R_{\xi\vartheta\xi\vartheta} = \sin^2\xi, \quad R_{\xi\varphi\xi\varphi} = \sin^2\xi\sin^2\vartheta, \quad R_{\vartheta\varphi\vartheta\varphi} = \sin^4\xi\sin^2\vartheta. \quad (2.1.20)$$

Ricci-Tensor:

$$R_{\xi\xi} = 2, \quad R_{\vartheta\vartheta} = 2\sin^2\xi, \quad R_{\varphi\varphi} = 2\sin^2\xi\sin^2\vartheta. \quad (2.1.21)$$

Ricci and Kretschmann scalars:

$$\mathcal{R} = 6, \quad \mathcal{K} = 12. \quad (2.1.22)$$

The Weyl tensor vanishes identically.

Local tetrad:

$$\mathbf{e}_{(\psi)} = \partial_{\psi}, \quad \mathbf{e}_{(\xi)} = \partial_{\xi}, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\sin\xi}\partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\sin\xi\sin\vartheta}\partial_{\varphi}. \quad (2.1.23)$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(\xi)(\vartheta)} = \gamma_{(\varphi)(\xi)(\varphi)} = \cot\xi, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{\sin\xi}. \quad (2.1.24)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\xi)} = 2\cot\xi, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{\sin\xi}. \quad (2.1.25)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\xi)(\vartheta)(\xi)(\vartheta)} = R_{(\xi)(\varphi)(\xi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = 1. \quad (2.1.26)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\xi)(\xi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = 2. \quad (2.1.27)$$

2.1.5 Rotating coordinates

The transformation $d\varphi \mapsto d\varphi + \omega dt$ brings the Minkowski metric (2.1.4) into the rotating form[Rin01] with coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \varphi \in [0, 2\pi), z \in \mathbb{R}\}$,

$$ds^2 = -\left(1 - \frac{\omega^2 r^2}{c^2}\right) [cdt - \Omega(r)d\varphi]^2 + dr^2 + \frac{r^2}{1 - \omega^2 r^2/c^2} d\varphi^2 + dz^2 \quad (2.1.28)$$

with $\Omega(r) = (r^2\omega/c)/(1 - \omega^2 r^2/c^2)$.

Metric-Tensor:

$$g_{tt} = -c^2 + \omega^2 r^2, \quad g_{t\varphi} = \omega r^2, \quad g_{rr} = g_{zz} = 1, \quad g_{\varphi\varphi} = r^2. \quad (2.1.29)$$

Christoffel symbols:

$$\Gamma_{tt}^r = -\omega^2 r, \quad \Gamma_{tr}^\varphi = \frac{\omega}{r}, \quad \Gamma_{t\varphi}^r = -\omega r, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\varphi\varphi}^r = -r. \quad (2.1.30)$$

Partial derivatives

$$\Gamma_{tt,r}^r = -\omega^2, \quad \Gamma_{tr,r}^\varphi = -\frac{\omega}{r^2}, \quad \Gamma_{t\varphi,r}^r = -\omega, \quad \Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\varphi\varphi,r}^r = -1. \quad (2.1.31)$$

The local tetrad of the comoving observer is

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t - \frac{\omega}{c} \partial_\varphi, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r} \partial_\varphi, \quad \mathbf{e}_{(z)} = \partial_z, \quad (2.1.32)$$

whereas the static observer has the local tetrad

$$\mathbf{e}_{(t)} = \frac{1}{c \sqrt{1 - \omega^2 r^2 / c^2}} \partial_t, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(z)} = \partial_z, \quad (2.1.33a)$$

$$\mathbf{e}_{(\varphi)} = \frac{\omega r}{c^2 \sqrt{1 - \omega^2 r^2 / c^2}} \partial_t + \frac{\sqrt{1 - \omega^2 r^2 / c^2}}{r} \partial_\varphi. \quad (2.1.33b)$$

2.1.6 Rindler coordinates

The worldline of an observer in the Minkowski spacetime who moves with constant proper acceleration α along the x direction reads

$$x = \frac{c^2}{\alpha} \cosh \frac{\alpha t'}{c}, \quad ct = \frac{c^2}{\alpha} \sinh \frac{\alpha t'}{c}, \quad (2.1.34)$$

where t' is the observer's proper time. The observer starts at $x = 1$ with zero velocity.

However, such an observer could also be described with Rindler coordinates. With the coordinate transformation

$$(ct, x) \mapsto (\tau, \rho): \quad ct = \frac{1}{\rho} \sinh \tau, \quad x = \frac{1}{\rho} \cosh \tau, \quad (2.1.35)$$

where $\rho = \alpha / c^2$, the Rindler metric reads

$$ds^2 = -\frac{1}{\rho^2} d\tau^2 + \frac{1}{\rho^4} d\rho^2 + dy^2 + dz^2. \quad (2.1.36)$$

Christoffel symbols:

$$\Gamma_{\tau\tau}^\rho = -\rho, \quad \Gamma_{\tau\rho}^\tau = -\frac{1}{\rho}, \quad \Gamma_{\rho\rho}^\rho = -\frac{2}{\rho}. \quad (2.1.37)$$

Partial derivatives

$$\Gamma_{\tau\tau,\rho}^\rho = -1, \quad \Gamma_{\tau\rho,\rho}^\tau = \frac{1}{\rho^2}, \quad \Gamma_{\rho\rho,\rho}^\rho = \frac{2}{\rho^2}. \quad (2.1.38)$$

The Riemann and Ricci tensors as well as the Ricci and Kretschmann scalar vanish identically.

Local tetrad:

$$\mathbf{e}_{(\tau)} = \rho \partial_\tau, \quad \mathbf{e}_{(\rho)} = \rho^2 \partial_\rho, \quad \mathbf{e}_{(y)} = \partial_y, \quad \mathbf{e}_{(z)} = \partial_z. \quad (2.1.39)$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(\rho)(\tau)} = \rho, \quad \text{and} \quad \gamma_{(\rho)} = -\rho. \quad (2.1.40)$$

2.2 Schwarzschild spacetime

2.2.1 Schwarzschild coordinates

In Schwarzschild coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$, the Schwarzschild metric reads

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{1}{1 - r_s/r} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.2.1)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, and M is the mass of the black hole. The critical point $r = 0$ is a real curvature singularity while the event horizon, $r = r_s$, is only a coordinate singularity, see e.g. the Kretschmann scalar.

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{c^2 r_s (r - r_s)}{2r^3}, \quad \Gamma_{tr}^r = \frac{r_s}{2r(r - r_s)}, \quad \Gamma_{rr}^r = -\frac{r_s}{2r(r - r_s)}, \quad (2.2.2a)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^r = -(r - r_s), \quad (2.2.2b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^r = -(r - r_s) \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.2c)$$

Partial derivatives

$$\Gamma_{tt,r}^r = -\frac{(2r - 3r_s)c^2 r_s}{2r^4}, \quad \Gamma_{tr,r}^r = -\frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \quad \Gamma_{rr,r}^r = \frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \quad (2.2.3a)$$

$$\Gamma_{r\vartheta,r}^\vartheta = -\frac{1}{r^2}, \quad \Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\vartheta\vartheta,r}^r = -1, \quad (2.2.3b)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,r}^r = -\sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\cos(2\vartheta), \quad (2.2.3c)$$

$$\Gamma_{\varphi\varphi,\vartheta}^r = -(r - r_s) \sin(2\vartheta). \quad (2.2.3d)$$

Riemann-Tensor:

$$R_{ttrr} = -\frac{c^2 r_s}{r^3}, \quad R_{t\vartheta t\vartheta} = \frac{1}{2} \frac{c^2 (r - r_s) r_s}{r^2}, \quad R_{t\varphi t\varphi} = \frac{1}{2} \frac{c^2 (r - r_s) r_s \sin^2 \vartheta}{r^2}, \quad (2.2.4a)$$

$$R_{r\vartheta r\vartheta} = -\frac{1}{2} \frac{r_s}{r - r_s}, \quad R_{r\varphi r\varphi} = -\frac{1}{2} \frac{r_s \sin^2 \vartheta}{r - r_s}, \quad R_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2 \vartheta. \quad (2.2.4b)$$

As expected, the Ricci tensor as well as the Ricci scalar vanish identically because the Schwarzschild spacetime is a vacuum solution of the field equations. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathcal{K} = 12 \frac{r_s^2}{r^6}. \quad (2.2.5)$$

Here, it becomes clear that at $r = r_s$ there is no real singularity.

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1 - r_s/r}} \partial_t, \quad \mathbf{e}_{(r)} = \sqrt{1 - \frac{r_s}{r}} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_\varphi. \quad (2.2.6)$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{1 - \frac{r_s}{r}} dt, \quad \theta^{(r)} = \frac{dr}{\sqrt{1 - r_s/r}}, \quad \theta^{(\vartheta)} = r d\vartheta, \quad \theta^{(\varphi)} = r \sin \vartheta d\varphi. \quad (2.2.7)$$

Ricci rotation coefficients:

$$\gamma_{(t)(t)(t)} = \frac{r_s}{2r^2 \sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r} \sqrt{1 - \frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{r}. \quad (2.2.8)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s}{2r^2 \sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{r}. \quad (2.2.9)$$

Structure coefficients:

$$c_{(t)(r)}^{(t)} = \frac{r_s}{2r^2 \sqrt{1 - r_s/r}}, \quad c_{(r)(\vartheta)}^{(\vartheta)} = c_{(r)(\varphi)}^{(\varphi)} = -\frac{1}{r} \sqrt{1 - \frac{r_s}{r}}, \quad c_{(\vartheta)(\varphi)}^{(\varphi)} = \frac{\cot \vartheta}{r}. \quad (2.2.10)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.2.11a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.2.11b)$$

The covariant derivatives of the Riemann tensor read

$$R_{(t)(r)(t)(r);(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi);(r)} = \frac{3r_s}{r^5} \sqrt{r(r - r_s)}, \quad (2.2.12a)$$

$$\begin{aligned} R_{(t)(r)(r)(\vartheta);(\vartheta)} &= R_{(t)(r)(t)(\varphi);(\varphi)} = R_{(t)(\vartheta)(t)(\vartheta);(r)} = R_{(t)(\varphi)(t)(\varphi);(r)} = \\ &= R_{(r)(\varphi)(\vartheta)(\varphi);(\vartheta)} = -\frac{3r_s}{2r^5} \sqrt{r(r - r_s)}, \end{aligned} \quad (2.2.12b)$$

$$R_{(r)(\vartheta)(r)(\vartheta);(r)} = R_{(r)(\vartheta)(\vartheta)(\varphi);(\varphi)} = R_{(r)(\varphi)(r)(\varphi);(r)} = \frac{3r_s}{2r^5} \sqrt{r(r - r_s)}. \quad (2.2.12c)$$

Newman-Penrose tetrad:

$$\mathbf{l} = \frac{1}{\sqrt{2}} (\mathbf{e}_{(t)} + \mathbf{e}_{(r)}), \quad \mathbf{n} = \frac{1}{\sqrt{2}} (\mathbf{e}_{(t)} - \mathbf{e}_{(r)}), \quad \mathbf{m} = \frac{1}{\sqrt{2}} (\mathbf{e}_{(\vartheta)} + i\mathbf{e}_{(\varphi)}). \quad (2.2.13)$$

Non-vanishing spin coefficients:

$$\rho = \mu = -\frac{1}{\sqrt{2}r} \sqrt{1 - \frac{r_s}{r}}, \quad \gamma = \varepsilon = \frac{r_s}{4\sqrt{2}r^2 \sqrt{1 - r_s/r}}, \quad \alpha = -\beta = -\frac{\cot \vartheta}{2\sqrt{2}r}. \quad (2.2.14)$$

Embedding:

The embedding function reads

$$z = 2\sqrt{r_s} \sqrt{r - r_s}. \quad (2.2.15)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2} \dot{r}^2 + V_{\text{eff}} = \frac{1}{2} \frac{k^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2} \left(1 - \frac{r_s}{r}\right) \left(\frac{h^2}{r^2} - \kappa c^2\right) \quad (2.2.16)$$

with the constants of motion $k = (1 - r_s/r)c^2 i$, $h = r^2 \dot{\varphi}$, and κ as in Eq. (1.8.2). For timelike geodesics, the effective potential has the extremal points

$$r_{\pm} = \frac{h^2 \pm h \sqrt{h^2 - 3c^2 r_s^2}}{c^2 r_s}, \quad (2.2.17)$$

where r_+ is a maximum and r_- is a minimum. The innermost timelike circular geodesic follows from $h^2 = 3c^2 r_s^2$ and reads $r_{\text{itcg}} = 3r_s$. Null geodesics, however, have only a maximum at $r_{\text{po}} = \frac{3}{2}r_s$. The corresponding circular orbit is called photon orbit.

Further reading:

Schwarzschild[Sch16, Sch03], MTW[MTW73], Rindler[Rin01], Wald[Wal84], Chandrasekhar[Cha06], Müller[Mül08b, Mül09].

2.2.2 Schwarzschild in pseudo-Cartesian coordinates

The Schwarzschild spacetime in pseudo-Cartesian coordinates (t, x, y, z) reads

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \left(\frac{x^2}{1 - r_s/r} + y^2 + z^2\right) \frac{dx^2}{r^2} + \left(x^2 + \frac{y^2}{1 - r_s/r} + z^2\right) \frac{dy^2}{r^2} + \left(x^2 + y^2 + \frac{z^2}{1 - r_s/r}\right) \frac{dz^2}{r^2} + \frac{2r_s}{r^2(r - r_s)} (xy dx dy + xz dx dz + yz dy dz), \quad (2.2.18)$$

where $r^2 = x^2 + y^2 + z^2$. For a natural local tetrad that is adapted to the x-axis, we make the following ansatz:

$$\mathbf{e}_{(0)} = \frac{1}{c\sqrt{1 - r_s/r}} \partial_t, \quad \mathbf{e}_{(1)} = A \partial_x, \quad \mathbf{e}_{(2)} = B \partial_x + C \partial_y, \quad \mathbf{e}_{(3)} = D \partial_x + E \partial_y + F \partial_z. \quad (2.2.19)$$

$$A = \frac{1}{\sqrt{g_{xx}}}, \quad B = \frac{-g_{xy}}{g_{xx}\sqrt{-g_{xy}^2/g_{xx} + g_{yy}}}, \quad C = \frac{1}{\sqrt{-g_{xy}^2/g_{xx} + g_{yy}}}, \quad (2.2.20a)$$

$$D = \frac{g_{xy}g_{yz} - g_{xz}g_{yy}}{\sqrt{NW}}, \quad E = \frac{g_{xz}g_{xy} - g_{xx}g_{yz}}{\sqrt{NW}}, \quad F = \frac{\sqrt{N}}{\sqrt{W}}, \quad (2.2.20b)$$

with

$$N = g_{xx}g_{yy} - g_{xy}^2, \quad (2.2.21a)$$

$$W = g_{xx}g_{yy}g_{zz} - g_{xz}^2g_{yy} + 2g_{xz}g_{xy}g_{yz} - g_{xy}^2g_{zz} - g_{xx}g_{yz}^2. \quad (2.2.21b)$$

2.2.3 Isotropic coordinates

Spherical isotropic coordinates

The Schwarzschild metric (2.2.1) in spherical isotropic coordinates $(t, \rho, \vartheta, \varphi)$ reads

$$ds^2 = -\left(\frac{1 - \rho_s/\rho}{1 + \rho_s/\rho}\right)^2 c^2 dt^2 + \left(1 + \frac{\rho_s}{\rho}\right)^4 [d\rho^2 + \rho^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)], \quad (2.2.22)$$

where

$$r = \rho \left(1 + \frac{\rho_s}{\rho}\right)^2 \quad \text{or} \quad \rho = \frac{1}{4} (2r - r_s \pm 2\sqrt{r(r - r_s)}) \quad (2.2.23)$$

is the coordinate transformation between the Schwarzschild radial coordinate r and the isotropic radial coordinate ρ , see e.g. MTW[MTW73] page 840. The event horizon is given by $\rho_s = r_s/4$. The photon orbit and the innermost timelike circular geodesic read

$$\rho_{\text{po}} = (2 + \sqrt{3}) \rho_s \quad \text{and} \quad \rho_{\text{itcg}} = (5 + 2\sqrt{6}) \rho_s. \quad (2.2.24)$$

Christoffel symbols:

$$\Gamma_{tt}^\rho = \frac{2(\rho - \rho_s)\rho^4 \rho_s c^2}{(\rho + \rho_s)^7}, \quad \Gamma_{t\rho}^t = \frac{2\rho_s}{\rho^2 - \rho_s^2}, \quad \Gamma_{\rho\rho}^\rho = -\frac{2\rho_s}{(\rho + \rho_s)\rho}, \quad (2.2.25a)$$

$$\Gamma_{\rho\vartheta}^\vartheta = \frac{\rho - \rho_s}{(\rho + \rho_s)\rho}, \quad \Gamma_{\rho\varphi}^\varphi = \frac{\rho - \rho_s}{(\rho + \rho_s)\rho}, \quad \Gamma_{\vartheta\vartheta}^\rho = -\rho \frac{\rho - \rho_s}{\rho + \rho_s}, \quad (2.2.25b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^\rho = -\frac{(\rho - \rho_s)\rho \sin^2 \vartheta}{\rho + \rho_s}, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.25c)$$

Riemann-Tensor:

$$R_{t\rho t\rho} = -4 \frac{(\rho - \rho_s)^2 \rho_s c^2}{(\rho + \rho_s)^4 \rho}, \quad R_{t\vartheta t\vartheta} = 2 \frac{(\rho - \rho_s)^2 \rho \rho_s c^2}{(\rho + \rho_s)^4}, \quad (2.2.26a)$$

$$R_{t\varphi t\varphi} = 2 \frac{(\rho - \rho_s)^2 \rho c^2 \rho_s \sin^2 \vartheta}{(\rho + \rho_s)^4}, \quad R_{\rho\vartheta\rho\vartheta} = -2 \frac{(\rho + \rho_s)^2 \rho_s}{\rho^3}, \quad (2.2.26b)$$

$$R_{\rho\varphi\rho\varphi} = -2 \frac{(\rho + \rho_s)^2 \rho_s \sin^2 \vartheta}{\rho^3}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{4(\rho + \rho_s)^2 \rho_s \sin^2 \vartheta}{\rho}. \quad (2.2.26c)$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathcal{K} = 192 \frac{r_s^2}{\rho^6 (1 + \rho_s/\rho)^{12}} = 12 \frac{r_s^2}{r(\rho)^6}. \quad (2.2.27)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1 + \rho_s/\rho}{1 - \rho_s/\rho} \frac{\partial}{\partial t}, \quad \mathbf{e}_{(r)} = \frac{1}{[1 + \rho_s/\rho]^2} \frac{\partial}{\partial \rho}, \quad (2.2.28a)$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{\rho [1 + \rho_s/\rho]^2} \frac{\partial}{\partial \vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\rho [1 + \rho_s/\rho]^2 \sin^2 \vartheta} \frac{\partial}{\partial \varphi}. \quad (2.2.28b)$$

Ricci rotation coefficients:

$$\gamma_{(\rho)(t)(t)} = \frac{2\rho_s \rho^2}{(\rho + \rho_s)^3 (\rho - \rho_s)}, \quad \gamma_{(\vartheta)(\rho)(\vartheta)} = \gamma_{(\varphi)(\rho)(\varphi)} = \frac{\rho(\rho - \rho_s)}{(\rho + \rho_s)^3}, \quad (2.2.29a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\rho \cot \vartheta}{(\rho + \rho_s)^2}. \quad (2.2.29b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\rho)} = \frac{2\rho(\rho^2 - \rho\rho_s + \rho_s^2)}{(\rho + \rho_s)^3 (\rho - \rho_s)}, \quad \gamma_{(\vartheta)} = \frac{\rho \cot \vartheta}{(\rho + \rho_s)^2}. \quad (2.2.30)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\rho)(t)(\rho)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r(\rho)^3}, \quad (2.2.31a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(\rho)(\vartheta)(\rho)(\vartheta)} = -R_{(\rho)(\varphi)(\rho)(\varphi)} = \frac{r_s}{2r(\rho)^3}. \quad (2.2.31b)$$

Further reading:

Buchdahl[Buc85].

Cartesian isotropic coordinates

The Schwarzschild metric (2.2.1) in Cartesian isotropic coordinates (t, x, y, z) reads,

$$ds^2 = - \left(\frac{1 - \rho_s/\rho}{1 + \rho_s/\rho} \right)^2 c^2 dt^2 + \left(1 + \frac{\rho_s}{\rho} \right)^4 [dx^2 + dy^2 + dz^2], \quad (2.2.32)$$

where $\rho^2 = x^2 + y^2 + z^2$ and, as before,

$$r = \rho \left(1 + \frac{\rho_s}{\rho} \right)^2. \quad (2.2.33)$$

Christoffel symbols:

$$\Gamma_{tt}^x = \frac{2c^2\rho^3\rho_s(\rho - \rho_s)x}{(\rho + \rho_s)^7}, \quad \Gamma_{tt}^y = \frac{2c^2\rho^3\rho_s(\rho - \rho_s)y}{(\rho + \rho_s)^7}, \quad \Gamma_{tt}^z = \frac{2c^2\rho^3\rho_s(\rho - \rho_s)z}{(\rho + \rho_s)^7}, \quad (2.2.34a)$$

$$\Gamma_{tx}^r = \frac{2\rho_s x}{\rho^3 [1 - \rho_s^2/\rho^2]}, \quad \Gamma_{ty}^r = \frac{2\rho_s y}{\rho^3 [1 - \rho_s^2/\rho^2]}, \quad \Gamma_{tz}^r = \frac{2\rho_s z}{\rho^3 [1 - \rho_s^2/\rho^2]}, \quad (2.2.34b)$$

$$\Gamma_{xx}^x = \Gamma_{xy}^y = \Gamma_{xz}^z = -\Gamma_{yy}^x = -\Gamma_{zz}^x = -\frac{2\rho_s}{\rho^3} \frac{x}{1 + \rho_s/\rho}, \quad (2.2.34c)$$

$$\Gamma_{xx}^y = -\Gamma_{xy}^x = -\Gamma_{yy}^y = -\Gamma_{yz}^z = \Gamma_{zz}^y = \frac{2\rho_s}{\rho^3} \frac{y}{1 + \rho_s/\rho}, \quad (2.2.34d)$$

$$\Gamma_{xx}^z = -\Gamma_{xz}^x = \Gamma_{yy}^z = -\Gamma_{yz}^y = -\Gamma_{zz}^z = \frac{2\rho_s}{\rho^3} \frac{z}{1 + \rho_s/\rho}. \quad (2.2.34e)$$

2.2.4 Eddington-Finkelstein

The transformation of the Schwarzschild metric (2.2.1) from the usual Schwarzschild time coordinate t to the advanced null coordinate v with

$$cv = ct + r + r_s \ln(r - r_s) \quad (2.2.35)$$

leads to the ingoing Eddington-Finkelstein[Edd24, Fin58] metric with coordinates $(v, r, \vartheta, \varphi)$,

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dv^2 + 2cdvdr + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (2.2.36)$$

Metric-Tensor:

$$g_{vv} = -c^2 \left(1 - \frac{r_s}{r}\right), \quad g_{vr} = c, \quad g_{\vartheta\vartheta} = r^2, \quad g_{\varphi\varphi} = r^2 \sin^2 \vartheta. \quad (2.2.37)$$

Christoffel symbols:

$$\Gamma_{vv}^v = \frac{cr_s}{2r^2}, \quad \Gamma_{vv}^r = \frac{c^2 r_s (r - r_s)}{2r^3}, \quad \Gamma_{vr}^r = -\frac{cr_s}{2r^2}, \quad \Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad (2.2.38a)$$

$$\Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^v = -\frac{r}{c}, \quad \Gamma_{\vartheta\vartheta}^r = -(r - r_s), \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad (2.2.38b)$$

$$\Gamma_{\varphi\varphi}^v = -\frac{r \sin^2 \vartheta}{c}, \quad \Gamma_{\varphi\varphi}^r = -(r - r_s) \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.38c)$$

Partial derivatives

$$\Gamma_{vv,r}^v = -\frac{cr_s}{r^3}, \quad \Gamma_{vv,r}^r = -\frac{(2r - 3r_s)c^2 r_s}{2r^4}, \quad \Gamma_{vr,r}^r = \frac{cr_s}{r^3}, \quad (2.2.39a)$$

$$\Gamma_{r\vartheta,r}^\vartheta = -\frac{1}{r^2}, \quad \Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\vartheta\vartheta,r}^v = -\frac{1}{c}, \quad (2.2.39b)$$

$$\Gamma_{\vartheta\vartheta,r}^r = -1, \quad \Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,r}^v = -\frac{\sin^2 \vartheta}{c}, \quad (2.2.39c)$$

$$\Gamma_{\varphi\varphi,\vartheta}^v = -\frac{r \sin(2\vartheta)}{c}, \quad \Gamma_{\varphi\varphi,r}^r = -\sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\cos(2\vartheta), \quad (2.2.39d)$$

$$\Gamma_{\varphi\varphi,\vartheta}^r = -(r - r_s) \sin(2\vartheta). \quad (2.2.39e)$$

Riemann-Tensor:

$$R_{vrvr} = -\frac{c^2 r_s}{r^3}, \quad R_{v\vartheta v\vartheta} = \frac{c^2 r_s (r - r_s)}{2r^2}, \quad R_{v\vartheta r\vartheta} = -\frac{cr_s}{2r}, \quad (2.2.40a)$$

$$R_{v\varphi v\varphi} = \frac{c^2 r_s (r - r_s) \sin^2 \vartheta}{2r^2}, \quad R_{v\varphi r\varphi} = -\frac{cr_s \sin^2 \vartheta}{2r}, \quad R_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2 \vartheta. \quad (2.2.40b)$$

While the Ricci tensor and the Ricci scalar vanish identically, the Kretschmann scalar is $\mathcal{K} = 12r_s^2/r^6$.

Static local tetrad:

$$\mathbf{e}_{(v)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_v, \quad \mathbf{e}_{(r)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_r + \sqrt{1-\frac{r_s}{r}}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_{\varphi}. \quad (2.2.41)$$

Dual tetrad:

$$\theta^{(v)} = c\sqrt{1-\frac{r_s}{r}}dv - \frac{dr}{\sqrt{1-r_s/r}}, \quad \theta^{(r)} = \frac{dr}{\sqrt{1-r_s/r}}, \quad \theta^{(\vartheta)} = rd\vartheta, \quad \theta^{(\varphi)} = r\sin\vartheta d\varphi. \quad (2.2.42)$$

Ricci rotation coefficients:

$$\gamma_{(r)(v)(v)} = \frac{r_s}{2r^2\sqrt{1-r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}\sqrt{1-\frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \quad (2.2.43)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r-3r_s}{2r^2\sqrt{1-r_s/r}}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \quad (2.2.44)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(v)(r)(v)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.2.45a)$$

$$R_{(v)(\vartheta)(v)(\vartheta)} = R_{(v)(\varphi)(v)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.2.45b)$$

2.2.5 Kruskal-Szekeres

The Schwarzschild metric in Kruskal-Szekeres [Kru60, Wal84] coordinates $(T, X, \vartheta, \varphi)$ reads

$$ds^2 = \frac{4r_s^3}{r} e^{-r/r_s} (-dT^2 + dX^2) + r^2 d\Omega^2, \quad (2.2.46)$$

where $r \in \mathbb{R}_+ \setminus \{0\}$ is given by means of the LambertW-function \mathcal{W} ,

$$\left(\frac{r}{r_s} - 1\right) e^{r/r_s} = X^2 - T^2 \quad \text{or} \quad r = r_s \left[\mathcal{W} \left(\frac{X^2 - T^2}{e} \right) + 1 \right]. \quad (2.2.47)$$

The Schwarzschild coordinate time t in terms of the Kruskal coordinates T and X reads

$$t = 2r_s \operatorname{arctanh} \frac{T}{X}, \quad r > r_s, \quad (2.2.48a)$$

$$t = 2r_s \operatorname{arctanh} \frac{X}{T}, \quad r < r_s, \quad (2.2.48b)$$

$$t = \infty, \quad r = r_s. \quad (2.2.48c)$$

The transformations between Kruskal- and Schwarzschild coordinates read

$$X = \sqrt{1-\frac{r}{r_s}} e^{r/(2r_s)} \sinh \frac{ct}{2r_s}, \quad T = \sqrt{1-\frac{r}{r_s}} e^{r/(2r_s)} \cosh \frac{ct}{2r_s}, \quad 0 < r < r_s, \quad (2.2.49a)$$

$$X = \sqrt{\frac{r}{r_s} - 1} e^{r/(2r_s)} \cosh \frac{ct}{2r_s}, \quad T = \sqrt{\frac{r}{r_s} - 1} e^{r/(2r_s)} \sinh \frac{ct}{2r_s}, \quad r \geq r_s. \quad (2.2.49b)$$

Christoffel symbols:

$$\Gamma_{TT}^T = \Gamma_{TX}^X = \Gamma_{XX}^T = \frac{Tr_s(r+r_s)}{r^2} e^{-r/r_s}, \quad (2.2.50a)$$

$$\Gamma_{TT}^X = \Gamma_{TX}^T = \Gamma_{XX}^X = -\frac{Xr_s(r+r_s)}{r^2} e^{-r/r_s}, \quad (2.2.50b)$$

$$\Gamma_{T\vartheta}^{\vartheta} = -\frac{2r_s^2 T}{r^2} e^{-r/r_s}, \quad \Gamma_{X\vartheta}^{\vartheta} = \frac{2r_s^2 X}{r^2} e^{-r/r_s}, \quad (2.2.50c)$$

$$\Gamma_{\vartheta\vartheta}^T = -\frac{r}{2r_s} T, \quad \Gamma_{\vartheta\vartheta}^X = \frac{r}{2r_s} X, \quad (2.2.50d)$$

$$\Gamma_{\vartheta\vartheta}^T = -\frac{r}{2r_s} T \sin^2 \vartheta, \quad \Gamma_{\vartheta\vartheta}^X = \frac{r}{2r_s} X \sin^2 \vartheta, \quad (2.2.50e)$$

$$\Gamma_{\vartheta\varphi}^{\varphi} = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^{\vartheta} = -\sin \vartheta \cos \vartheta. \quad (2.2.50f)$$

Riemann-Tensor:

$$R_{TXXT} = -16 \frac{r_s^7}{r^5} e^{-2r/r_s}, \quad R_{T\vartheta T\vartheta} = \frac{2r_s^4}{r^2} e^{-r/r_s}, \quad (2.2.51a)$$

$$R_{T\varphi T\varphi} = \frac{2r_s^4}{r^2} e^{-r/r_s} \sin^2 \vartheta, \quad R_{X\vartheta X\vartheta} = -\frac{2r_s^4}{r^2} e^{-r/r_s}, \quad (2.2.51b)$$

$$R_{X\varphi X\varphi} = -\frac{2r_s^4}{r^2} e^{-r/r_s} \sin^2 \vartheta, \quad R_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2 \vartheta. \quad (2.2.51c)$$

The *Ricci-Tensor* as well as the *Ricci-scalar* vanish identically.

Kretschmann scalar:

$$\mathcal{K} = \frac{12r_s^2}{r^6}. \quad (2.2.52)$$

Local tetrad:

$$\mathbf{e}_{(T)} = \frac{\sqrt{r}}{2r_s\sqrt{r_s}} e^{r/(2r_s)} \partial_T, \quad \mathbf{e}_{(X)} = \frac{\sqrt{r}}{2r_s\sqrt{r_s}} e^{r/(2r_s)} \partial_X, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_{\varphi} \quad (2.2.53)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(T)(X)(T)(X)} = R_{(X)(\vartheta)(X)(\vartheta)} = R_{(X)(\varphi)(X)(\varphi)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.2.54a)$$

$$R_{(T)(\vartheta)(T)(\vartheta)} = R_{(T)(\varphi)(T)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.2.54b)$$

2.2.6 Tortoise coordinates

The Schwarzschild metric represented by tortoise coordinates $(t, \rho, \vartheta, \varphi)$ reads

$$ds^2 = -\left(1 - \frac{r_s}{r(\rho)}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r(\rho)}\right) d\rho^2 + r(\rho)^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.2.55)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, and M is the mass of the black hole. The tortoise radial coordinate ρ and the Schwarzschild radial coordinate r are related by

$$\rho = r + r_s \ln \left(\frac{r}{r_s} - 1 \right) \quad \text{or} \quad r = r_s \left\{ 1 + \mathcal{W} \left[\exp \left(\frac{\rho}{r_s} - 1 \right) \right] \right\}. \quad (2.2.56)$$

Christoffel symbols:

$$\Gamma_{tt}^\rho = \frac{c^2 r_s}{2r(\rho)^2}, \quad \Gamma_{t\rho}^t = \frac{r_s}{2r(\rho)^2}, \quad \Gamma_{\rho\rho}^\rho = \frac{r_s}{2r(\rho)^2}, \quad (2.2.57a)$$

$$\Gamma_{\rho\vartheta}^\vartheta = \frac{1}{r(\rho)} - \frac{1}{r_s}, \quad \Gamma_{\rho\varphi}^\varphi = \frac{1}{r(\rho)} - \frac{1}{r_s}, \quad \Gamma_{\vartheta\vartheta}^\rho = -r(\rho), \quad (2.2.57b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^\rho = -r(\rho) \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.57c)$$

Riemann-Tensor:

$$R_{t\rho t\rho} = -\frac{c^2 r_s}{r(\rho)^3} \left(1 - \frac{r_s}{r(\rho)}\right)^2, \quad R_{t\vartheta t\vartheta} = \frac{c^2}{2} \left(1 - \frac{r_s}{r(\rho)}\right) \frac{r_s}{r(\rho)}, \quad (2.2.58a)$$

$$R_{t\varphi t\varphi} = \frac{c^2 \sin^2 \vartheta}{2} \left(1 - \frac{r_s}{r(\rho)}\right) \frac{r_s}{r(\rho)}, \quad R_{\rho\vartheta\rho\vartheta} = -\frac{1}{2} \left(1 - \frac{r_s}{r(\rho)}\right) \frac{r_s}{r(\rho)} \quad (2.2.58b)$$

$$R_{\rho\varphi\rho\varphi} = -\frac{\sin^2 \vartheta}{2} \left(1 - \frac{r_s}{r(\rho)}\right) \frac{r_s}{r(\rho)}, \quad R_{\vartheta\varphi\vartheta\varphi} = r(\rho) r_s \sin^2 \vartheta. \quad (2.2.58c)$$

The Ricci tensor as well as the Ricci scalar vanish identically because the Schwarzschild spacetime is a vacuum solution of the field equations. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathcal{K} = 12 \frac{r_s^2}{r(\rho)^6}. \quad (2.2.59)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1-r_s/r(\rho)}} \partial_t, \quad \mathbf{e}_{(\rho)} = \frac{1}{\sqrt{1-r_s/r(\rho)}} \partial_\rho, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r(\rho)} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r(\rho) \sin \vartheta} \partial_\varphi. \quad (2.2.60)$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{1-\frac{r_s}{r(\rho)}} dt, \quad \theta^{(\rho)} = \sqrt{1-\frac{r_s}{r(\rho)}} d\rho, \quad \theta^{(\vartheta)} = r(\rho) d\vartheta, \quad \theta^{(\varphi)} = r(\rho) \sin \vartheta d\varphi. \quad (2.2.61)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\rho)(t)(\rho)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r(\rho)^3}, \quad (2.2.62a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(\rho)(\vartheta)(\rho)(\vartheta)} = -R_{(\rho)(\varphi)(\rho)(\varphi)} = \frac{r_s}{2r(\rho)^3}. \quad (2.2.62b)$$

Further reading:

MTW[MTW73]

2.2.7 Painlevé-Gullstrand

The Schwarzschild metric expressed in Painlevé-Gullstrand coordinates[MP01] reads

$$ds^2 = -c^2 dT^2 + \left(dr + \sqrt{\frac{r_s}{r}} c dT \right)^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.2.63)$$

where the new time coordinate T follows from the Schwarzschild time t in the following way:

$$cT = ct + 2r_s \left(\sqrt{\frac{r}{r_s}} + \frac{1}{2} \ln \left| \frac{\sqrt{r/r_s} - 1}{\sqrt{r/r_s} + 1} \right| \right). \quad (2.2.64)$$

Metric-Tensor:

$$g_{TT} = -c^2 \left(1 - \frac{r_s}{r}\right), \quad g_{Tr} = c\sqrt{\frac{r_s}{r}}, \quad g_{rr} = 1, \quad g_{\vartheta\vartheta} = r^2, \quad g_{\varphi\varphi} = r^2 \sin^2 \vartheta. \quad (2.2.65)$$

Christoffel symbols:

$$\Gamma_{TT}^T = \frac{cr_s}{2r^2} \sqrt{\frac{r_s}{r}}, \quad \Gamma_{TT}^r = \frac{c^2 r_s (r - r_s)}{2r^3}, \quad \Gamma_{Tr}^T = \frac{r_s}{2r^2}, \quad (2.2.66a)$$

$$\Gamma_{Tr}^r = -\frac{cr_s}{2r^2} \sqrt{\frac{r_s}{r}}, \quad \Gamma_{rr}^T = \frac{r_s}{2cr^2} \sqrt{\frac{r}{r_s}}, \quad \Gamma_{rr}^r = -\frac{r_s}{2r^2}, \quad (2.2.66b)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^T = -\frac{r}{c} \sqrt{\frac{r_s}{r}}, \quad (2.2.66c)$$

$$\Gamma_{\vartheta\vartheta}^r = -(r - r_s), \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^T = -\frac{r}{c} \sqrt{\frac{r_s}{r}} \sin^2 \vartheta, \quad (2.2.66d)$$

$$\Gamma_{\varphi\varphi}^r = -(r - r_s) \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.2.66e)$$

Riemann-Tensor:

$$R_{TrTr} = -\frac{c^2 r_s}{r^3}, \quad R_{T\vartheta T\vartheta} = \frac{c^2 r_s (r - r_s)}{2r^2}, \quad R_{T\vartheta r\vartheta} = -\frac{cr_s}{2r} \sqrt{\frac{r_s}{r}}, \quad (2.2.67a)$$

$$R_{T\varphi T\varphi} = \frac{c^2 r_s (r - r_s) \sin^2 \vartheta}{2r^2}, \quad R_{T\varphi r\varphi} = -\frac{cr_s}{2r} \sqrt{\frac{r_s}{r}} \sin^2 \vartheta, \quad R_{r\vartheta r\vartheta} = -\frac{r_s}{2r}, \quad (2.2.67b)$$

$$R_{r\varphi r\varphi} = -\frac{r_s \sin^2 \vartheta}{2r}, \quad R_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2 \vartheta. \quad (2.2.67c)$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathcal{K} = 12r_s^2/r^6. \quad (2.2.68)$$

For the Painlevé-Gullstrand coordinates, we can define two natural local tetrads.

Static local tetrad:

$$\hat{\mathbf{e}}_{(T)} = \frac{1}{c\sqrt{1-r_s/r}} \partial_T, \quad \hat{\mathbf{e}}_{(r)} = \frac{\sqrt{r_s}}{c\sqrt{r-r_s}} \partial_T + \sqrt{1-\frac{r_s}{r}} \partial_r, \quad \hat{\mathbf{e}}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \hat{\mathbf{e}}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_\varphi, \quad (2.2.69)$$

Dual tetrad:

$$\hat{\theta}^{(T)} = c\sqrt{1-\frac{r_s}{r}} dT - \frac{dr}{\sqrt{r/r_s-1}}, \quad \hat{\theta}^{(r)} = \frac{dr}{\sqrt{1-r_s/r}}, \quad \hat{\theta}^{(\vartheta)} = rd\vartheta, \quad \hat{\theta}^{(\varphi)} = r \sin \vartheta d\varphi. \quad (2.2.70)$$

Freely falling local tetrad:

$$\mathbf{e}_{(T)} = \frac{1}{c} \partial_T - \sqrt{\frac{r_s}{r}} \partial_r, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_\varphi. \quad (2.2.71)$$

Dual tetrad:

$$\theta^{(T)} = c dT, \quad \theta^{(r)} = c\sqrt{\frac{r_s}{r}} dT + dr, \quad \theta^{(\vartheta)} = rd\vartheta, \quad \theta^{(\varphi)} = r \sin \vartheta d\varphi. \quad (2.2.72)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(T)(r)(T)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.2.73a)$$

$$R_{(T)(\vartheta)(T)(\vartheta)} = R_{(T)(\varphi)(T)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.2.73b)$$

2.2.8 Israel coordinates

The Schwarzschild metric in Israel coordinates $(x, y, \vartheta, \varphi)$ reads[SKM+03]

$$ds^2 = r_s^2 \left[4dx \left(dy + \frac{y^2 dx}{1+xy} \right) + (1+xy)^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right], \quad (2.2.74)$$

where the coordinates x and y follow from the Schwarzschild coordinates via

$$t = r_s \left(1 + xy + \ln \frac{y}{x} \right) \quad \text{and} \quad r = r_s (1 + xy). \quad (2.2.75)$$

Christoffel symbols:

$$\Gamma_{xx}^x = -\frac{y(2+xy)}{(1+xy)^2}, \quad \Gamma_{xx}^y = \frac{y^3(3+xy)}{(1+xy)^3}, \quad \Gamma_{xy}^y = \frac{y(2+xy)}{(1+xy)^2}, \quad (2.2.76a)$$

$$\Gamma_{x\vartheta}^{\vartheta} = \frac{y}{1+xy}, \quad \Gamma_{x\varphi}^{\varphi} = \frac{y}{1+xy}, \quad \Gamma_{y\vartheta}^{\vartheta} = \frac{x}{1+xy}, \quad (2.2.76b)$$

$$\Gamma_{x\varphi}^{\varphi} = \frac{x}{1+xy}, \quad \Gamma_{\vartheta\vartheta}^x = -\frac{x}{2}(1+xy), \quad \Gamma_{\vartheta\vartheta}^y = -\frac{y}{2}(1-xy), \quad (2.2.76c)$$

$$\Gamma_{\vartheta\varphi}^{\varphi} = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^x = -\frac{x}{2}(1+xy) \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^y = -\frac{y}{2}(1-xy) \sin^2 \vartheta, \quad (2.2.76d)$$

$$\Gamma_{\varphi\varphi}^{\vartheta} = -\sin \vartheta \cos \vartheta. \quad (2.2.76e)$$

Riemann-Tensor:

$$R_{xyxy} = -4 \frac{r_s^2}{(1+xy)^3}, \quad R_{x\vartheta x\vartheta} = -2 \frac{y^2 r_s^2}{(1+xy)^2}, \quad R_{x\vartheta y\vartheta} = -\frac{r_s^2}{1+xy}, \quad (2.2.77a)$$

$$R_{x\varphi x\varphi} = -2 \frac{r_s^2 y^2 \sin^2 \vartheta}{(1+xy)^2}, \quad R_{x\varphi y\varphi} = -\frac{r_s^2 \sin^2 \vartheta}{1+xy}, \quad R_{\vartheta\varphi\vartheta\varphi} = (1+xy) r_s^2 \sin^2 \vartheta. \quad (2.2.77b)$$

The Ricci tensor as well as the Ricci scalar vanish identically. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathcal{K} = \frac{12}{r_s^4 (1+xy)^6}. \quad (2.2.78)$$

Local tetrad:

$$\mathbf{e}_{(0)} = -\frac{\sqrt{1+xy}}{2r_s y} \partial_x + \frac{y}{r_s \sqrt{1+xy}} \partial_y, \quad \mathbf{e}_{(1)} = \frac{\sqrt{1+xy}}{2r_s y} \partial_x, \quad (2.2.79a)$$

$$\mathbf{e}_{(2)} = \frac{1}{r_s (1+xy)} \partial_{\vartheta}, \quad \mathbf{e}_{(3)} = \frac{1}{r_s (1+xy) \sin \vartheta} \partial_{\varphi}. \quad (2.2.79b)$$

Dual tetrad:

$$\theta^{(0)} = \frac{r_s \sqrt{1+xy}}{y} dy, \quad \theta^{(1)} = \frac{2r_s y}{\sqrt{1+xy}} dx + \frac{r_s \sqrt{1+xy}}{y} dy, \quad (2.2.80a)$$

$$\theta^{(2)} = r_s (1+xy) d\vartheta, \quad \theta^{(3)} = r_s (1+xy) \sin \vartheta d\varphi. \quad (2.2.80b)$$

2.3 Alcubierre Warp

The Warp metric given by Miguel Alcubierre[Alc94] reads

$$ds^2 = -c^2 dt^2 + (dx - v_s f(r_s) dt)^2 + dy^2 + dz^2 \quad (2.3.1)$$

where

$$v_s = \frac{dx_s(t)}{dt}, \quad (2.3.2a)$$

$$r_s(t) = \sqrt{(x - x_s(t))^2 + y^2 + z^2}, \quad (2.3.2b)$$

$$f(r_s) = \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2 \tanh(\sigma R)}. \quad (2.3.2c)$$

The parameter $R > 0$ defines the radius of the warp bubble and the parameter $\sigma > 0$ its thickness.

Metric-Tensor:

$$g_{tt} = -c^2 + v_s^2 f(r_s)^2, \quad g_{tx} = -v_s f(r_s), \quad g_{xx} = g_{yy} = g_{zz} = 1. \quad (2.3.3)$$

Christoffel symbols:

$$\Gamma_{tt}^t = \frac{f^2 f_x v_s^3}{c^2}, \quad \Gamma_{tt}^z = -f f_z v_s^2, \quad \Gamma_{tt}^y = -f f_y v_s^2, \quad (2.3.4a)$$

$$\Gamma_{tt}^x = \frac{f^3 f_x v_s^4 - c^2 f f_x v_s^2 - c^2 f_t v_s}{c^2}, \quad \Gamma_{tx}^t = -\frac{f f_x v_s^2}{c^2}, \quad \Gamma_{tx}^x = -\frac{f^2 f_x v_s^3}{c^2}, \quad (2.3.4b)$$

$$\Gamma_{tx}^y = \frac{f_y v_s}{2}, \quad \Gamma_{tx}^z = \frac{f_z v_s}{2}, \quad \Gamma_{ty}^t = -\frac{f f_y v_s^2}{2c^2}, \quad (2.3.4c)$$

$$\Gamma_{ty}^x = -\frac{f^2 f_y v_s^3 + c^2 f_y v_s}{2c^2}, \quad \Gamma_{tz}^t = -\frac{f f_z v_s^2}{2c^2}, \quad \Gamma_{tz}^x = -\frac{f^2 f_z v_s^3 + c^2 f_z v_s}{2c^2}, \quad (2.3.4d)$$

$$\Gamma_{xx}^t = \frac{f_x v_s}{c^2}, \quad \Gamma_{xx}^x = \frac{f f_x v_s^2}{c^2}, \quad \Gamma_{xy}^t = \frac{f_y v_s}{2c^2}, \quad (2.3.4e)$$

$$\Gamma_{xy}^x = \frac{f f_y v_s^2}{2c^2}, \quad \Gamma_{xz}^t = \frac{f_z v_s}{2c^2}, \quad \Gamma_{xz}^x = \frac{f f_z v_s^2}{2c^2}, \quad (2.3.4f)$$

with derivatives

$$f_t = \frac{df(r_s)}{dt} = \frac{-v_s \sigma (x - x_s(t))}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R)) \right] \quad (2.3.5a)$$

$$f_x = \frac{df(r_s)}{dx} = \frac{\sigma (x - x_s(t))}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R)) \right] \quad (2.3.5b)$$

$$f_y = \frac{df(r_s)}{dy} = \frac{\sigma y}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R)) \right] \quad (2.3.5c)$$

$$f_z = \frac{df(r_s)}{dz} = \frac{\sigma z}{2r_s \tanh(\sigma R)} \left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R)) \right] \quad (2.3.5d)$$

Riemann- and Ricci-tensor as well as Ricci- and Kretschman-scalar are shown only in the Maple worksheet.

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{c} (\partial_t + v_s f \partial_x), \quad \mathbf{e}_{(1)} = \partial_x, \quad \mathbf{e}_{(2)} = \partial_y, \quad \mathbf{e}_{(3)} = \partial_z. \quad (2.3.6)$$

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{\sqrt{c^2 - v_s^2 f^2}} \partial_t, \quad \mathbf{e}_{(1)} = \frac{v_s f}{c \sqrt{c^2 - v_s^2 f^2}} \partial_t + \frac{\sqrt{c^2 - v_s^2 f^2}}{c} \partial_x, \quad \mathbf{e}_{(2)} = \partial_y, \quad \mathbf{e}_{(3)} = \partial_z. \quad (2.3.7)$$

Further reading:

Pfenning[PF97], Clark[CHL99], Van Den Broeck[Bro99]

2.4 Barriola-Vilenkin monopole

The Barriola-Vilenkin metric describes the gravitational field of a global monopole[BV89]. In spherical coordinates $(t, r, \vartheta, \varphi)$, the metric reads

$$ds^2 = -c^2 dt^2 + dr^2 + k^2 r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.4.1)$$

where k is the scaling factor responsible for the deficit/surplus angle.

Christoffel symbols:

$$\Gamma_{\vartheta\vartheta}^r = -k^2 r, \quad \Gamma_{\varphi\varphi}^r = -k^2 r \sin^2 \vartheta, \quad \Gamma_{r\vartheta}^{\vartheta} = \frac{1}{r}, \quad (2.4.2a)$$

$$\Gamma_{\varphi\varphi}^{\vartheta} = -\sin \vartheta \cos \vartheta, \quad \Gamma_{r\varphi}^{\varphi} = \frac{1}{r}, \quad \Gamma_{\vartheta\varphi}^{\varphi} = \cot \vartheta. \quad (2.4.2b)$$

Partial derivatives

$$\Gamma_{r\vartheta,r}^{\vartheta} = -\frac{1}{r^2}, \quad \Gamma_{r\varphi,r}^{\varphi} = -\frac{1}{r^2}, \quad \Gamma_{\vartheta\vartheta,r}^r = -k^2, \quad (2.4.3a)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^{\varphi} = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,r}^r = -k^2 \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^{\vartheta} = -\cos(2\vartheta), \quad (2.4.3b)$$

$$\Gamma_{\varphi\varphi,\vartheta}^r = -k^2 r \sin(2\vartheta). \quad (2.4.3c)$$

Riemann-Tensor:

$$R_{\vartheta\varphi\vartheta\varphi} = (1 - k^2) k^2 r^2 \sin^2 \vartheta. \quad (2.4.4)$$

Ricci tensor, Ricci and Kretschmann scalar:

$$R_{\vartheta\vartheta} = (1 - k^2), \quad R_{\varphi\varphi} = (1 - k^2) \sin^2 \vartheta, \quad \mathcal{R} = 2 \frac{1 - k^2}{k^2 r^2}, \quad \mathcal{K} = 4 \frac{(1 - k^2)^2}{k^4 r^4}. \quad (2.4.5)$$

Weyl-Tensor:

$$C_{trrr} = -\frac{c^2(1 - k^2)}{3k^2 r^2}, \quad C_{t\vartheta t\vartheta} = \frac{c^2}{6}(1 - k^2), \quad C_{t\varphi t\varphi} = \frac{c^2}{6}(1 - k^2) \sin^2 \vartheta, \quad (2.4.6a)$$

$$C_{r\vartheta r\vartheta} = -\frac{1}{6}(1 - k^2), \quad C_{r\varphi r\varphi} = -\frac{1}{6}(1 - k^2) \sin^2 \vartheta, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{k^2 r^2}{3}(1 - k^2) \sin^2 \vartheta. \quad (2.4.6b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(r)} = \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{kr} \partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{kr \sin \vartheta} \partial_{\varphi}. \quad (2.4.7)$$

Dual tetrad:

$$\theta^{(t)} = c dt, \quad \theta^{(r)} = dr, \quad \theta^{(\vartheta)} = kr d\vartheta, \quad \theta^{(\varphi)} = kr \sin \vartheta d\varphi. \quad (2.4.8)$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{kr}. \quad (2.4.9)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2}{r}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{kr}. \quad (2.4.10)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{1-k^2}{k^2 r^2}. \quad (2.4.11)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{1-k^2}{k^2 r^2}. \quad (2.4.12)$$

Weyl-Tensor with respect to local tetrad:

$$C_{(r)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{1-k^2}{3k^2 r^2}, \quad (2.4.13a)$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{1-k^2}{6k^2 r^2}. \quad (2.4.13b)$$

Embedding:

The embedding function, see Sec. 1.7, for $k < 1$ reads

$$z = \sqrt{1-k^2} r. \quad (2.4.14)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2} \dot{r}^2 + V_{\text{eff}} = \frac{1}{2} \frac{h_1^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2} \left(\frac{h_2^2}{k^2 r^2} - \kappa c^2 \right), \quad (2.4.15)$$

with the constants of motion $h_1 = c^2 i$ and $h_2 = k^2 r^2 \dot{\varphi}$.

The point of closest approach r_{pca} for a null geodesic that starts at $r = r_i$ with $\mathbf{y} = \pm \mathbf{e}_{(t)} + \cos \xi \mathbf{e}_{(r)} + \sin \xi \mathbf{e}_{(\varphi)}$ is given by $r = r_i \sin \xi$. Hence, the r_{pca} is independent of k . The same is also true for timelike geodesics.

Further reading:

Barriola and Vilenkin[BV89], Perlick[Per04].

2.5 Bertotti-Kasner

The Bertotti-Kasner spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ reads[Rin98]

$$ds^2 = -c^2 dt^2 + e^{2\sqrt{\Lambda}ct} dr^2 + \frac{1}{\Lambda} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.5.1)$$

where the cosmological constant Λ must be positive.

Christoffel symbols:

$$\Gamma_{tr}^r = c\sqrt{\Lambda}, \quad \Gamma_{rr}^r = \frac{\sqrt{\Lambda}}{c} e^{2\sqrt{\Lambda}ct}, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.5.2)$$

Partial derivatives

$$\Gamma_{rr,t}^r = 2\Lambda e^{2\sqrt{\Lambda}ct}, \quad \Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\cos(2\vartheta). \quad (2.5.3)$$

Riemann-Tensor:

$$R_{trtr} = -\Lambda c^2 e^{2\sqrt{\Lambda}ct}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{\sin^2 \vartheta}{\Lambda}. \quad (2.5.4)$$

Ricci-Tensor:

$$R_{tt} = -\Lambda c^2, \quad R_{rr} = \Lambda e^{2\sqrt{\Lambda}ct}, \quad R_{\vartheta\vartheta} = 1, \quad R_{\varphi\varphi} = \sin^2 \vartheta. \quad (2.5.5)$$

The Ricci and Kretschmann scalars read

$$\mathcal{R} = 4\Lambda, \quad \mathcal{K} = 8\Lambda^2. \quad (2.5.6)$$

Weyl-Tensor:

$$C_{trtr} = -\frac{2}{3}\Lambda c^2 e^{2\sqrt{\Lambda}ct}, \quad C_{t\vartheta t\vartheta} = \frac{c^2}{3}, \quad C_{t\varphi t\varphi} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct}, \quad (2.5.7a)$$

$$C_{r\vartheta r\vartheta} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct}, \quad C_{r\varphi r\varphi} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct} \sin^2 \vartheta, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{2}{3} \frac{\sin^2 \vartheta}{\Lambda}. \quad (2.5.7b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(r)} = e^{-\sqrt{\Lambda}ct} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \sqrt{\Lambda} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{\sqrt{\Lambda}}{\sin \vartheta} \partial_\varphi. \quad (2.5.8)$$

Dual tetrad:

$$\theta^{(t)} = c dt, \quad \theta^{(r)} = e^{\sqrt{\Lambda}ct} dr, \quad \theta^{(\vartheta)} = \frac{1}{\sqrt{\Lambda}} d\vartheta, \quad \theta^{(\varphi)} = \frac{\sin \vartheta}{\sqrt{\Lambda}} d\varphi. \quad (2.5.9)$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(r)} = \sqrt{\Lambda}, \quad \gamma_{(\vartheta)(\varphi)(\varphi)} = -\sqrt{\Lambda} \cot \vartheta. \quad (2.5.10)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = -\sqrt{\Lambda}, \quad \gamma_{(\vartheta)} = \sqrt{\Lambda} \cot \vartheta. \quad (2.5.11)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\Lambda. \quad (2.5.12)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -R_{(r)(r)} = -R_{(\vartheta)(\vartheta)} = -R_{(\varphi)(\varphi)} = -\Lambda. \quad (2.5.13)$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{2\Lambda}{3}, \quad (2.5.14a)$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{\Lambda}{3}. \quad (2.5.14b)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$c^2 \dot{t}^2 = h_1^2 e^{-2\sqrt{\Lambda}ct} + \Lambda h_2^2 - \kappa \quad (2.5.15)$$

with the constants of motion $h_1 = \dot{r}e^{2\sqrt{\Lambda}ct}$ and $h_2 = \dot{\varphi}/\Lambda$. Thus,

$$\lambda = \frac{1}{c\sqrt{\Lambda}\sqrt{\Lambda h_2^2 - \kappa}} \ln \left(\frac{1+q(t)}{1-q(t)} \frac{1-q(t_i)}{1+q(t_i)} \right), \quad q(t) = \frac{h_1^2 e^{-2\sqrt{\Lambda}ct}}{\Lambda h_2^2 - \kappa} + 1, \quad (2.5.16)$$

where t_i is the initial time. We can also solve the orbital equation:

$$r(t) = w(t) - w(t_i) + r_i, \quad w(t) = -\frac{\sqrt{h_1^2 e^{-2\sqrt{\Lambda}ct} + \Lambda h_2^2 - \kappa}}{h_1 \sqrt{\Lambda}}, \quad (2.5.17)$$

where r_i is the initial radial position.

Further reading:

Rindler[Rin98]: “Every spherically symmetric solution of the generalized vacuum field equations $R_{ij} = \Lambda g_{ij}$ is either equivalent to Kottler’s generalization of Schwarzschild space or to the [...] Bertotti-Kasner space (for which Λ must be necessarily be positive).”

2.6 Bessel gravitational wave

D. Kramer introduced in [Kra99] an exact gravitational wave solution of Einstein's vacuum field equations. According to [Ste03] we execute the substitution $x \rightarrow t$ and $y \rightarrow z$.

2.6.1 Cylindrical coordinates

The metric of the Bessel wave in cylindrical coordinates reads

$$ds^2 = e^{-2U} [e^{2K} (d\rho^2 - dt^2) + \rho^2 d\varphi^2] + e^{2U} dz^2. \quad (2.6.1)$$

The functions U and K are given by

$$U := C J_0(\rho) \cos(t), \quad (2.6.2)$$

$$K := \frac{1}{2} C^2 \rho \left\{ \rho [J_0(\rho)^2 + J_1(\rho)^2] - 2J_0(\rho) J_1(\rho) \cos^2(t) \right\}, \quad (2.6.3)$$

where $J_n(\rho)$ are the Bessel functions of the first kind.

Christoffel symbols:

$$\Gamma_{tt}^t = \Gamma_{t\rho}^\rho = \Gamma_{\rho\rho}^\rho = -\frac{\partial U}{\partial t} + \frac{\partial K}{\partial t}, \quad \Gamma_{t\varphi}^\varphi = \Gamma_{tz}^z = -\frac{\partial U}{\partial t}, \quad \Gamma_{\varphi\varphi}^\varphi = -e^{-2K} \rho^2 \frac{\partial U}{\partial t}, \quad (2.6.4a)$$

$$\Gamma_{tt}^\rho = \Gamma_{t\rho}^t = \Gamma_{\rho\rho}^\rho = -\frac{\partial U}{\partial \rho} + \frac{\partial K}{\partial \rho}, \quad \Gamma_{\rho\varphi}^\varphi = \frac{1}{\rho} - \frac{\partial U}{\partial \rho}, \quad \Gamma_{zz}^\rho = -e^{4U-2K} \frac{\partial U}{\partial \rho}, \quad (2.6.4b)$$

$$\Gamma_{\varphi\varphi}^\rho = \rho e^{-2K} \left(\rho \frac{\partial U}{\partial \rho} - 1 \right), \quad \Gamma_{\rho z}^z = \frac{\partial U}{\partial \rho}, \quad \Gamma_{zz}^t = e^{4U-2K} \frac{\partial U}{\partial t}. \quad (2.6.4c)$$

Local tetrad:

$$\mathbf{e}_{(t)} = e^{U-K} \partial_t, \quad \mathbf{e}_{(\rho)} = e^{U-K} \partial_\rho, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\rho} e^U \partial_\varphi, \quad \mathbf{e}_{(z)} = e^{-U} \partial_z. \quad (2.6.5)$$

Dual tetrad:

$$\theta^{(t)} = e^{K-U} dt, \quad \theta^{(\rho)} = e^{K-U} d\rho, \quad \theta^{(\varphi)} = \rho e^{-U} d\varphi, \quad \theta^{(z)} = e^U dz. \quad (2.6.6)$$

2.6.2 Cartesian coordinates

In Cartesian coordinates with $\rho = \sqrt{x^2 + y^2}$ the metric (2.6.1) reads

$$ds^2 = -e^{2(K-U)} dt^2 + \frac{e^{-2U}}{x^2 + y^2} \left[(e^{2K} x^2 + y^2) dx^2 + 2xy (e^{2K} - 1) dx dy + (x^2 + e^{2K} y^2) dy^2 \right] + e^{2U} dz^2. \quad (2.6.7)$$

Local tetrad:

$$\mathbf{e}_{(t)} = e^{U-K} \partial_t, \quad \mathbf{e}_{(x)} = e^U \sqrt{\frac{x^2 + y^2}{e^{2K} x^2 + y^2}} \partial_x, \quad (2.6.8)$$

$$\mathbf{e}_{(y)} = e^{U-K} \sqrt{\frac{e^{2K} x^2 + y^2}{x^2 + y^2}} \partial_y + xy \frac{e^{U-K} (e^{2K} - 1)}{\sqrt{(x^2 + y^2) (e^{2K} x^2 + y^2)}} \partial_x, \quad \mathbf{e}_{(z)} = e^{-U} \partial_z$$

2.7 Cosmic string in Schwarzschild spacetime

A cosmic string in the Schwarzschild spacetime represented by Schwarzschild coordinates $(t, r, \vartheta, \varphi)$ reads

$$ds^2 = - \left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{1}{1 - r_s/r} dr^2 + r^2 (d\vartheta^2 + \beta^2 \sin^2 \vartheta d\varphi^2), \quad (2.7.1)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, M is the mass of the black hole, and β is the string parameter, compare Aryal et al [AFV86].

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{c^2 r_s (r - r_s)}{2r^3}, \quad \Gamma_{tr}^t = \frac{r_s}{2r(r - r_s)}, \quad \Gamma_{rr}^r = -\frac{r_s}{2r(r - r_s)}, \quad (2.7.2a)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^r = -(r - r_s), \quad (2.7.2b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^r = -(r - r_s) \beta^2 \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\beta^2 \sin \vartheta \cos \vartheta. \quad (2.7.2c)$$

Partial derivatives

$$\Gamma_{tt,r}^r = -\frac{(2r - 3r_s)c^2 r_s}{2r^4}, \quad \Gamma_{tr,r}^t = -\frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \quad \Gamma_{rr,r}^r = \frac{(2r - r_s)r_s}{2r^2(r - r_s)^2}, \quad (2.7.3a)$$

$$\Gamma_{r\vartheta,r}^\vartheta = -\frac{1}{r^2}, \quad \Gamma_{r\varphi,r}^\varphi = -\frac{1}{r^2}, \quad \Gamma_{\vartheta\vartheta,r}^r = -1, \quad (2.7.3b)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^\varphi = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,r}^r = -\beta^2 \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^\vartheta = -\beta^2 \cos(2\vartheta), \quad (2.7.3c)$$

$$\Gamma_{\varphi\varphi,\vartheta}^r = -(r - r_s) \beta^2 \sin(2\vartheta). \quad (2.7.3d)$$

Riemann-Tensor:

$$R_{ttrt} = -\frac{c^2 r_s}{r^3}, \quad R_{t\vartheta t\vartheta} = \frac{1}{2} \frac{c^2 (r - r_s) r_s}{r^2}, \quad R_{t\varphi t\varphi} = \frac{1}{2} \frac{c^2 (r - r_s) r_s \beta^2 \sin^2 \vartheta}{r^2}, \quad (2.7.4a)$$

$$R_{r\vartheta r\vartheta} = -\frac{1}{2} \frac{r_s}{r - r_s}, \quad R_{r\varphi r\varphi} = -\frac{1}{2} \frac{r_s \beta^2 \sin^2 \vartheta}{r - r_s}, \quad R_{\vartheta\varphi\vartheta\varphi} = r r_s \beta^2 \sin^2 \vartheta. \quad (2.7.4b)$$

The Ricci tensor as well as the Ricci scalar vanish identically. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschmann scalar reads

$$\mathcal{K} = 12 \frac{r_s^2}{r^6}. \quad (2.7.5)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1 - r_s/r}} \partial_t, \quad \mathbf{e}_{(r)} = \sqrt{1 - \frac{r_s}{r}} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\beta \sin \vartheta} \partial_\varphi. \quad (2.7.6)$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{1 - \frac{r_s}{r}} dt, \quad \theta^{(r)} = \frac{dr}{\sqrt{1 - r_s/r}}, \quad \theta^{(\vartheta)} = r d\vartheta, \quad \theta^{(\varphi)} = r\beta \sin \vartheta d\varphi. \quad (2.7.7)$$

Ricci rotation coefficients:

$$\gamma_{(t)(t)(t)} = \frac{r_s}{2r^2 \sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r} \sqrt{1 - \frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{r}. \quad (2.7.8)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s}{2r^2\sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{r}. \quad (2.7.9)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.7.10a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.7.10b)$$

Embedding:

The embedding function for $\beta^2 < 1$ reads

$$z = (r - r_s) \sqrt{\frac{r}{r - r_s} - \beta^2} - \frac{r_s}{2\sqrt{1 - \beta^2}} \ln \frac{\sqrt{r/(r - r_s) - \beta^2} - \sqrt{1 - \beta^2}}{\sqrt{r/(r - r_s) - \beta^2} + \sqrt{1 - \beta^2}}. \quad (2.7.11)$$

If $\beta^2 = 1$, we have the embedding function of the standard Schwarzschild metric, compare Eq.(2.2.15).

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2} \frac{k^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2} \left(1 - \frac{r_s}{r}\right) \left(\frac{h^2}{r^2\beta^2} - \kappa c^2\right) \quad (2.7.12)$$

with the constants of motion $k = (1 - r_s/r)c^2 i$ and $h = r^2\beta^2\dot{\varphi}$. The maxima of the effective potential V_{eff} lead to the same critical orbits $r_{\text{po}} = \frac{3}{2}r_s$ and $r_{\text{itcg}} = 3r_s$ as in the standard Schwarzschild metric.

2.8 Ernst spacetime

“The Ernst metric is a static, axially symmetric, electro-vacuum solution of the Einstein-Maxwell equations with a black hole immersed in a magnetic field.”[KV92]

In spherical coordinates $(t, r, \vartheta, \varphi)$, the Ernst metric reads[Ern76] ($G = c = 1$)

$$ds^2 = \Lambda^2 \left[- \left(1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - 2M/r} + r^2 d\vartheta^2 \right] + \frac{r^2 \sin^2 \vartheta}{\Lambda^2} d\varphi^2, \quad (2.8.1)$$

where $\Lambda = 1 + B^2 r^2 \sin^2 \vartheta$. Here, M is the mass of the black hole and B the magnetic field strength.

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{(2B^2 r^3 \sin^2 \vartheta - 3MB^2 r^2 \sin^2 \vartheta + M)(r - 2M)}{r^3 \Lambda}, \quad \Gamma_{tt}^{\vartheta} = \frac{2(r - 2M)B^2 \sin \vartheta \cos \vartheta}{r\Lambda}, \quad (2.8.2a)$$

$$\Gamma_{tr}^r = \frac{2B^2 r^3 \sin^2 \vartheta - 3MB^2 r^2 \sin^2 \vartheta + M}{r(r - 2M)\Lambda}, \quad \Gamma_{t\vartheta}^r = \frac{2B^2 r^2 \sin \vartheta \cos \vartheta}{\Lambda}, \quad (2.8.2b)$$

$$\Gamma_{rr}^r = \frac{2B^2 r^3 \sin^2 \vartheta - 5MB^2 r^2 \sin^2 \vartheta - M}{r(r - 2M)\Lambda}, \quad \Gamma_{rr}^{\vartheta} = -\frac{2B^2 r \sin \vartheta \cos \vartheta}{(r - 2M)\Lambda}, \quad (2.8.2c)$$

$$\Gamma_{r\vartheta}^r = \frac{2B^2 r^2 \sin \vartheta \cos \vartheta}{\Lambda}, \quad \Gamma_{r\vartheta}^{\vartheta} = \frac{3B^2 r^2 \sin^2 \vartheta + 1}{r\Lambda}, \quad (2.8.2d)$$

$$\Gamma_{r\varphi}^{\varphi} = \frac{1 - B^2 r^2 \sin^2 \vartheta}{r\Lambda}, \quad \Gamma_{\vartheta\vartheta}^r = \frac{(3B^2 r^2 \sin^2 \vartheta + 1)(r - 2M)}{\Lambda}, \quad (2.8.2e)$$

$$\Gamma_{\vartheta\vartheta}^{\vartheta} = \frac{2B^2 r^2 \sin \vartheta \cos \vartheta}{\Lambda}, \quad \Gamma_{\vartheta\varphi}^{\varphi} = \frac{\Xi \cos \vartheta}{\Lambda}, \quad (2.8.2f)$$

$$\Gamma_{\varphi\varphi}^r = \frac{(r - 2M)\Xi \sin^2 \vartheta}{\Lambda^5}, \quad (2.8.2g)$$

$$\Gamma_{\varphi\varphi}^{\vartheta} = \frac{\Xi \sin \vartheta \cos \vartheta}{\Lambda^5}. \quad (2.8.2h)$$

with $\Xi = 1 - B^2 r^2 \sin^2 \vartheta$.

Riemann-Tensor:

$$R_{ttrr} = \frac{2}{r^3} \left[B^4 r^4 \sin^4 \vartheta (3M - r) - M + 2r^5 B^4 \sin^2 \vartheta \cos^2 \vartheta + B^2 r^2 \sin^2 \vartheta (r - 2M) \right], \quad (2.8.3a)$$

$$R_{trr\vartheta} = 2B^2 \sin \vartheta \cos \vartheta \left[(3B^2 r^2 \sin^2 \vartheta (2M - 3r) + r - 2M) \right], \quad (2.8.3b)$$

$$R_{t\vartheta t\vartheta} = \frac{1}{r^2} \left[B^4 r^4 (r - 2M)(4r - 9M) \sin^4 \vartheta + 2\Xi B^2 r^3 (r - 2M) \cos^2 \vartheta + M(r - 2M) \right], \quad (2.8.3c)$$

$$R_{t\varphi t\varphi} = \frac{1}{\Lambda^4 r^2} \left[(2B^2 r^3 - 3B^2 M r^2 \sin^2 \vartheta + M)\Xi (r - 2M) \sin^2 \vartheta \right], \quad (2.8.3d)$$

$$R_{r\vartheta r\vartheta} = -\frac{(2B^2 r^3 - 3B^2 M r^2 \sin^2 \vartheta + M)\Xi}{r - 2M}, \quad (2.8.3e)$$

$$R_{r\varphi r\varphi} = -\frac{\sin^2 \vartheta}{\Lambda^4 (r - 2M)} \left[B^4 r^4 (4r - 9M) \sin^4 \vartheta + 2B^2 r^2 (8M - 4r\vartheta) \sin^2 \vartheta + 2\Xi B^2 r^3 \cos^2 \vartheta + M \right], \quad (2.8.3f)$$

$$R_{r\varphi\vartheta\varphi} = -\frac{2B^2 r^3 \sin^3 \vartheta \cos \vartheta (3B^2 r^2 \sin^2 \vartheta - 5)}{\Lambda^4}, \quad (2.8.3g)$$

$$R_{\vartheta\varphi\vartheta\varphi} = \frac{r \sin^2 \vartheta}{\Lambda^4} \left[2B^4 r^4 (r - 3M) \sin^4 \vartheta + 4B^2 r^3 \cos^2 \vartheta (1 + \Xi) + 2B^2 r^2 \sin^2 \vartheta (2M - r) + 2M \right]. \quad (2.8.3h)$$

Ricci-Tensor:

$$R_{tt} = \frac{4B^2(r-2M)(r+2M\sin^2\vartheta)}{r^2\Lambda^2}, \quad R_{rr} = -\frac{4B^2[r\cos^2\vartheta - (r-2M)\sin^2\vartheta]}{(r-2M)\Lambda^2}, \quad (2.8.4a)$$

$$R_{r\vartheta} = \frac{8B^2r\sin\vartheta\cos\vartheta}{\Lambda^2}, \quad R_{\vartheta\vartheta} = \frac{4B^2r[r\cos^2\vartheta + (r-2M)\sin^2\vartheta]}{\Lambda^2}, \quad (2.8.4b)$$

$$R_{\varphi\varphi} = \frac{4B^2r\sin^2\vartheta(r+2M\sin^2\vartheta)}{\Lambda^6}. \quad (2.8.4c)$$

Ricci and Kretschmann scalars:

$$R = 0, \quad (2.8.5a)$$

$$\begin{aligned} \mathcal{K} = \frac{16}{r^6\Lambda^8} & \left[3B^8r^8(4r^2 - 18Mr + 21M^2)\sin^8\vartheta \right. \\ & + 2B^4r^4(31M^2 - 37Mr - 24B^2r^4\cos^2\vartheta + 42B^2Mr^3\cos^2\vartheta + 10r^2 + 6B^4r^6\cos^4\vartheta)\sin^6\vartheta \\ & + 2B^2r^2(-3Mr + 20B^2r^4\cos^2\vartheta + 6M^2 - 46B^2Mr^3\cos^2\vartheta - 12B^4r^6\cos^4\vartheta)\sin^4\vartheta \\ & - 6B^6r^6(6B^2Mr^3\cos^2\vartheta + 4r^2 - 4B^2r^4\cos^2\vartheta + 18M^2 - 17Mr) \\ & \left. + 20B^4r^6\cos^4\vartheta + 12B^2Mr^3\cos^2\vartheta + 3M^2 \right]. \quad (2.8.5b) \end{aligned}$$

Static local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{\Lambda\sqrt{1-2m/r}}\partial_t, \quad \mathbf{e}_{(r)} = \frac{\sqrt{1-2m/r}}{\Lambda}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\Lambda r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{\Lambda}{r\sin\vartheta}\partial_\varphi. \quad (2.8.6)$$

Dual tetrad:

$$\theta^{(t)} = \Lambda\sqrt{1-\frac{2m}{r}}dt, \quad \theta^{(r)} = \frac{\Lambda}{\sqrt{1-2m/r}}dr, \quad \theta^{(\vartheta)} = \Lambda r d\vartheta, \quad \theta^{(\varphi)} = \frac{r\sin\vartheta}{\Lambda}d\varphi. \quad (2.8.7)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\dot{r}^2 + \frac{h^2(1-r_s/r)}{r^2} - \frac{k^2}{\Lambda^4} + \kappa\frac{1-r_s/r}{\Lambda^2} = 0 \quad (2.8.8)$$

with constants of motion $k = \Lambda^2(1-r_s/r)\dot{t}$ and $h = (r^2/\Lambda^2)\dot{\varphi}$.

Further reading:

Ernst[Ern76], Dhurandhar and Sharma[DS83], Karas and Vokrouhlicky[KV92], Stuchlík and Hledík[SH99].

2.9 Friedman-Robertson-Walker

The Friedman-Robertson-Walker metric describes a general homogeneous and isotropic universe. In a general form it reads:

$$ds^2 = -c^2 dt^2 + R^2 d\sigma^2 \quad (2.9.1)$$

with $R = R(t)$ being an arbitrary function of time only and $d\sigma^2$ being a metric of a 3-space of constant curvature for which three explicit forms will be described here.

In all formulas in this section a dot denotes differentiation with respect to t , e.g. $\dot{R} = dR(t)/dt$.

2.9.1 Form 1

$$ds^2 = -c^2 dt^2 + R^2 \left\{ \frac{d\eta^2}{1 - k\eta^2} + \eta^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right\} \quad (2.9.2)$$

Christoffel symbols:

$$\Gamma_{t\eta}^\eta = \frac{\dot{R}}{R}, \quad \Gamma_{t\vartheta}^\vartheta = \frac{\dot{R}}{R}, \quad \Gamma_{t\varphi}^\varphi = \frac{\dot{R}}{R}, \quad (2.9.3a)$$

$$\Gamma_{\eta\eta}^t = \frac{R\dot{R}}{c^2(1 - k\eta^2)}, \quad \Gamma_{\eta\eta}^\eta = \frac{k\eta}{1 - k\eta^2}, \quad \Gamma_{\eta\vartheta}^\vartheta = \frac{1}{\eta}, \quad (2.9.3b)$$

$$\Gamma_{\eta\varphi}^\varphi = \frac{1}{\eta}, \quad \Gamma_{\vartheta\vartheta}^t = \frac{R\eta^2\dot{R}}{c^2}, \quad \Gamma_{\vartheta\vartheta}^\eta = (k\eta^2 - 1)\eta, \quad (2.9.3c)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^t = \frac{R\eta^2 \sin^2 \vartheta \dot{R}}{c^2}, \quad \Gamma_{\varphi\varphi}^\eta = (k\eta^2 - 1)\eta \sin^2 \vartheta, \quad (2.9.3d)$$

$$\Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.9.3e)$$

Riemann-Tensor:

$$R_{t\eta t\eta} = \frac{R\ddot{R}}{k\eta^2 - 1}, \quad R_{t\vartheta t\vartheta} = -R\eta^2\ddot{R}, \quad (2.9.4a)$$

$$R_{t\varphi t\varphi} = -R\eta^2 \sin^2 \vartheta \ddot{R}, \quad R_{\eta\vartheta\eta\vartheta} = -\frac{R^2\eta^2(\dot{R}^2 + kc^2)}{c^2(k\eta^2 - 1)}, \quad (2.9.4b)$$

$$R_{\eta\varphi\eta\varphi} = -\frac{R^2\eta^2 \sin^2 \vartheta (\dot{R}^2 + kc^2)}{c^2(k\eta^2 - 1)}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\eta^4 \sin^2 \vartheta (\dot{R}^2 + kc^2)}{c^2}. \quad (2.9.4c)$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \quad R_{\eta\eta} = \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2(1 - k\eta^2)}, \quad (2.9.5a)$$

$$R_{\vartheta\vartheta} = \eta^2 \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2}, \quad R_{\varphi\varphi} = \eta^2 \sin^2 \vartheta \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2}. \quad (2.9.5b)$$

The Ricci scalar and Kretschmann scalar read:

$$\mathcal{R} = 6\frac{R\ddot{R} + \dot{R}^2 + kc^2}{R^2c^2}, \quad \mathcal{K} = 12\frac{\ddot{R}^2R^2 + \dot{R}^4 + 2\dot{R}^2kc^2 + k^2c^4}{R^4c^4}. \quad (2.9.6)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \quad e_{(\eta)} = \frac{\sqrt{1 - k\eta^2}}{R}\partial_\eta, \quad e_{(\vartheta)} = \frac{1}{R\eta}\partial_\vartheta, \quad e_{(\varphi)} = \frac{1}{R\eta \sin \vartheta}\partial_\varphi. \quad (2.9.7)$$

Ricci rotation coefficients:

$$\begin{aligned} \gamma_{(\eta)(t)(\eta)} &= \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} & \gamma_{(\vartheta)(\eta)(\vartheta)} &= \gamma_{(\varphi)(\eta)(\varphi)} = \frac{\sqrt{1-k\eta^2}}{R\eta}, \\ \gamma_{(\varphi)(\vartheta)(\varphi)} &= \frac{\cot \vartheta}{R\eta}. \end{aligned} \quad (2.9.8)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \quad \gamma_{(r)} = \frac{2\sqrt{1-k\eta^2}}{R\eta}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{R\eta}. \quad (2.9.9)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\eta)(t)(\eta)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2} \quad (2.9.10a)$$

$$R_{(\eta)(\vartheta)(\eta)(\vartheta)} = R_{(\eta)(\varphi)(\eta)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2 + kc^2}{R^2c^2}. \quad (2.9.10b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \quad R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2\dot{R}^2 + 2kc^2}{R^2c^2}. \quad (2.9.11)$$

2.9.2 Form 2

$$ds^2 = -c^2 dt^2 + \frac{R^2}{(1 + \frac{k}{4}r^2)^2} \{ dr^2 + r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \} \quad (2.9.12)$$

Christoffel symbols:

$$\Gamma_{tr}^r = \frac{\dot{R}}{R}, \quad \Gamma_{t\vartheta}^{\vartheta} = \frac{\dot{R}}{R}, \quad \Gamma_{t\varphi}^{\varphi} = \frac{\dot{R}}{R}, \quad (2.9.13a)$$

$$\Gamma_{rr}^r = 16 \frac{R\dot{R}}{c^2(4+kr^2)^2}, \quad \Gamma_{rr}^r = -\frac{2kr}{4+kr^2}, \quad \Gamma_{r\vartheta}^{\vartheta} = \frac{4-kr^2}{(4+kr^2)r}, \quad (2.9.13b)$$

$$\Gamma_{r\varphi}^{\varphi} = \frac{4-kr^2}{(4+kr^2)r}, \quad \Gamma_{\vartheta\vartheta}^{\vartheta} = 16 \frac{Rr^2\dot{R}}{c^2(4+kr^2)^2}, \quad \Gamma_{\vartheta\vartheta}^r = \frac{r(kr^2-4)}{4+kr^2}, \quad (2.9.13c)$$

$$\Gamma_{\vartheta\varphi}^{\varphi} = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^r = 16 \frac{Rr^2 \sin^2 \vartheta \dot{R}}{c^2(4+kr^2)^2}, \quad \Gamma_{\varphi\varphi}^{\vartheta} = -\sin \vartheta \cos \vartheta, \quad (2.9.13d)$$

$$\Gamma_{\varphi\varphi}^r = \frac{r \sin^2 \vartheta (kr^2 - 4)}{4 + kr^2}. \quad (2.9.13e)$$

Riemann-Tensor:

$$R_{trtr} = -16 \frac{R\ddot{R}}{(4+kr^2)^2}, \quad R_{t\vartheta t\vartheta} = -16 \frac{Rr^2\ddot{R}}{(4+kr^2)^2}, \quad (2.9.14a)$$

$$R_{t\varphi t\varphi} = -16 \frac{Rr^2 \sin^2 \vartheta \ddot{R}}{(4+kr^2)^2}, \quad R_{r\vartheta r\vartheta} = 256 \frac{R^2 r^2 (\dot{R}^2 + kc^2)}{c^2(4+kr^2)^4}, \quad (2.9.14b)$$

$$R_{r\varphi r\varphi} = 256 \frac{R^2 r^2 \sin^2 \vartheta (\dot{R}^2 + kc^2)}{c^2(4+kr^2)^4}, \quad R_{\vartheta\varphi\vartheta\varphi} = 256 \frac{R^2 r^4 \sin^2 \vartheta (\dot{R}^2 + kc^2)}{c^2(4+kr^2)^4}. \quad (2.9.14c)$$

Ricci-Tensor:

$$R_{tt} = -3 \frac{\ddot{R}}{R}, \quad R_{rr} = 16 \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2(4+kr^2)^2}, \quad (2.9.15a)$$

$$R_{\vartheta\vartheta} = 16r^2 \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2(4+kr^2)^2}, \quad R_{\varphi\varphi} = 16r^2 \sin^2 \vartheta \frac{R\ddot{R} + 2(\dot{R}^2 + kc^2)}{c^2(4+kr^2)^2}. \quad (2.9.15b)$$

The Ricci scalar and Kretschmann scalar read:

$$\mathcal{R} = 6 \frac{R\ddot{R} + \dot{R}^2 + kc^2}{R^2 c^2}, \quad \mathcal{K} = 12 \frac{\ddot{R}^2 R^2 + \dot{R}^4 + 2\dot{R}^2 kc^2 + k^2 c^4}{R^4 c^4}. \quad (2.9.16)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c} \partial_t, \quad e_{(r)} = \frac{1 + \frac{k}{4} r^2}{R} \partial_r, \quad e_{(\vartheta)} = \frac{1 + \frac{k}{4} r^2}{Rr} \partial_{\vartheta}, \quad e_{(\varphi)} = \frac{1 + k/4r^2}{Rr \sin \vartheta} \partial_{\varphi}. \quad (2.9.17)$$

Ricci rotation coefficients:

$$\gamma_{(t)(t)(r)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = -\frac{\frac{k}{4} r^2 - 1}{Rr}, \quad (2.9.18a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{(\frac{k}{4} r^2 + 1) \cot \vartheta}{Rr}. \quad (2.9.18b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \quad \gamma_{(r)} = 2 \frac{1 - \frac{k}{4} r^2}{Rr}, \quad \gamma_{(\vartheta)} = \frac{(\frac{k}{4} r^2 + 1) \cot \vartheta}{Rr}. \quad (2.9.19)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\eta)(t)(\eta)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2} \quad (2.9.20a)$$

$$R_{(\eta)(\vartheta)(\eta)(\vartheta)} = R_{(\eta)(\varphi)(\eta)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2 + kc^2}{R^2 c^2}. \quad (2.9.20b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \quad R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2\dot{R}^2 + 2kc^2}{R^2 c^2}. \quad (2.9.21)$$

2.9.3 Form 3

The following forms of the metric are obtained from 2.9.2 by setting $\eta = \sin \psi$, ψ , $\sinh \psi$ for $k = 1, 0, -1$ respectively.

Positive Curvature

$$\boxed{ds^2 = -c^2 dt^2 + R^2 \{d\psi^2 + \sin^2 \psi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)\}} \quad (2.9.22)$$

Christoffel symbols:

$$\Gamma_{t\psi}^{\psi} = \frac{\dot{R}}{R}, \quad \Gamma_{t\vartheta}^{\vartheta} = \frac{\dot{R}}{R}, \quad \Gamma_{t\varphi}^{\varphi} = \frac{\dot{R}}{R}, \quad (2.9.23a)$$

$$\Gamma_{\psi\psi}^r = \frac{R\dot{R}}{c^2}, \quad \Gamma_{\psi\vartheta}^{\vartheta} = \cot \psi, \quad \Gamma_{\psi\varphi}^{\varphi} = \cot \psi, \quad (2.9.23b)$$

$$\Gamma_{\vartheta\vartheta}^r = \frac{R \sin^2 \psi \dot{R}}{c^2}, \quad \Gamma_{\vartheta\vartheta}^{\psi} = -\sin \psi \cos \psi, \quad \Gamma_{\vartheta\varphi}^{\varphi} = \cot(\vartheta), \quad (2.9.23c)$$

$$\Gamma_{\varphi\varphi}^r = \frac{R \sin^2 \psi \sin^2 \vartheta \dot{R}}{c^2}, \quad \Gamma_{\varphi\varphi}^{\psi} = -\sin \psi \cos \psi \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^{\vartheta} = -\sin \vartheta \cos \vartheta. \quad (2.9.23d)$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \quad R_{t\vartheta t\vartheta} = -R\sin^2\psi\ddot{R}, \quad (2.9.24a)$$

$$R_{t\varphi t\varphi} = -R\sin^2\psi\sin^2\vartheta\ddot{R}, \quad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\sin^2\psi(\dot{R}^2 + c^2)}{c^2}, \quad (2.9.24b)$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2\sin^2\psi\sin^2\vartheta(\dot{R}^2 + c^2)}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\sin^4\psi\sin^2\vartheta(\dot{R}^2 + c^2)}{c^2}. \quad (2.9.24c)$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \quad R_{\psi\psi} = \frac{R\ddot{R} + 2(\dot{R}^2 + c^2)}{c^2}, \quad (2.9.25a)$$

$$R_{\vartheta\vartheta} = \sin^2\psi\frac{R\ddot{R} + 2(\dot{R}^2 + c^2)}{c^2}, \quad R_{\varphi\varphi} = \sin^2\vartheta\sin^2\psi\frac{R\ddot{R} + 2(\dot{R}^2 + c^2)}{c^2}. \quad (2.9.25b)$$

The Ricci scalar and Kretschmann read

$$\mathcal{R} = 6\frac{R\ddot{R} + \dot{R}^2 + c^2}{R^2c^2}, \quad \mathcal{K} = 12\frac{\ddot{R}^2R^2 + \dot{R}^4 + 2\dot{R}^2c^2 + c^4}{R^4c^4}. \quad (2.9.26)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \quad e_{(\psi)} = \frac{1}{R}\partial_\psi, \quad e_{(\vartheta)} = \frac{1}{R\sin\psi}\partial_\vartheta, \quad e_{(\varphi)} = \frac{1}{R\sin\psi\sin\vartheta}\partial_\varphi. \quad (2.9.27)$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc}, \quad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{\cot\psi}{R}, \quad (2.9.28a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{R\sin\psi}. \quad (2.9.28b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \quad \gamma_{(\psi)} = 2\frac{\cot\psi}{R}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\sin\psi}. \quad (2.9.29)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \quad (2.9.30a)$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2 + c^2}{R^2c^2}. \quad (2.9.30b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \quad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2(\dot{R}^2 + c^2)}{R^2c^2}. \quad (2.9.31)$$

Vanishing Curvature

$$\boxed{ds^2 = -c^2dt^2 + R^2\{d\psi^2 + \psi^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\}} \quad (2.9.32)$$

Christoffel symbols:

$$\Gamma_{t\psi}^\psi = \frac{\dot{R}}{R}, \quad \Gamma_{t\vartheta}^\vartheta = \frac{\dot{R}}{R}, \quad \Gamma_{t\varphi}^\varphi = \frac{\dot{R}}{R}, \quad (2.9.33a)$$

$$\Gamma_{\psi\psi}^t = \frac{R\dot{R}}{c^2}, \quad \Gamma_{\psi\vartheta}^\vartheta = \frac{1}{\psi}, \quad \Gamma_{\psi\varphi}^\varphi = \frac{1}{\psi}, \quad (2.9.33b)$$

$$\Gamma_{\vartheta\vartheta}^t = \frac{R\psi^2\dot{R}}{c^2}, \quad \Gamma_{\vartheta\vartheta}^\psi = -\psi, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot(\vartheta), \quad (2.9.33c)$$

$$\Gamma_{\varphi\varphi}^t = \frac{R\psi^2\sin^2\vartheta\dot{R}}{c^2}, \quad \Gamma_{\varphi\varphi}^\psi = -\psi\sin^2\vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin\vartheta\cos\vartheta. \quad (2.9.33d)$$

Riemann-Tensor:

$$R_{t\psi r\psi} = -R\ddot{R}, \quad R_{t\vartheta t\vartheta} = -R\psi^2\ddot{R}, \quad (2.9.34a)$$

$$R_{t\varphi t\varphi} = -R\psi^2 \sin^2 \vartheta \ddot{R}, \quad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\psi^2\ddot{R}^2}{c^2}, \quad (2.9.34b)$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2\psi^2 \sin^2 \vartheta \ddot{R}^2}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\psi^4 \sin^2 \vartheta \ddot{R}^2}{c^2}. \quad (2.9.34c)$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \quad R_{\psi\psi} = \frac{R\ddot{R} + 2\dot{R}^2}{c^2}, \quad (2.9.35a)$$

$$R_{\vartheta\vartheta} = \psi^2 \frac{R\ddot{R} + 2\dot{R}^2}{c^2}, \quad R_{\varphi\varphi} = \sin^2 \vartheta \psi^2 \frac{R\ddot{R} + 2\dot{R}^2}{c^2}. \quad (2.9.35b)$$

The Ricci scalar and Kretschmann read

$$\mathcal{R} = 6\frac{R\ddot{R} + \dot{R}^2}{R^2c^2}, \quad \mathcal{K} = 12\frac{\ddot{R}^2R^2 + \dot{R}^4}{R^4c^4}. \quad (2.9.36)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \quad e_{(\psi)} = \frac{1}{R}\partial_\psi, \quad e_{(\vartheta)} = \frac{1}{R\psi}\partial_\vartheta, \quad e_{(\varphi)} = \frac{1}{R\psi \sin \vartheta}\partial_\varphi. \quad (2.9.37)$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc}, \quad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{1}{R\psi}, \quad (2.9.38a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot(\vartheta)}{R\psi}. \quad (2.9.38b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \quad \gamma_{(r)} = \frac{2}{R\psi}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{R\psi}. \quad (2.9.39)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \quad (2.9.40a)$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2}{R^2c^2}. \quad (2.9.40b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \quad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2\dot{R}^2}{R^2c^2}. \quad (2.9.41)$$

Negative Curvature

$$ds^2 = -c^2 dt^2 + R^2 \{ d\psi^2 + \sinh^2 \psi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \} \quad (2.9.42)$$

Christoffel symbols:

$$\Gamma_{t\psi}^\psi = \frac{\dot{R}}{R}, \quad \Gamma_{t\vartheta}^\vartheta = \frac{\dot{R}}{R}, \quad \Gamma_{t\varphi}^\varphi = \frac{\dot{R}}{R}, \quad (2.9.43a)$$

$$\Gamma_{\psi\psi}^r = \frac{R\dot{R}}{c^2}, \quad \Gamma_{\psi\vartheta}^\vartheta = \coth \psi, \quad \Gamma_{\psi\varphi}^\varphi = \coth \psi, \quad (2.9.43b)$$

$$\Gamma_{\vartheta\vartheta}^r = \frac{R \sinh^2 \psi \dot{R}}{c^2}, \quad \Gamma_{\vartheta\vartheta}^\psi = -\sinh \psi \cosh \psi, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad (2.9.43c)$$

$$\Gamma_{\varphi\varphi}^r = \frac{R \sinh^2 \psi \sin^2 \vartheta \dot{R}}{c^2}, \quad \Gamma_{\varphi\varphi}^\psi = -\sinh \psi \cosh \psi \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.9.43d)$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \quad R_{t\vartheta t\vartheta} = -R \sinh^2 \psi \ddot{R}, \quad (2.9.44a)$$

$$R_{t\varphi t\varphi} = -R \sinh^2 \psi \sin^2 \vartheta \ddot{R}, \quad R_{\psi\vartheta\psi\vartheta} = \frac{R^2 \sinh^2 \psi (\dot{R}^2 - c^2)}{c^2}, \quad (2.9.44b)$$

$$R_{\psi\varphi\psi\varphi} = \frac{R^2 \sinh^2 \psi \sin^2 \vartheta (\dot{R}^2 - c^2)}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2 \sinh^4 \psi \sin^2 \vartheta (\dot{R}^2 - c^2)}{c^2}. \quad (2.9.44c)$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \quad R_{\psi\psi} = \frac{R\ddot{R} + 2(\dot{R}^2 - c^2)}{c^2}, \quad (2.9.45a)$$

$$R_{\vartheta\vartheta} = \sinh^2 \psi \frac{R\ddot{R} + 2(\dot{R}^2 - c^2)}{c^2}, \quad R_{\varphi\varphi} = \sin^2 \vartheta \sin^2 \psi \frac{R\ddot{R} + 2(\dot{R}^2 - c^2)}{c^2}. \quad (2.9.45b)$$

The Ricci scalar and Kretschmann read

$$\mathcal{R} = 6\frac{R\ddot{R} + \dot{R}^2 - c^2}{R^2 c^2}, \quad \mathcal{K} = 12\frac{\ddot{R}^2 R^2 + \dot{R}^4 - 2\dot{R}^2 c^2 + c^4}{R^4 c^4}. \quad (2.9.46)$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \quad e_{(\psi)} = \frac{1}{R}\partial_\psi, \quad e_{(\vartheta)} = \frac{1}{R \sinh \psi}\partial_\vartheta, \quad e_{(\varphi)} = \frac{1}{R \sinh \psi \sin \vartheta}\partial_\varphi. \quad (2.9.47)$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc}, \quad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{\coth \psi}{R}, \quad (2.9.48a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{R \sinh \psi}. \quad (2.9.48b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \quad \gamma_{(r)} = 2\frac{\coth \psi}{R}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{R \sinh \psi}. \quad (2.9.49)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \quad (2.9.50a)$$

$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2 - c^2}{R^2 c^2}. \quad (2.9.50b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \quad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R} + 2(\dot{R}^2 - c^2)}{R^2 c^2}. \quad (2.9.51)$$

Further reading:

Rindler[Rin01]

2.10 Gödel Universe

Gödel introduced a homogeneous and rotating universe model in [Göd49]. We follow the notation of [KWSD04]

2.10.1 Cylindrical coordinates

The Gödel metric in cylindrical coordinates is

$$ds^2 = -c^2 dt^2 + \frac{dr^2}{1 + [r/(2a)]^2} + r^2 \left[1 - \left(\frac{r}{2a} \right)^2 \right] d\varphi^2 + dz^2 - 2r^2 \frac{c}{\sqrt{2a}} dt d\varphi, \quad (2.10.1)$$

where $2a$ is the Gödel radius.

Christoffel symbols:

$$\Gamma_{tr}^t = \frac{r}{2a^2} \frac{1}{1 + [r/(2a)]^2}, \quad \Gamma_{tr}^\varphi = -\frac{c}{\sqrt{2a}r} \frac{1}{1 + [r/(2a)]^2}, \quad (2.10.2a)$$

$$\Gamma_{t\varphi}^r = \frac{cr}{\sqrt{2a}} \left[1 + \left(\frac{r}{2a} \right)^2 \right], \quad \Gamma_{rr}^r = -\frac{r}{4a^2} \frac{1}{1 + [r/(2a)]^2}, \quad (2.10.2b)$$

$$\Gamma_{r\varphi}^t = \frac{r^3}{4\sqrt{2ca}^3} \frac{1}{1 + [r/(2a)]^2}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r} \frac{1}{1 + [r/(2a)]^2}, \quad (2.10.2c)$$

$$\Gamma_{\varphi\varphi}^r = r \left[1 + \left(\frac{r}{2a} \right)^2 \right] \left[1 - \frac{1}{2} \left(\frac{r}{a} \right)^2 \right]. \quad (2.10.2d)$$

Riemann-Tensor:

$$R_{trtr} = \frac{c^2}{2a^2} \frac{1}{1 + [r/(2a)]^2}, \quad R_{trr\varphi} = -\frac{cr^2}{2\sqrt{2a}^3} \frac{1}{1 + [r/(2a)]^2}, \quad (2.10.3a)$$

$$R_{t\varphi t\varphi} = \frac{c^2 r^2}{2a^2} \frac{1}{1 + [r/(2a)]^2}, \quad R_{r\varphi r\varphi} = \frac{r^2}{2a^2} \frac{1 + 3[r/(2a)]^2}{1 + [r/(2a)]^2}. \quad (2.10.3b)$$

Ricci-Tensor:

$$R_{tt} = \frac{c^2}{a^2}, \quad R_{t\varphi} = \frac{r^2 c}{\sqrt{2a}^3}, \quad R_{\varphi\varphi} = \frac{r^4}{2a^4}. \quad (2.10.4)$$

Ricci and Kretschmann scalar

$$\mathcal{R} = -\frac{1}{a^2}, \quad \mathcal{K} = \frac{3}{a^4}. \quad (2.10.5)$$

cosmological constant:

$$\Lambda = \frac{R}{2} \quad (2.10.6)$$

Killing vectors:

An infinitesimal isometric transformation $x'^\mu = x^\mu + \varepsilon \xi^\mu(x^\nu)$ leaves the metric unchanged, that is $g'_{\mu\nu}(x'^\sigma) = g_{\mu\nu}(x'^\sigma)$. A killing vector field ξ^μ is solution to the killing equation $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$. There exist five killing vector fields in Gödel's spacetime:

$$\xi_a^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_b^\mu = \frac{1}{\sqrt{1 + [r/(2a)]^2}} \begin{pmatrix} \frac{r}{\sqrt{2c}} \cos \varphi \\ a \left(1 + [r/(2a)]^2 \right) \sin \varphi \\ \frac{a}{r} \left(1 + 2[r/(2a)]^2 \right) \cos \varphi \\ 0 \end{pmatrix}, \quad \xi_c^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (2.10.7a)$$

$$\xi_d^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \xi_e^\mu = \frac{1}{\sqrt{1 + [r/(2a)]^2}} \begin{pmatrix} \frac{r}{\sqrt{2c}} \sin \varphi \\ -a \left(1 + [r/(2a)]^2 \right) \cos \varphi \\ \frac{a}{r} \left(1 + 2[r/(2a)]^2 \right) \sin \varphi \\ 0 \end{pmatrix}. \quad (2.10.7b)$$

An arbitrary linear combination of killing vector fields is again a killing vector field.

Local tetrad:

For the local tetrad in Gödel's spacetime an ansatz similar to the local tetrad of a rotating spacetime in spherical coordinates (Sec. 1.4.7) can be used. After substituting $\vartheta \rightarrow z$ and swapping base vectors $\mathbf{e}_{(2)}$ and $\mathbf{e}_{(3)}$ an orthonormalized and right-handed local tetrad is obtained.

$$\mathbf{e}_{(0)} = \Gamma(\partial_t + \zeta\partial_\phi), \quad \mathbf{e}_{(1)} = \sqrt{1 + [r/(2a)]^2}\partial_r, \quad \mathbf{e}_{(2)} = \Delta\Gamma(A\partial_t + B\partial_\phi), \quad \mathbf{e}_{(3)} = \partial_z, \quad (2.10.8a)$$

where

$$A = -\frac{r^2c}{\sqrt{2a}} + \zeta r^2(1 - [r/(2a)]^2), \quad B = c^2 + \frac{\zeta r^2c}{\sqrt{2a}}, \quad (2.10.9a)$$

$$\Gamma = \frac{1}{\sqrt{c^2 + \zeta r^2c\sqrt{2/a} - \zeta^2 r^2(1 - [r/(2a)]^2)}}, \quad \Delta = \frac{1}{rc\sqrt{1 + [r/(2a)]^2}}. \quad (2.10.9b)$$

Transformation between local direction $y^{(i)}$ and coordinate direction y^μ :

$$y^0 = y^{(0)}\Gamma + y^{(2)}\Delta\Gamma A, \quad y^1 = y^{(1)}\sqrt{1 + [r/(2a)]^2}, \quad y^2 = y^{(0)}\Gamma\zeta + y^{(2)}\Delta\Gamma B, \quad y^3 = y^{(3)}. \quad (2.10.10)$$

with the above abbreviations.

2.10.2 Scaled cylindrical coordinates

If we apply the simple transformation

$$T = \frac{t}{r_G}, \quad R = \frac{r}{r_G}, \quad \phi = \phi, \quad Z = \frac{z}{r_G}, \quad (2.10.11)$$

with $r_G = 2a$, we find a formulation for the metric scaling with r_G , which is

$$ds^2 = r_G^2 \left(-c^2 dT^2 + \frac{dR^2}{1+R^2} + R^2(1-R^2)D\phi^2 + dZ^2 - 2\sqrt{2}cR^2 dT d\phi \right). \quad (2.10.12)$$

Christoffel symbols:

$$\Gamma_{TR}^T = \frac{2R}{1+R^2}, \quad \Gamma_{TR}^\phi = -\frac{\sqrt{2}c}{R(1+R^2)}, \quad (2.10.13a)$$

$$\Gamma_{T\phi}^R = \sqrt{2}cR(1+R^2), \quad \Gamma_{RR}^R = -\frac{R}{1+R^2}, \quad (2.10.13b)$$

$$\Gamma_{R\phi}^T = \frac{\sqrt{2}R^3}{c(1+R^2)}, \quad \Gamma_{R\phi}^\phi = \frac{1}{R(1+R^2)}, \quad (2.10.13c)$$

$$\Gamma_{\phi\phi}^R = R(1+R^2)(2R^2 - 1). \quad (2.10.13d)$$

Riemann-Tensor:

$$R_{TRTR} = \frac{2r_G^2c^2}{1+R^2}, \quad R_{TRR\phi} = -\frac{2\sqrt{2}r_G^2cR^2}{1+R^2}, \quad (2.10.14a)$$

$$R_{T\phi T\phi} = 2c^2r_G^2R^2(1+R^2), \quad R_{R\phi R\phi} = \frac{2r_G^2R^2(1+3R^2)}{1+R^2}. \quad (2.10.14b)$$

Ricci-Tensor:

$$R_{TT} = 4c^2, \quad R_{T\phi} = 4\sqrt{2}cR^2, \quad R_{\phi\phi} = 8R^4. \quad (2.10.15)$$

Ricci and Kretschmann scalar

$$\mathcal{R} = -\frac{4}{r_G^2}, \quad \mathcal{K} = \frac{48}{r_G^4}. \quad (2.10.16)$$

cosmological constant:

$$\Lambda = \frac{R}{2} \quad (2.10.17)$$

Killing vectors:

The Killing vectors read

$$\xi_a^\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_b^\mu = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} \frac{R}{\sqrt{2c}} \cos \varphi \\ \frac{1}{2}(1+R^2) \sin \varphi \\ \frac{1}{2R}(1+2R^2) \cos \varphi \\ 0 \end{pmatrix}, \quad \xi_c^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (2.10.18a)$$

$$\xi_d^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \xi_e^\mu = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} \frac{R}{\sqrt{2c}} \sin \varphi \\ -\frac{1}{2}(1+R^2) \cos \varphi \\ \frac{1}{2R}(1+2R^2) \sin \varphi \\ 0 \end{pmatrix}. \quad (2.10.18b)$$

Local tetrad:

After the transformation to scaled cylindrical coordinates, the local tetrad reads

$$\mathbf{e}_{(0)} = \frac{\Gamma}{r_G} (\partial_T + \zeta \partial_\phi), \quad \mathbf{e}_{(1)} = \frac{1}{r_G} \sqrt{1+R^2} \partial_R, \quad \mathbf{e}_{(2)} = \frac{\Delta \Gamma}{r_G} (A \partial_T + B \partial_\phi), \quad \mathbf{e}_{(3)} = \frac{1}{r_G} \partial_Z, \quad (2.10.19a)$$

where

$$A = R^2 [-\sqrt{2c} + (1-R^2)\zeta], \quad B = c^2 + \sqrt{2}R^2 c \zeta, \quad (2.10.20a)$$

$$\Gamma = \frac{1}{\sqrt{c^2 + 2\sqrt{2}R^2 c \zeta - R^2(1-R^2)\zeta^2}}, \quad \Delta = \frac{1}{Rc\sqrt{1+R^2}}. \quad (2.10.20b)$$

Transformation between local direction $y^{(i)}$ and coordinate direction y^μ :

$$y^0 = \frac{\Gamma}{r_G} y^{(0)} + \frac{\Delta \Gamma A}{r_G} y^{(2)}, \quad y^1 = \frac{1}{r_G} \sqrt{1+R^2} y^{(1)}, \quad y^2 = \frac{\Gamma \zeta}{r_G} y^{(0)} + \frac{\Delta \Gamma B}{r_G} y^{(2)}, \quad y^3 = \frac{1}{r_G} y^{(3)}, \quad (2.10.21)$$

and the back transformation is given by

$$y^{(0)} = \frac{r_G}{\Gamma} \frac{By^0 - Ay^2}{B - \zeta A}, \quad y^{(1)} = \frac{r_G}{\sqrt{1+R^2}} y^1, \quad y^{(2)} = \frac{r_G}{\Delta \Gamma} \frac{y^2 - \zeta y^0}{B - \zeta A}, \quad y^{(3)} = r_G y^3. \quad (2.10.22a)$$

2.11 Halilsoy standing wave

The standing wave metric by Halilsoy[Hal88] reads

$$ds^2 = V [e^{2K} (d\rho^2 - dt^2) + \rho^2 d\varphi^2] + \frac{1}{V} (dz + A d\varphi)^2, \quad (2.11.1)$$

where

$$V = \cosh^2 \alpha e^{-2CJ_0(\rho) \cos(t)} + \sinh^2 \alpha e^{2CJ_0(\rho) \cos(t)}, \quad (2.11.2a)$$

$$K = \frac{C^2}{2} [\rho^2 (J_0(\rho)^2 + J_1(\rho)^2) - 2\rho J_0(\rho) J_1(\rho) \cos^2 t], \quad (2.11.2b)$$

$$A = -2C \sinh(2\alpha) \rho J_1(\rho) \sin(t). \quad (2.11.2c)$$

with spherical Bessel functions $J_{1,2}$ and parameters α and C .

Local tetrad:

$$\mathbf{e}_{(0)} = \frac{e^{-K}}{\sqrt{V}} \partial_t, \quad \mathbf{e}_{(1)} = \frac{e^{-K}}{\sqrt{V}} \partial_\rho, \quad \mathbf{e}_{(2)} = \frac{1}{\rho\sqrt{V}} \partial_\varphi - \frac{A}{\rho\sqrt{V}} \partial_z, \quad \mathbf{e}_{(3)} = \sqrt{V} \partial_z. \quad (2.11.3)$$

dual tetrad:

$$\theta^{(0)} = \sqrt{V} e^K dt, \quad \theta^{(1)} = \sqrt{V} e^K d\rho, \quad \theta^{(2)} = \sqrt{V} \rho d\varphi, \quad \theta^{(3)} = \frac{1}{\sqrt{V}} (dz + A d\varphi). \quad (2.11.4)$$

2.12 Janis-Newman-Winicour

The Janis-Newman-Winicour [JNW68] spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ is represented by the line element

$$ds^2 = -\alpha^\gamma c^2 dt^2 + \alpha^{-\gamma} dr^2 + r^2 \alpha^{-\gamma+1} (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.12.1)$$

where $\alpha = 1 - r_s/(\gamma r)$. The Schwarzschild radius $r_s = 2GM/c^2$ is defined by Newton's constant G , the speed of light c , and the mass parameter M . For $\gamma = 1$, we obtain the Schwarzschild metric (2.2.1).

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{r_s c^2}{2r^2} \alpha^{2\gamma-1}, \quad \Gamma_{tr}^r = \frac{r_s}{2\gamma r^2 \alpha}, \quad \Gamma_{rr}^r = -\frac{r_s}{2\gamma r^2 \alpha}, \quad (2.12.2a)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2 \alpha}, \quad \Gamma_{r\varphi}^\varphi = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2 \alpha}, \quad \Gamma_{\vartheta\vartheta}^r = -\frac{2\gamma r - r_s(\gamma+1)}{2\gamma}, \quad (2.12.2b)$$

$$\Gamma_{\varphi\varphi}^r = \Gamma_{\vartheta\vartheta}^r \sin^2 \vartheta, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.12.2c)$$

Riemann-Tensor:

$$R_{ttrr} = -\frac{r_s c^2 [2\gamma r - r_s(\gamma+1)] \alpha^{\gamma-2}}{2\gamma r^4}, \quad R_{t\vartheta t\vartheta} = \frac{r_s c^2 [2\gamma r - r_s(\gamma+1)] \alpha^{\gamma-1}}{4\gamma r^2}, \quad (2.12.3a)$$

$$R_{t\varphi t\varphi} = \frac{r_s c^2 [2\gamma r - r_s(\gamma+1)] \alpha^{\gamma-1} \sin^2 \vartheta}{4\gamma r^2}, \quad R_{r\vartheta r\vartheta} = -\frac{r_s [2\gamma^2 r - r_s(\gamma+1)]}{4\gamma^2 r^2 \alpha^{\gamma-1}}, \quad (2.12.3b)$$

$$R_{r\varphi r\varphi} = -\frac{r_s [2\gamma^2 r - r_s(\gamma+1)] \sin^2 \vartheta}{4\gamma^2 r^2 \alpha^{\gamma-1}}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{r_s [4\gamma^2 r - r_s(\gamma+1)^2] \sin^2 \vartheta}{4\gamma^2 \alpha^\gamma}. \quad (2.12.3c)$$

Weyl-Tensor:

$$C_{ttrr} = -\frac{r_s c^2 \alpha^{\gamma-2} \beta}{6\gamma^2 r^4}, \quad C_{t\vartheta t\vartheta} = \frac{r_s c^2 \alpha^{\gamma-1} \beta}{12\gamma^2 r^2}, \quad (2.12.4a)$$

$$C_{t\varphi t\varphi} = \frac{r_s c^2 \alpha^{\gamma-1} \beta \sin^2 \vartheta}{12\gamma^2 r^2}, \quad C_{r\vartheta r\vartheta} = -\frac{r_s \beta}{12\gamma^2 r^2 \alpha^{\gamma-1}}, \quad (2.12.4b)$$

$$C_{r\varphi r\varphi} = -\frac{r_s \beta \sin^2 \vartheta}{12\gamma^2 r^2 \alpha^{\gamma-1}}, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{r_s \beta \sin^2 \vartheta}{6\gamma^2 \alpha^\gamma}, \quad (2.12.4c)$$

where $\beta = 6\gamma^2 r - r_s(\gamma+1)(2\gamma+1)$.

Ricci-Tensor:

$$R_{rr} = \frac{r_s^2 (1 - \gamma^2)}{2\gamma^2 r^4 \alpha^2}. \quad (2.12.5)$$

The Ricci scalar reads

$$\mathcal{R} = \frac{r_s^2 (1 - \gamma^2) \alpha^{\gamma-2}}{2\gamma^2 r^4}, \quad (2.12.6)$$

whereas the Kretschmann scalar is given by

$$\mathcal{K} = \frac{r_s^2 \alpha^{2\gamma-4}}{4\gamma^4 r^8} [7\gamma^2 r_s^2 (2 + \gamma^2) + 48\gamma^4 r^2 \alpha + 8\gamma r_s (2\gamma^2 + 1)(r_s - 2\gamma r) + 3r_s^2]. \quad (2.12.7)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\alpha^{\gamma/2}} \partial_t, \quad \mathbf{e}_{(r)} = \alpha^{\gamma/2} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{\alpha^{(\gamma-1)/2}}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{\alpha^{(\gamma-1)/2}}{r \sin \vartheta} \partial_\varphi. \quad (2.12.8)$$

Dual tetrad:

$$\theta^{(t)} = c\alpha^{\gamma/2}dt, \quad \theta^{(r)} = \frac{dr}{\alpha^{\gamma/2}}, \quad \theta^{(\vartheta)} = \frac{r}{\alpha^{(\gamma-1)/2}}d\vartheta, \quad \theta^{(\varphi)} = \frac{r \sin \vartheta}{\alpha^{(\gamma-1)/2}}d\varphi. \quad (2.12.9)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2}\alpha^{(\gamma-2)/2}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2}\alpha^{(\gamma-2)/2}, \quad (2.12.10a)$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{r}\alpha^{(\gamma-1)/2}. \quad (2.12.10b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4\gamma r - r_s(2+\gamma)}{2\gamma r^2}\alpha^{(\gamma-1)/2}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{r}\alpha^{(\gamma-1)/2}. \quad (2.12.11)$$

Structure coefficients:

$$c_{(t)(r)}^{(t)} = \frac{r_s}{2r^2}\alpha^{(\gamma-2)/2}, \quad c_{(r)(\vartheta)}^{(\vartheta)} = c_{(r)(\varphi)}^{(\varphi)} = -\frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2}\alpha^{(\gamma-2)/2}, \quad (2.12.12a)$$

$$c_{(\vartheta)(\varphi)}^{(\varphi)} = -\frac{\cot \vartheta}{r}\alpha^{(\gamma-1)/2}. \quad (2.12.12b)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields the effective potential

$$V_{\text{eff}} = \frac{1}{2}\alpha^\gamma \left(\frac{h^2 \alpha^{\gamma-1}}{r^2} - \kappa c^2 \right) \quad (2.12.13)$$

with the constants of motion $h = r^2 \alpha^{-\gamma+1} \dot{\varphi}$ and $k = \alpha^\gamma c^2 i$. For null geodesics ($\kappa = 0$) and $\gamma > \frac{1}{2}$, there is an extremum at

$$r = r_s \frac{1+2\gamma}{2\gamma}. \quad (2.12.14)$$

Embedding:

The embedding function $z = z(r)$ for $r \in [r_s(\gamma+1)^2/(4\gamma^2), \infty)$ follows from

$$\frac{dz}{dr} = \sqrt{\frac{r_s [4r\gamma^2 - r_s(1+\gamma)^2]}{4r^2\gamma^2\alpha^{\gamma+1}}}. \quad (2.12.15)$$

However, the analytic solution

$$z(r) = 2\sqrt{r_s r} F_1 \left(-\frac{1}{2}; \frac{\gamma+1}{2}, -\frac{1}{2}; \frac{1}{2}, \frac{r_s}{r\gamma}, \frac{r_s(1+\gamma)^2}{4r\gamma^2} \right) - \frac{2\pi\gamma}{\gamma+1} {}_2F_1 \left(-\frac{1}{2}, \frac{\gamma+1}{2}; 1; \frac{4\gamma}{(\gamma+1)^2} \right), \quad (2.12.16)$$

depends on the Appell- F_1 - and the Hypergeometric- ${}_2F_1$ -function.

2.13 Kasner

The Kasner spacetime in Cartesian coordinates (t, x, y, z) is represented by the line element [MTW73, Kas21] ($c = 1$)

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2, \quad (2.13.1)$$

where p_1, p_2, p_3 have to fulfill the two conditions

$$p_1 + p_2 + p_3 = 1 \quad \text{and} \quad p_1^2 + p_2^2 + p_3^2 = 1. \quad (2.13.2)$$

These two conditions can also be represented by the Khalatnikov-Lifshitz parameter u with

$$p_1 = -\frac{u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}. \quad (2.13.3)$$

Christoffel symbols:

$$\Gamma_{tx}^x = \frac{p_1}{t}, \quad \Gamma_{ty}^y = \frac{p_2}{t}, \quad \Gamma_{tz}^z = \frac{p_3}{t}, \quad (2.13.4a)$$

$$\Gamma_{xx}^t = \frac{p_1 t^{2p_1}}{t}, \quad \Gamma_{yy}^t = \frac{p_2 t^{2p_2}}{t}, \quad \Gamma_{zz}^t = \frac{p_3 t^{2p_3}}{t}. \quad (2.13.4b)$$

Partial derivatives

$$\Gamma_{tx,t}^x = -\frac{p_1}{t^2}, \quad \Gamma_{ty,t}^y = -\frac{p_2}{t^2}, \quad \Gamma_{tz,t}^z = -\frac{p_3}{t^2}, \quad (2.13.5a)$$

$$\Gamma_{xx,t}^t = p_1(2p_1 - 1)t^{2p_1-2}, \quad \Gamma_{yy,t}^t = p_2(2p_2 - 1)t^{2p_2-2}, \quad \Gamma_{zz,t}^t = p_3(2p_3 - 1)t^{2p_3-2}. \quad (2.13.5b)$$

Riemann-Tensor:

$$R_{txtx} = \frac{p_1(1-p_1)t^{2p_1}}{t^2}, \quad R_{tyty} = \frac{p_2(1-p_2)t^{2p_2}}{t^2}, \quad R_{tztz} = \frac{p_3(1-p_3)t^{2p_3}}{t^2}, \quad (2.13.6a)$$

$$R_{xyxy} = \frac{p_1 p_2 t^{2p_1} t^{2p_2}}{t^2}, \quad R_{xzxz} = \frac{p_1 p_3 t^{2p_1} t^{2p_3}}{t^2}, \quad R_{yzyz} = \frac{p_2 p_3 t^{2p_2} t^{2p_3}}{t^2}. \quad (2.13.6b)$$

The Ricci tensor as well as the Ricci scalar vanish identically. The Kretschmann scalar reads

$$\mathcal{K} = \frac{4}{t^4} (p_1^2 - 2p_1^3 + p_1^4 + p_2^2 - 2p_2^3 + p_2^4 + p_1^2 p_3^2 + p_3^2 - 2p_3^3 + p_3^4 + p_1^2 p_2^2 + p_2^2 p_3^2) \quad (2.13.7a)$$

$$= \frac{16u^2(1+u)^2}{t^4(1+u+u^2)^3}. \quad (2.13.7b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \partial_t, \quad \mathbf{e}_{(x)} = t^{-p_1} \partial_x, \quad \mathbf{e}_{(y)} = t^{-p_2} \partial_y, \quad \mathbf{e}_{(z)} = t^{-p_3} \partial_z. \quad (2.13.8)$$

Dual tetrad:

$$\theta^{(t)} = dt, \quad \theta^{(x)} = t^{p_1} dx, \quad \theta^{(y)} = t^{p_2} dy, \quad \theta^{(z)} = t^{p_3} dz. \quad (2.13.9)$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(r)} = \frac{p_1}{t}, \quad \gamma_{(t)(\vartheta)(\vartheta)} = \frac{p_2}{t}, \quad \gamma_{(t)(\varphi)(\varphi)} = \frac{p_3}{t}. \quad (2.13.10)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = -\frac{1}{t}. \quad (2.13.11)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(x)(y)(x)} = \frac{p_1(1-p_1)}{t^2}, \quad R_{(t)(y)(t)(y)} = \frac{p_2(1-p_2)}{t^2}, \quad R_{(t)(z)(t)(z)} = \frac{p_3(1-p_3)}{t^2}, \quad (2.13.12a)$$

$$R_{(x)(y)(x)(y)} = \frac{p_1 p_2}{t^2}, \quad R_{(x)(z)(x)(z)} = \frac{p_1 p_3}{t^2}, \quad R_{(y)(z)(y)(z)} = \frac{p_2 p_3}{t^2}. \quad (2.13.12b)$$

2.14 Kerr

The Kerr spacetime, found by Roy Kerr in 1963[Ker63], describes a rotating black hole.

2.14.1 Boyer-Lindquist coordinates

The Kerr metric in Boyer-Lindquist coordinates

$$\begin{aligned}
 ds^2 = & - \left(1 - \frac{r_s r}{\Sigma}\right) c^2 dt^2 - \frac{2r_s a r \sin^2 \vartheta}{\Sigma} c dt d\varphi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\vartheta^2 \\
 & + \left(r^2 + a^2 + \frac{r_s a^2 r \sin^2 \vartheta}{\Sigma}\right) \sin^2 \vartheta d\varphi^2,
 \end{aligned}
 \tag{2.14.1}$$

with $\Sigma = r^2 + a^2 \cos^2 \vartheta$, $\Delta = r^2 - r_s r + a^2$, and $r_s = 2GM/c^2$, is taken from Bardeen[BPT72]. M is the mass and a is the angular momentum per unit mass of the black hole. The contravariant form of the metric reads

$$\partial_s^2 = -\frac{A}{c^2 \Sigma \Delta} \partial_t^2 - \frac{2r_s a r}{c \Sigma \Delta} \partial_t \partial_\varphi + \frac{\Delta}{\Sigma} \partial_r^2 + \frac{1}{\Sigma} \partial_\vartheta^2 + \frac{\Delta - a^2 \sin^2 \vartheta}{\Sigma \Delta \sin^2 \vartheta} \partial_\varphi^2,
 \tag{2.14.2}$$

where $A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \vartheta = (r^2 + a^2) \Sigma + r_s a^2 r \sin^2 \vartheta$.

The event horizon r_+ is defined by the outer root of Δ ,

$$r_+ = \frac{r_s}{2} + \sqrt{\frac{r_s^2}{4} - a^2},
 \tag{2.14.3}$$

whereas the outer boundary r_0 of the ergosphere follows from the outer root of $\Sigma - r_s r$,

$$r_0 = \frac{r_s}{2} + \sqrt{\frac{r_s^2}{4} - a^2 \cos^2 \vartheta},
 \tag{2.14.4}$$

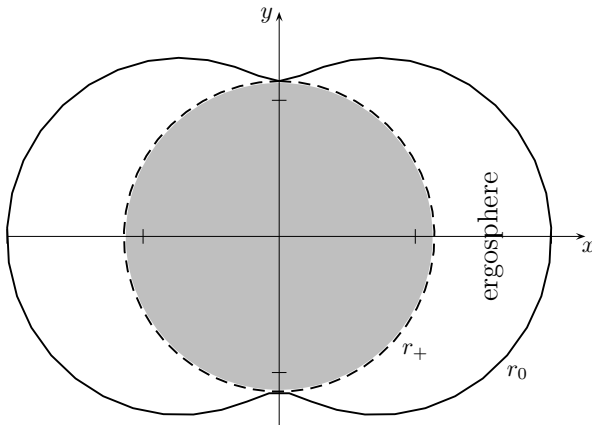


Figure 2.1: Ergosphere and horizon (dashed circle) for $a = 0.99 \frac{r_s}{2}$.

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{c^2 r_s \Delta (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^3}, \quad \Gamma_{tt}^\vartheta = -\frac{c^2 r_s a^2 r \sin \vartheta \cos \vartheta}{\Sigma^3}, \quad (2.14.5a)$$

$$\Gamma_{tr}^r = \frac{r_s (r^2 + a^2) (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^2 \Delta}, \quad \Gamma_{tr}^\varphi = \frac{c r_s a (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^2 \Delta}, \quad (2.14.5b)$$

$$\Gamma_{t\vartheta}^r = -\frac{r_s a^2 r \sin \vartheta \cos \vartheta}{\Sigma^2}, \quad \Gamma_{t\vartheta}^\varphi = -\frac{c r_s a r \cot \vartheta}{\Sigma^2}, \quad (2.14.5c)$$

$$\Gamma_{t\varphi}^r = -\frac{c \Delta r_s a \sin^2 \vartheta (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma^3}, \quad \Gamma_{t\varphi}^\vartheta = \frac{c r_s a r (r^2 + a^2) \sin \vartheta \cos \vartheta}{\Sigma^3}, \quad (2.14.5d)$$

$$\Gamma_{rr}^r = \frac{2r a^2 \sin^2 \vartheta - r_s (r^2 - a^2 \cos^2 \vartheta)}{2\Sigma \Delta}, \quad \Gamma_{rr}^\vartheta = \frac{a^2 \sin \vartheta \cos \vartheta}{\Sigma \Delta}, \quad (2.14.5e)$$

$$\Gamma_{r\vartheta}^r = -\frac{a^2 \sin \vartheta \cos \vartheta}{\Sigma}, \quad \Gamma_{r\vartheta}^\vartheta = \frac{r}{\Sigma}, \quad (2.14.5f)$$

$$\Gamma_{\vartheta\vartheta}^r = -\frac{r \Delta}{\Sigma}, \quad \Gamma_{\vartheta\vartheta}^\vartheta = -\frac{a^2 \sin \vartheta \cos \vartheta}{\Sigma}, \quad (2.14.5g)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \frac{\cot \vartheta}{\Sigma^2} [\Sigma^2 + r_s a^2 r \sin^2 \vartheta], \quad \Gamma_{\vartheta\varphi}^r = \frac{r_s a^3 r \sin^3 \vartheta \cos \vartheta}{c \Sigma^2}, \quad (2.14.5h)$$

$$\Gamma_{r\varphi}^r = \frac{r_s a \sin^2 \vartheta [a^2 \cos^2 \vartheta (a^2 - r^2) - r^2 (a^2 + 3r^2)]}{2c \Sigma^2 \Delta}, \quad (2.14.5i)$$

$$\Gamma_{r\varphi}^\varphi = \frac{2r \Sigma^2 + r_s [a^4 \sin^2 \vartheta \cos^2 \vartheta - r^2 (\Sigma + r^2 + a^2)]}{2\Sigma^2 \Delta}, \quad (2.14.5j)$$

$$\Gamma_{\varphi\varphi}^r = \frac{\Delta \sin^2 \vartheta}{2\Sigma^3} [-2r \Sigma^2 + r_s a^2 \sin^2 \vartheta (r^2 - a^2 \cos^2 \vartheta)], \quad (2.14.5k)$$

$$\Gamma_{\varphi\varphi}^\vartheta = -\frac{\sin \vartheta \cos \vartheta}{\Sigma^3} [A \Sigma + (r^2 + a^2) r_s a^2 r \sin^2 \vartheta], \quad (2.14.5l)$$

General local tetrad:

$$\mathbf{e}_{(0)} = \Gamma (\partial_t + \zeta \partial_\varphi), \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}} \partial_r, \quad (2.14.6a)$$

$$\mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}} \partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{\Gamma}{c} \left(\mp \frac{g_{t\varphi} + \zeta g_{\varphi\varphi}}{\sqrt{\Delta} \sin \vartheta} \partial_t \pm \frac{g_{tt} + \zeta g_{t\varphi}}{\sqrt{\Delta} \sin \vartheta} \partial_\varphi \right), \quad (2.14.6b)$$

where $-\Gamma^{-2} = g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi}$,

$$\Gamma^{-2} = \left(1 - \frac{r_s r}{\Sigma}\right) + \frac{2r_s a r \sin^2 \vartheta}{\Sigma} \frac{\zeta}{c} - \left(r^2 + a^2 + \frac{r_s a^2 r \sin^2 \vartheta}{\Sigma}\right) \frac{\zeta^2}{c^2} \sin^2 \vartheta \quad (2.14.7)$$

Non-rotating local tetrad ($\zeta = \omega$):

$$\mathbf{e}_{(0)} = \sqrt{\frac{A}{\Sigma \Delta}} \left(\frac{1}{c} \partial_t + \omega \partial_\varphi \right), \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}} \partial_\vartheta, \quad \mathbf{e}_{(3)} = \sqrt{\frac{\Sigma}{A}} \frac{1}{\sin \vartheta} \partial_\varphi, \quad (2.14.8)$$

where $\omega = -g_{t\varphi}/g_{\varphi\varphi} = r_s a r/A$.

Dual tetrad:

$$\theta^{(2)} = \sqrt{\frac{\Sigma \Delta}{A}} c dt, \quad \theta^{(1)} = \sqrt{\frac{\Sigma}{\Delta}} dr, \quad \theta^{(2)} = \sqrt{\Sigma} d\vartheta, \quad \theta^{(3)} = \sqrt{\frac{A}{\Sigma}} \sin \vartheta (d\varphi - \omega d\varphi). \quad (2.14.9)$$

The relation between the constants of motion E , L , Q , and μ (defined in Bardeen[BPT72]) and the initial direction v , compare Sec. (1.4.5), with respect to the LNRF reads ($c = 1$)

$$v^{(0)} = \sqrt{\frac{A}{\Sigma\Delta}} E - \frac{r_s r a}{\sqrt{A\Sigma\Delta}} L, \quad v^{(1)} = \sqrt{\frac{\Delta}{\Sigma}} p_r, \quad (2.14.10a)$$

$$v^{(2)} = \frac{1}{\sqrt{\Sigma}} \sqrt{Q - \cos^2 \vartheta \left[a^2 (\mu^2 - E^2) + \frac{L^2}{\sin^2 \vartheta} \right]}, \quad v^{(3)} = \sqrt{\frac{\Sigma}{A}} \frac{L}{\sin \vartheta}. \quad (2.14.10b)$$

Static local tetrad ($\zeta = 0$):

$$\mathbf{e}_{(0)} = \frac{1}{c\sqrt{1 - r_s r/\Sigma}} \partial_t, \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}} \partial_\vartheta, \quad (2.14.11a)$$

$$\mathbf{e}_{(3)} = \pm \frac{r_s a r \sin \vartheta}{c\sqrt{1 - r_s r/\Sigma} \sqrt{\Delta\Sigma}} \partial_t \mp \frac{\sqrt{1 - r_s r/\Sigma}}{\sqrt{\Delta} \sin \vartheta} \partial_\varphi. \quad (2.14.11b)$$

Photon orbits:

The direct(-) and retrograd(+) photon orbits have radius

$$r_{\text{po}} = r_s \left[1 + \cos \left(\frac{2}{3} \arccos \frac{\mp 2a}{r_s} \right) \right]. \quad (2.14.12)$$

Marginally stable timelike circular orbits

are defined via

$$r_{\text{ms}} = \frac{r_s}{2} \left(3 + Z_2 \mp \sqrt{(3 - Z_1)(2 + Z_1 + 2Z_2)} \right), \quad (2.14.13)$$

where

$$Z_1 = 1 + \left(1 - \frac{4a^2}{r_s^2} \right)^{1/3} \left[\left(1 + \frac{2a}{r_s} \right)^{1/3} + \left(1 - \frac{2a}{r_s} \right)^{1/3} \right], \quad (2.14.14a)$$

$$Z_2 = \sqrt{\frac{12a^2}{r_s^2} + Z_1^2}. \quad (2.14.14b)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2} \dot{r}^2 + V_{\text{eff}} = 0 \quad (2.14.15)$$

with the effective potential

$$V_{\text{eff}} = \frac{1}{2r^3} \left\{ h^2 (r - r_s) + 2 \frac{ahk}{c} r_s - \frac{k^2}{c^2} [r^3 + a^2 (r + r_s)] \right\} - \frac{\kappa c^2 \Delta}{r^2} \quad (2.14.16)$$

and the constants of motion

$$k = \left(1 - \frac{r_s}{r} \right) c^2 \dot{t} + \frac{c r_s a}{r} \dot{\varphi}, \quad h = \left(r^2 + a^2 + \frac{r_s a^2}{r} \right) \dot{\varphi} - \frac{c r_s a}{r} \dot{t}. \quad (2.14.17)$$

Further reading:

Boyer and Lindquist[BL67], Wilkins[Wil72], Brill[BC66].

2.15 Kottler spacetime

The Kottler spacetime is represented in spherical coordinates $(t, r, \vartheta, \varphi)$ by the line element[Per04]

$$ds^2 = - \left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3} \right) c^2 dt^2 + \frac{1}{1 - r_s/r - \Lambda r^2/3} dr^2 + r^2 d\Omega^2, \quad (2.15.1)$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, M is the mass of the black hole, and Λ is the cosmological constant. If $\Lambda > 0$ the metric is also known as Schwarzschild-deSitter metric, whereas if $\Lambda < 0$ it is called Schwarzschild-anti-deSitter.

For the following, we define the two abbreviations

$$\alpha = 1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3} \quad \text{and} \quad \beta = \frac{r_s}{r} - \frac{2\Lambda}{3} r^2. \quad (2.15.2)$$

The critical points of the Kottler metric follow from the roots of the cubic equation $\alpha = 0$. These can be found by means of the parameters $p = -1/\Lambda$ and $q = 3r_s/(2\Lambda)$. If $\Lambda < 0$, we have only one real root

$$r_1 = \frac{2}{\sqrt{-\Lambda}} \sinh \left[\frac{1}{3} \operatorname{arsinh} \left(\frac{3r_s}{2} \sqrt{-\Lambda} \right) \right]. \quad (2.15.3)$$

If $\Lambda > 0$, we have to distinguish whether $D \equiv q^2 + p^3 = 9r_s^2/(4\Lambda^2) - \Lambda^{-3}$ is positive or negative. If $D > 0$, there is no real positive root. For $D < 0$, the two real positive roots read

$$r_{\pm} = \frac{2}{\sqrt{\Lambda}} \cos \left[\frac{\pi}{3} \pm \frac{1}{3} \arccos \left(\frac{3r_s}{2} \sqrt{\Lambda} \right) \right] \quad (2.15.4)$$

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{c^2 \alpha \beta}{2r}, \quad \Gamma_{tr}^t = \frac{\beta}{2r\alpha}, \quad \Gamma_{rr}^r = -\frac{\beta}{2r\alpha}, \quad (2.15.5a)$$

$$\Gamma_{r\vartheta}^{\vartheta} = \frac{1}{r}, \quad \Gamma_{r\varphi}^{\varphi} = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^r = -\alpha r, \quad (2.15.5b)$$

$$\Gamma_{\vartheta\varphi}^{\varphi} = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^r = -\alpha r \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^{\vartheta} = -\sin \vartheta \cos \vartheta. \quad (2.15.5c)$$

Riemann-Tensor:

$$R_{ttrr} = -\frac{c^2 (3r_s + \Lambda r^3)}{3r^3}, \quad R_{t\vartheta t\vartheta} = \frac{1}{2} c^2 \alpha \beta, \quad (2.15.6a)$$

$$R_{t\varphi t\varphi} = \frac{1}{2} c^2 \alpha \beta \sin^2 \vartheta, \quad R_{r\vartheta r\vartheta} = -\frac{\beta}{2\alpha}, \quad (2.15.6b)$$

$$R_{r\varphi r\varphi} = -\frac{\beta}{2\alpha} \sin^2 \vartheta, \quad R_{\vartheta\varphi\vartheta\varphi} = r \left(r_s + \frac{\Lambda r^3}{3} \right) \sin^2 \vartheta. \quad (2.15.6c)$$

Ricci-Tensor:

$$R_{tt} = -c^2 \alpha \Lambda, \quad R_{rr} = \frac{\Lambda}{\alpha}, \quad R_{\vartheta\vartheta} = \Lambda r^2, \quad R_{\varphi\varphi} = \Lambda r^2 \sin^2 \vartheta. \quad (2.15.7)$$

The Ricci scalar and the Kretschmann scalar read

$$\mathcal{R} = 4\Lambda, \quad \mathcal{K} = 12 \frac{r_s^2}{r^6} + \frac{8\Lambda^2}{3}. \quad (2.15.8)$$

Weyl-Tensor:

$$C_{ttrr} = -\frac{c^2 r_s}{r^3}, \quad C_{t\vartheta t\vartheta} = \frac{c^2 \alpha r_s}{2r}, \quad C_{t\varphi t\varphi} = \frac{c^2 \alpha r_s \sin^2 \vartheta}{2r}, \quad (2.15.9a)$$

$$C_{r\vartheta r\vartheta} = -\frac{r_s}{2r\alpha}, \quad C_{r\varphi r\varphi} = -\frac{r_s \sin^2 \vartheta}{2r\alpha}, \quad C_{\vartheta\varphi\vartheta\varphi} = r r_s \sin^2 \vartheta. \quad (2.15.9b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{\alpha}}\partial_t, \quad \mathbf{e}_{(r)} = \sqrt{\alpha}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_{\varphi}. \quad (2.15.10)$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{\alpha}dt, \quad \theta^{(r)} = \frac{dr}{\sqrt{\alpha}}, \quad \theta^{(\vartheta)} = rd\vartheta, \quad \theta^{(\varphi)} = r\sin\vartheta d\varphi. \quad (2.15.11)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s - \frac{2}{3}\Lambda r^3}{2r^2\sqrt{\alpha}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{\sqrt{\alpha}}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \quad (2.15.12)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s - 2\Lambda r^3}{2r^2\sqrt{\alpha}}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \quad (2.15.13)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{\Lambda r^3 + 3r_s}{3r^3}, \quad (2.15.14a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{3r_s - 2\Lambda r^3}{6r^3}. \quad (2.15.14b)$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \quad (2.15.15a)$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \quad (2.15.15b)$$

Embedding:

The embedding function follows from the numerical integration of

$$\frac{dz}{dr} = \sqrt{\frac{r_s/r + \Lambda r^2/3}{1 - r_s/r - \Lambda r^2/3}}. \quad (2.15.16)$$

Euler-Lagrange:

The Euler-Lagrangian formalism[Rin01] yields the effective potential

$$V_{\text{eff}} = \frac{1}{2} \left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3} \right) \left(\frac{h^2}{r^2} - \kappa c^2 \right) \quad (2.15.17)$$

with the constants of motion $k = (1 - r_s/r - \Lambda r^2/3)c^2i$, $h = r^2\dot{\varphi}$, and κ as in Eq. (1.8.2).

As in the Schwarzschild metric, the effective potential has only one extremum for null geodesics, the so called photon orbit at $r = \frac{3}{2}r_s$. For timelike geodesics, however, we have

$$\frac{dV_{\text{eff}}}{dr} = \frac{h^2(-6r + 9r_s) + c^2 r^2(3r_s - 2r^3\Lambda)}{3r^4} \stackrel{!}{=} 0. \quad (2.15.18)$$

This polynomial of fifth order might have up to five extrema.

Further reading:

Kottler[Kot18], Weyl[Wey19], Hackmann[HL08], Cruz[COV05].

2.16 Morris-Thorne

The most simple wormhole geometry is represented by the metric of Morris and Thorne[MT88],

$$ds^2 = -c^2 dt^2 + dl^2 + (b_0^2 + l^2) (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.16.1)$$

where b_0 is the throat radius and l is the proper radial coordinate; and $\{t \in \mathbb{R}, l \in \mathbb{R}, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$.

Christoffel symbols:

$$\Gamma_{l\vartheta}^{\vartheta} = \frac{l}{b_0^2 + l^2}, \quad \Gamma_{l\varphi}^{\varphi} = \frac{l}{b_0^2 + l^2}, \quad \Gamma_{\vartheta\vartheta}^l = -l, \quad (2.16.2a)$$

$$\Gamma_{\vartheta\varphi}^{\varphi} = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^l = -l \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^{\vartheta} = -\sin \vartheta \cos \vartheta. \quad (2.16.2b)$$

Partial derivatives

$$\Gamma_{l\vartheta,l}^{\vartheta} = -\frac{l^2 - b_0^2}{(b_0^2 + l^2)^2}, \quad \Gamma_{l\varphi,l}^{\varphi} = -\frac{l^2 - b_0^2}{(b_0^2 + l^2)^2}, \quad \Gamma_{\vartheta\vartheta,l}^l = -1, \quad (2.16.3a)$$

$$\Gamma_{\vartheta\varphi,\vartheta}^{\varphi} = -\frac{1}{\sin^2 \vartheta}, \quad \Gamma_{\varphi\varphi,l}^l = -\sin^2 \vartheta, \quad \Gamma_{\varphi\varphi,\vartheta}^l = -l \sin(2\vartheta), \quad (2.16.3b)$$

$$\Gamma_{\varphi\varphi,\vartheta}^{\vartheta} = -\cos(2\vartheta). \quad (2.16.3c)$$

Riemann-Tensor:

$$R_{l\vartheta l\vartheta} = -\frac{b_0^2}{b_0^2 + l^2}, \quad R_{l\varphi l\varphi} = -\frac{b_0^2 \sin^2 \vartheta}{b_0^2 + l^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = b_0^2 \sin^2 \vartheta. \quad (2.16.4)$$

Ricci tensor, Ricci and Kretschmann scalar:

$$R_{ll} = -2 \frac{b_0^2}{(b_0^2 + l^2)^2}, \quad \mathcal{R} = -2 \frac{b_0^2}{(b_0^2 + l^2)^2}, \quad \mathcal{K} = \frac{12b_0^4}{(b_0^2 + l^2)^4}. \quad (2.16.5)$$

Weyl-Tensor:

$$C_{llll} = -\frac{2}{3} \frac{c^2 b_0^2}{(b_0^2 + l^2)^2}, \quad C_{l\vartheta l\vartheta} = \frac{1}{3} \frac{c^2 b_0^2}{b_0^2 + l^2}, \quad C_{l\varphi l\varphi} = \frac{1}{3} \frac{c^2 b_0^2 \sin^2 \vartheta}{b_0^2 + l^2}, \quad (2.16.6a)$$

$$C_{l\vartheta l\vartheta} = -\frac{1}{3} \frac{b_0^2}{b_0^2 + l^2}, \quad C_{l\varphi l\varphi} = -\frac{1}{3} \frac{b_0^2 \sin^2 \vartheta}{b_0^2 + l^2}, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{2}{3} b_0^2 \sin^2 \vartheta. \quad (2.16.6b)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(l)} = \partial_l, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\sqrt{b_0^2 + l^2}} \partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\sqrt{b_0^2 + l^2} \sin \vartheta} \partial_{\varphi}. \quad (2.16.7)$$

Dual tetrad

$$\theta^{(t)} = c dt, \quad \theta^{(l)} = dl, \quad \theta^{(\vartheta)} = \sqrt{b_0^2 + l^2} d\vartheta, \quad \theta^{(\varphi)} = \sqrt{b_0^2 + l^2} \sin \vartheta d\varphi. \quad (2.16.8)$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{l}{b_0^2 + l^2}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot \vartheta}{\sqrt{b_0^2 + l^2}}. \quad (2.16.9)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2l}{b_0^2 + l^2}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{\sqrt{b_0^2 + l^2}}. \quad (2.16.10)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(l)(\vartheta)(l)(\vartheta)} = R_{(l)(\varphi)(l)(\varphi)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{b_0^2}{(b_0^2 + l^2)^2}. \quad (2.16.11)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(l)(l)} = -\frac{2b_0^2}{(b_0^2 + l^2)^2}. \quad (2.16.12)$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(l)(t)(l)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{2b_0^2}{3(b_0^2 + l^2)^2}, \quad (2.16.13a)$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(l)(\vartheta)(l)(\vartheta)} = -C_{(l)(\varphi)(l)(\varphi)} = \frac{b_0^2}{3(b_0^2 + l^2)^2}. \quad (2.16.13b)$$

Embedding:

The embedding function reads

$$z(r) = \pm b_0 \ln \left[\frac{r}{b_0} + \sqrt{\left(\frac{r}{b_0}\right)^2 - 1} \right] \quad (2.16.14)$$

with $r^2 = b_0^2 + l^2$.

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}l^2 + V_{\text{eff}} = \frac{1}{2} \frac{k^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2} \left(\frac{h^2}{b_0^2 + l^2} - \kappa c^2 \right), \quad (2.16.15)$$

with the constants of motion $k = c^2 i$ and $h = (b_0^2 + l^2)\dot{\varphi}$. The shape of the effective potential V_{eff} is independent of the geodesic type. The maximum of the effective potential is located at $l = 0$.

A geodesic that starts at $l = l_i$ with direction $\mathbf{y} = \pm \mathbf{e}_{(t)} + \cos \xi \mathbf{e}_{(l)} + \sin \xi \mathbf{e}_{(\varphi)}$ approaches the wormhole throat asymptotically for $\xi = \xi_{\text{crit}}$ with

$$\xi_{\text{crit}} = \arcsin \frac{b_0}{\sqrt{b_0^2 + l_i^2}}. \quad (2.16.16)$$

This critical angle is independent of the type of the geodesic.

Further reading:

Ellis[Ell73], Visser[Vis95], Müller[Mül04, Mül08a]

2.17 Oppenheimer-Snyder collapse

2.17.1 Outer metric

The metric of the outer spacetime, $R > R_b$, in comoving coordinates $(\tau, R, \vartheta, \varphi)$ with $(c = 1)$ is given by

$$ds^2 = -d\tau^2 + \frac{R}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}} dR^2 + \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{4/3} (d\vartheta^2 + \sin^2\vartheta d\varphi^2). \quad (2.17.1)$$

Christoffel symbols:

$$\Gamma_{\tau R}^R = \frac{1}{2} \frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \quad \Gamma_{\tau\vartheta}^\vartheta = -\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \quad (2.17.2a)$$

$$\Gamma_{\tau\varphi}^\varphi = -\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \quad \Gamma_{RR}^\tau = \frac{R\sqrt{r_s}}{2\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{5/3}}, \quad (2.17.2b)$$

$$\Gamma_{RR}^R = -\frac{3\sqrt{r_s}\tau}{4\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)R}, \quad \Gamma_{R\vartheta}^\vartheta = \frac{\sqrt{R}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \quad (2.17.2c)$$

$$\Gamma_{R\varphi}^\varphi = \frac{\sqrt{R}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \quad \Gamma_{\vartheta\vartheta}^\tau = -\sqrt{r_s} \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}, \quad (2.17.2d)$$

$$\Gamma_{\vartheta\vartheta}^R = -\frac{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}{\sqrt{R}}, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot\vartheta, \quad (2.17.2e)$$

$$\Gamma_{\varphi\varphi}^\tau = -\sqrt{r_s} \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3} \sin^2\vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin\vartheta \cos\vartheta, \quad (2.17.2f)$$

$$\Gamma_{\varphi\varphi}^R = -\frac{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right) \sin^2\vartheta}{\sqrt{R}}. \quad (2.17.2g)$$

Riemann-Tensor:

$$R_{\tau R \tau R} = -\frac{Rr_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{8/3}}, \quad R_{\tau\vartheta\tau\vartheta} = \frac{1}{2} \frac{r_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}}, \quad (2.17.3a)$$

$$R_{\tau\varphi\tau\varphi} = \frac{1}{2} \frac{r_s \sin^2\vartheta}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}}, \quad R_{R\vartheta R\vartheta} = -\frac{1}{2} \frac{Rr_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{4/3}}, \quad (2.17.3b)$$

$$R_{R\varphi R\varphi} = -\frac{1}{2} \frac{Rr_s \sin^2\vartheta}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{4/3}}, \quad R_{\vartheta\varphi\vartheta\varphi} = \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3} r_s \sin^2\vartheta. \quad (2.17.3c)$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschmann scalar:

$$\mathcal{K} = 12 \frac{r_s^2}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^4}. \quad (2.17.4)$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \quad \mathbf{e}_{(R)} = \frac{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}}{\sqrt{R}} \partial_R, \quad (2.17.5a)$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3} \sin\vartheta} \partial_\varphi. \quad (2.17.5b)$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(R)(R)} = -\frac{\sqrt{r_s}}{2R^{3/2} - 3\sqrt{r_s}\tau}, \quad \gamma_{(\tau)(\vartheta)(\vartheta)} = \gamma_{(\tau)(\varphi)(\varphi)} = \frac{2\sqrt{r_s}}{2R^{3/2} - 3\sqrt{r_s}\tau}, \quad (2.17.6a)$$

$$\gamma_{(R)(\varphi)(\varphi)} = \gamma_{(R)(\vartheta)(\vartheta)} = -\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}. \quad (2.17.6b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\tau)} = -\frac{3\sqrt{r_s}}{2R^{3/2} - 3\sqrt{r_s}\tau}, \quad \gamma_{(R)} = 2\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}, \quad \gamma_{(\vartheta)} = \cot\vartheta \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}. \quad (2.17.7)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\tau)(R)(\tau)(R)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{4r_s}{(2R^{3/2} - 3\sqrt{r_s}\tau)^2}, \quad (2.17.8a)$$

$$R_{(\tau)(\vartheta)(\tau)(\vartheta)} = R_{(\tau)(\varphi)(\tau)(\varphi)} = -R_{(R)(\vartheta)(R)(\vartheta)} = -R_{(R)(\varphi)(R)(\varphi)} = \frac{2r_s}{(2R^{3/2} - 3\sqrt{r_s}\tau)^2}. \quad (2.17.8b)$$

The Ricci tensor with respect to the local tetrad vanishes identically.

2.17.2 Inner metric

The metric of the inside, $R \leq R_b$, reads

$$ds^2 = -d\tau^2 + \left(1 - \frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{4/3} [dR^2 + R^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)]. \quad (2.17.9)$$

For the following components, we define

$$A_{\text{Oin}} := 1 - \frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau. \quad (2.17.10)$$

Christoffel symbols:

$$\Gamma_{\tau R}^R = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \quad \Gamma_{\tau\vartheta}^{\vartheta} = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \quad \Gamma_{\tau\varphi}^{\varphi} = -\frac{\sqrt{r_s}R_b^{-3/2}}{A_{\text{Oin}}}, \quad (2.17.11a)$$

$$\Gamma_{RR}^{\tau} = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}, \quad \Gamma_{R\vartheta}^{\vartheta} = \frac{1}{R}, \quad \Gamma_{R\varphi}^{\varphi} = \frac{1}{R}, \quad (2.17.11b)$$

$$\Gamma_{\vartheta\vartheta}^R = -R, \quad \Gamma_{\vartheta\varphi}^{\varphi} = \cot\vartheta, \quad \Gamma_{\vartheta\vartheta}^{\tau} = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}R^2, \quad (2.17.11c)$$

$$\Gamma_{\varphi\varphi}^R = -R\sin^2\vartheta, \quad \Gamma_{\varphi\varphi}^{\vartheta} = -\sin\vartheta\cos\vartheta, \quad \Gamma_{\varphi\varphi}^{\tau} = -A_{\text{Oin}}^{1/3}\sqrt{r_s}R_b^{-3/2}R^2\sin^2\vartheta. \quad (2.17.11d)$$

Riemann-Tensor:

$$R_{\tau R\tau R} = \frac{1}{2}\frac{r_s}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad R_{\tau\vartheta\tau\vartheta} = \frac{1}{2}\frac{r_s R^2}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad R_{\tau\varphi\tau\varphi} = \frac{1}{2}\frac{r_s R^2 \sin^2\vartheta}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad (2.17.12a)$$

$$R_{R\varphi R\varphi} = \frac{r_s R^2 \sin^2\vartheta}{R_b^3} A_{\text{Oin}}^{2/3}, \quad R_{R\vartheta R\vartheta} = \frac{r_s R^2}{R_b^3} A_{\text{Oin}}^{2/3}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{r_s R^4 \sin^2\vartheta}{R_b^3} A_{\text{Oin}}^{2/3}. \quad (2.17.12b)$$

Ricci-Tensor:

$$R_{\tau\tau} = \frac{3}{2}\frac{r_s}{R_b^3 A_{\text{Oin}}^2}, \quad R_{RR} = \frac{3}{2}\frac{r_s}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad R_{\vartheta\vartheta} = \frac{3}{2}\frac{r_s R^2}{R_b^3 A_{\text{Oin}}^{2/3}}, \quad R_{\varphi\varphi} = \frac{3}{2}\frac{r_s R^2 \sin^2\vartheta}{R_b^3 A_{\text{Oin}}^{2/3}}. \quad (2.17.13)$$

The Ricci and Kretschmann scalars read:

$$\mathcal{R} = \frac{3r_s}{R_b^3 A_{\text{Oin}}^2}, \quad \mathcal{K} = 15 \frac{r_s^2}{R_b^6 A_{\text{Oin}}^4}. \quad (2.17.14)$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \quad \mathbf{e}_{(R)} = \frac{1}{A_{\text{Oin}}^{2/3}} \partial_R, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{R A_{\text{Oin}}^{2/3}} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{A_{\text{Oin}}^{2/3} R \sin \vartheta} \partial_\varphi. \quad (2.17.15)$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(R)(R)} = \gamma_{(\tau)(\vartheta)(\vartheta)} = \gamma_{(\tau)(\varphi)(\varphi)} = \frac{\sqrt{r_s} R_b^{-3/2}}{A_{\text{Oin}}}, \quad (2.17.16a)$$

$$\gamma_{(R)(\vartheta)(\vartheta)} = \gamma_{(R)(\varphi)(\varphi)} = -\frac{1}{R A_{\text{Oin}}^{2/3}}, \quad (2.17.16b)$$

$$\gamma_{(\vartheta)(\varphi)(\varphi)} = -\frac{\cot \vartheta}{R A_{\text{Oin}}^{2/3}}. \quad (2.17.16c)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\tau)} = -\frac{3\sqrt{r_s} R_b^{-3/2}}{A_{\text{Oin}}}, \quad \gamma_{(R)} = \frac{2}{R A_{\text{Oin}}^{2/3}}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{R A_{\text{Oin}}^{2/3}}. \quad (2.17.17)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\tau)(R)(\tau)(R)} = R_{(\tau)(\vartheta)(\tau)(\vartheta)} = R_{(\tau)(\varphi)(\tau)(\varphi)} = \frac{r_s R_b^{-3}}{2A_{\text{Oin}}^2}, \quad (2.17.18a)$$

$$R_{(R)(\vartheta)(R)(\vartheta)} = R_{(R)(\varphi)(R)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{r_s R_b^{-3}}{A_{\text{Oin}}^2}. \quad (2.17.18b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\tau)(\tau)} = R_{(R)(R)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{3r_s R_b^{-3}}{2A_{\text{Oin}}^2}. \quad (2.17.19)$$

Further reading:

Oppenheimer and Snyder[OS39].

2.18 Petrov-Type D – Levi-Civita spacetimes

The Petrov type D static vacuum spacetimes AI-C are taken from Stephani et al.[SKM⁺03], Sec. 18.6, with the coordinate and parameter ranges given in "Exact solutions of the gravitational field equations" by Ehlers and Kundt [EK62].

2.18.1 Case AI

In spherical coordinates, $(t, r, \vartheta, \varphi)$, the metric is given by the line element

$$ds^2 = r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) + \frac{r}{r-b} dr^2 - \frac{r-b}{r} dt^2. \quad (2.18.1)$$

This is the well known Schwarzschild solution if $b = r_s$, cf. Eq. (2.2.1). Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad 0 < \vartheta < \pi, \quad \varphi \in [0, 2\pi), \quad (0 < b < r) \vee (b < 0 < r).$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{r}{r-b}} \partial_t, \quad \mathbf{e}_{(r)} = \sqrt{\frac{r-b}{r}} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r \sin \vartheta} \partial_{\varphi}. \quad (2.18.2)$$

Dual tetrad:

$$\theta^{(t)} = \sqrt{\frac{r-b}{r}} dt, \quad \theta^{(r)} = \sqrt{\frac{r}{r-b}} dr, \quad \theta^{(\vartheta)} = r d\vartheta, \quad \theta^{(\varphi)} = r \sin \vartheta d\varphi. \quad (2.18.3)$$

Effective potential:

With the Hamilton-Jacobi formalism it is possible to obtain an effective potential fulfilling $\frac{1}{2}\dot{r}^2 + \frac{1}{2}V_{\text{eff}}(r) = \frac{1}{2}C_0^2$ with

$$V_{\text{eff}}(r) = K \frac{r-b}{r^3} - \kappa \frac{r-b}{r} \quad (2.18.4)$$

and the constants of motion

$$C_0^2 = \dot{t}^2 \left(\frac{r-b}{r} \right)^2, \quad (2.18.5a)$$

$$K = \dot{\vartheta}^2 r^4 + \dot{\varphi}^2 r^4 \sin^2 \vartheta. \quad (2.18.5b)$$

2.18.2 Case AII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 (dr^2 + \sinh^2 r d\varphi^2) + \frac{z}{b-z} dz^2 - \frac{b-z}{z} dt^2. \quad (2.18.6)$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad 0 < r, \quad \varphi \in [0, 2\pi), \quad 0 < z < b.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{z}{b-z}} \partial_t, \quad \mathbf{e}_{(r)} = \frac{1}{z} \partial_r, \quad \mathbf{e}_{(\varphi)} = \frac{1}{z \sinh r} \partial_{\varphi}, \quad \mathbf{e}_{(z)} = \sqrt{\frac{b-z}{z}} \partial_z. \quad (2.18.7)$$

Dual tetrad:

$$\theta^{(t)} = \sqrt{\frac{b-z}{z}} dt, \quad \theta^{(r)} = z dr, \quad \theta^{(\varphi)} = z \sinh r d\varphi, \quad \theta^{(z)} = \sqrt{\frac{z}{b-z}} dz. \quad (2.18.8)$$

2.18.3 Case AIII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 (dr^2 + r^2 d\varphi^2) + zdz^2 - \frac{1}{z} dt^2. \quad (2.18.9)$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad 0 < r, \quad \varphi \in [0, 2\pi), \quad 0 < z.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{z} \partial_t, \quad \mathbf{e}_{(r)} = \frac{1}{z} \partial_r, \quad \mathbf{e}_{(\varphi)} = \frac{1}{zr} \partial_\varphi, \quad \mathbf{e}_{(z)} = \frac{1}{\sqrt{z}} \partial_z. \quad (2.18.10)$$

Dual tetrad:

$$\theta^{(t)} = \frac{1}{\sqrt{z}} dt, \quad \theta^{(r)} = z dr, \quad \theta^{(\varphi)} = zr d\varphi, \quad \theta^{(z)} = \sqrt{z} dz. \quad (2.18.11)$$

2.18.4 Case BI

In spherical coordinates, the metric is given by the line element

$$ds^2 = r^2 (d\vartheta^2 - \sin^2 \vartheta dt^2) + \frac{r}{r-b} dr^2 + \frac{r-b}{r} d\varphi^2. \quad (2.18.12)$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad 0 < \vartheta < \pi, \quad \varphi \in [0, 2\pi), \quad (0 < b < r) \vee (b < 0 < r).$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{r \sin \vartheta} \partial_t, \quad \mathbf{e}_{(r)} = \sqrt{\frac{r-b}{r}} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \sqrt{\frac{r}{r-b}} \partial_\varphi. \quad (2.18.13)$$

Dual tetrad:

$$\theta^{(t)} = r \sin \vartheta dt, \quad \theta^{(r)} = \sqrt{\frac{r}{r-b}} dr, \quad \theta^{(\vartheta)} = r d\vartheta, \quad \theta^{(\varphi)} = \sqrt{\frac{r-b}{r}} d\varphi. \quad (2.18.14)$$

Effective potential:

With the Hamilton-Jacobi formalism, an effective potential for the radial coordinate can be calculated fulfilling $\frac{1}{2}\dot{r}^2 + \frac{1}{2}V_{\text{eff}}(r) = \frac{1}{2}C_0^2$ with

$$V_{\text{eff}}(r) = K \frac{r-b}{r^3} - \kappa \frac{r-b}{r} \quad (2.18.15)$$

and the constants of motion

$$C_0^2 = \dot{\varphi}^2 \left(\frac{r-b}{r} \right)^2, \quad (2.18.16a)$$

$$K = \dot{\vartheta}^2 r^4 - \dot{t}^2 r^4 \sin^2 \vartheta. \quad (2.18.16b)$$

Note that the metric is not spherically symmetric. Particles or light rays fall into one of the poles if they are not moving in the $\vartheta = \frac{\pi}{2}$ plane.

2.18.5 Case BII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 (dr^2 - \sinh^2 r dt^2) + \frac{z}{b-z} dz^2 + \frac{b-z}{z} d\varphi^2. \quad (2.18.17)$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad \varphi \in [0, 2\pi), \quad 0 < z < b, \quad 0 < r.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{z \sinh r} \partial_t, \quad \mathbf{e}_{(r)} = \frac{1}{z} \partial_r, \quad \mathbf{e}_{(\varphi)} = \sqrt{\frac{z}{b-z}} \partial_\varphi, \quad \mathbf{e}_{(z)} = \sqrt{\frac{b-z}{z}} \partial_z. \quad (2.18.18)$$

Dual tetrad:

$$\theta^{(t)} = z \sinh r dt, \quad \theta^{(r)} = z dr, \quad \theta^{(\varphi)} = \sqrt{\frac{b-z}{z}} d\varphi, \quad \theta^{(z)} = \sqrt{\frac{z}{b-z}} dz. \quad (2.18.19)$$

2.18.6 Case BIII

In cylindrical coordinates, the metric is given by the line element

$$ds^2 = z^2 (dr^2 - r^2 dt^2) + z dz^2 + \frac{1}{z} d\varphi^2. \quad (2.18.20)$$

Coordinates and parameters are restricted to

$$t \in \mathbb{R}, \quad \varphi \in [0, 2\pi), \quad 0 < z, \quad 0 < r.$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{zr} \partial_t, \quad \mathbf{e}_{(r)} = \frac{1}{z} \partial_r, \quad \mathbf{e}_{(\varphi)} = \sqrt{z} \partial_\varphi, \quad \mathbf{e}_{(z)} = \frac{1}{\sqrt{z}} \partial_z. \quad (2.18.21)$$

Dual tetrad:

$$\theta^{(t)} = zr dt, \quad \theta^{(r)} = z dr, \quad \theta^{(\varphi)} = \frac{1}{\sqrt{z}} d\varphi, \quad \theta^{(z)} = \sqrt{z} dz. \quad (2.18.22)$$

2.18.7 Case C

The metric is given by the line element

$$ds^2 = \frac{1}{(x+y)^2} \left(\frac{1}{f(x)} dx^2 + f(x) d\varphi^2 - \frac{1}{f(-y)} dy^2 + f(-y) dt^2 \right) \quad (2.18.23)$$

with $f(u) := \pm(u^3 + au + b)$. Coordinates and parameters are restricted to

$$0 < x+y, \quad f(-y) > 0, \quad 0 > f(x).$$

Local tetrad:

$$\mathbf{e}_{(t)} = (x+y) \frac{1}{\sqrt{-y^3 - ay + b}} \partial_t, \quad \mathbf{e}_{(x)} = (x+y) \sqrt{x^3 + ax + b} \partial_x, \quad (2.18.24a)$$

$$\mathbf{e}_{(y)} = (x+y) \sqrt{-y^3 - ay + b} \partial_y, \quad \mathbf{e}_{(\varphi)} = (x+y) \frac{1}{\sqrt{x^3 + ax + b}} \partial_\varphi, \quad (2.18.24b)$$

Dual tetrad:

$$\theta^{(t)} = \frac{1}{x+y} \sqrt{-y^3 - ay + b} dt, \quad \theta^{(x)} = \frac{1}{x+y} \frac{1}{\sqrt{x^3 + ax + b}} dx, \quad (2.18.25a)$$

$$\theta^{(y)} = \frac{1}{x+y} \frac{1}{\sqrt{-y^3 - ay + b}} dy, \quad \theta^{(\varphi)} = \frac{1}{x+y} \sqrt{x^3 + ax + b} d\varphi, \quad (2.18.25b)$$

A coordinate change can eliminate the linear term in the polynomial f generating a quadratic term instead. This brings the line element to the form

$$ds^2 = \frac{1}{A(x+y)^2} \left[\frac{1}{f(x)} dx^2 + f(x) dp^2 - \frac{1}{f(-y)} dy^2 + f(-y) dq^2 \right] \quad (2.18.26)$$

with $f(u) := \pm(-2mAu^3 - u^2 + 1)$ given in [PP01].

Furthermore, coordinates can be adapted to the boost-rotation symmetry with the line element in [PP01] from in [Bon83]

$$ds^2 = \frac{1}{z^2 - t^2} \left[e^\rho r^2 (z dt - t dz)^2 - e^\lambda (z dz - t dt)^2 \right] - e^\lambda dr^2 - r^2 e^{-\rho} d\varphi^2 \quad (2.18.27)$$

with

$$e^\rho = \frac{R_3 + R + Z_3 - r^2}{4\alpha^2 (R_1 + R + Z_1 - r^2)},$$

$$e^\lambda = \frac{2\alpha^2 [R(R + R_1 + Z_1) - Z_1 r^2] [R_1 R_3 + (R + Z_1)(R + Z_3) - (Z_1 + Z_3)r^2]}{R_i R_3 [R(R + R_3 + Z_3) - Z_3 r^2]},$$

$$R = \frac{1}{2} (z^2 - t^2 + r^2),$$

$$R_i = \sqrt{(R + Z_i)^2 - 2Z_i r^2},$$

$$Z_i = z_i - z_2,$$

$$\alpha^2 = \frac{1}{4} \frac{m^2}{A^6 (z_2 - z_1)^2 (z_3 - z_1)^2},$$

$$q = \frac{1}{4\alpha^2},$$

and $z_3 < z_1 < z_2$ the roots of $2A^4 z^3 - A^2 z^2 + m^2$.

Local tetrad:

Case $z^2 - t^2 > 0$:

$$\mathbf{e}_{(t)} = \frac{1}{\sqrt{z^2 - t^2}} \left(qz e^{-\rho/2} \partial_t + t e^{-\lambda/2} \partial_z \right), \quad \mathbf{e}_{(r)} = e^{-\lambda/2} \partial_r, \quad (2.18.28a)$$

$$\mathbf{e}_{(z)} = \frac{1}{\sqrt{z^2 - t^2}} \left(q t e^{-\rho/2} \partial_t + z e^{-\lambda/2} \partial_z \right), \quad \mathbf{e}_{(\varphi)} = r e^{\rho/2} \partial_\varphi. \quad (2.18.28b)$$

Case $z^2 - t^2 < 0$:

$$\mathbf{e}_{(t)} = \frac{1}{\sqrt{t^2 - z^2}} \left(q t e^{-\rho/2} \partial_t + z e^{-\lambda/2} \partial_z \right), \quad \mathbf{e}_{(r)} = e^{-\lambda/2} \partial_r, \quad (2.18.29a)$$

$$\mathbf{e}_{(z)} = \frac{1}{\sqrt{t^2 - z^2}} \left(q z e^{-\rho/2} \partial_t + t e^{-\lambda/2} \partial_z \right), \quad \mathbf{e}_{(\varphi)} = r e^{\rho/2} \partial_\varphi. \quad (2.18.29b)$$

Dual tetrad:

Case $z^2 - t^2 > 0$:

$$\theta^{(t)} = \sqrt{\frac{e^\rho}{z^2 - t^2}} \frac{1}{q} (z dt + t dz), \quad \theta^{(r)} = e^\lambda dr, \quad (2.18.30a)$$

$$\theta^{(z)} = \sqrt{\frac{e^\lambda}{z^2 - t^2}} (t dt + z dz), \quad \theta^{(\varphi)} = \frac{1}{re^\rho} d\varphi. \quad (2.18.30b)$$

Case $t^2 - z^2 > 0$:

$$\theta^{(t)} = \sqrt{\frac{e^\lambda}{t^2 - z^2}} (t dt + z dz), \quad \theta^{(r)} = e^\lambda dr, \quad (2.18.31a)$$

$$\theta^{(z)} = \sqrt{\frac{e^\rho}{t^2 - z^2}} \frac{1}{q} (z dt + t dz), \quad \theta^{(\varphi)} = \frac{1}{re^\rho} d\varphi. \quad (2.18.31b)$$

2.19 Plane gravitational wave

W. Rindler described in [Rin01] an exact plane gravitational wave which is bounded between two planes. The metric of the so called 'sandwich wave' with $u := t - x$ reads

$$\boxed{ds^2 = -dt^2 + dx^2 + p^2(u) dy^2 + q^2(u) dz^2.} \quad (2.19.1)$$

The functions $p(u)$ and $q(u)$ are given by

$$p(u) := \begin{cases} p_0 = \text{const.} & u < -a \\ 1 - u & 0 < u \\ L(u)e^{m(u)} & \text{else} \end{cases} \quad \text{and} \quad q(u) := \begin{cases} q_0 = \text{const.} & u < -a \\ 1 - u & 0 < u \\ L(u)e^{-m(u)} & \text{else} \end{cases}, \quad (2.19.2)$$

where a is the longitudinal extension of the wave. The functions $L(u)$ and $m(u)$ are

$$L(u) = 1 - u + \frac{u^3}{a^2} + \frac{u^4}{2a^3}, \quad m(u) = \pm 2\sqrt{3} \int \sqrt{\frac{u^2 + au}{2a^3u - 2au^3 - u^4 - 2a^3}} du. \quad (2.19.3)$$

Christoffel symbols:

$$\Gamma_{ty}^y = -\Gamma_{xy}^y = \frac{1}{p} \frac{\partial p}{\partial u}, \quad \Gamma_{zz}^t = \Gamma_{zz}^x = q \frac{\partial q}{\partial u}, \quad \Gamma_{tz}^z = -\Gamma_{xz}^z = \frac{1}{q} \frac{\partial q}{\partial u}, \quad \Gamma_{yy}^t = \Gamma_{yy}^x = p \frac{\partial p}{\partial u}. \quad (2.19.4)$$

Riemann-Tensor:

$$R_{tyty} = R_{xyxy} = -R_{tyxy} = -p \frac{\partial^2 p}{\partial u^2}, \quad R_{tztz} = R_{xzxz} = -q \frac{\partial^2 q}{\partial u^2}. \quad (2.19.5)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \partial_t, \quad \mathbf{e}_{(x)} = \partial_x, \quad \mathbf{e}_{(y)} = \frac{1}{p} \partial_y, \quad \mathbf{e}_{(z)} = \frac{1}{q} \partial_z. \quad (2.19.6)$$

Dual tetrad:

$$\theta^{(t)} = dt, \quad \theta^{(x)} = dx, \quad \theta^{(y)} = p dy, \quad \theta^{(z)} = q dz. \quad (2.19.7)$$

2.20 Reissner-Nordstrøm

The Reissner-Nordstrøm black hole in spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ is defined by the metric[MTW73]

$$ds^2 = -A_{\text{RN}}c^2 dt^2 + A_{\text{RN}}^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (2.20.1)$$

where

$$A_{\text{RN}} = 1 - \frac{r_s}{r} + \frac{\rho Q^2}{r^2} \quad (2.20.2)$$

with $r_s = 2GM/c^2$, the charge Q , and $\rho = G/(\varepsilon_0 c^4) \approx 9.33 \cdot 10^{-34}$. As in the Schwarzschild case, there is a true curvature singularity at $r = 0$. However, for $Q^2 < r_s^2/(4\rho)$ there are also two critical points at

$$r = \frac{r_s}{2} \pm \frac{r_s}{2} \sqrt{1 - \frac{4\rho Q^2}{r_s^2}}. \quad (2.20.3)$$

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{A_{\text{RN}}c^2(r_s r - 2\rho Q^2)}{2r^3}, \quad \Gamma_{tr}^t = \frac{r_s r - 2\rho Q^2}{2r^3 A_{\text{RN}}}, \quad \Gamma_{rr}^r = -\frac{r_s r - 2\rho Q^2}{2r^3 A_{\text{RN}}}, \quad (2.20.4a)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^r = -r A_{\text{RN}}, \quad (2.20.4b)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^r = -r A_{\text{RN}} \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.20.4c)$$

Riemann-Tensor:

$$R_{ttrr} = -\frac{c^2(r_s r - 3\rho Q^2)}{r^4}, \quad R_{t\vartheta t\vartheta} = \frac{A_{\text{RN}}c^2(r_s r - 2\rho Q^2)}{2r^2}, \quad (2.20.5a)$$

$$R_{t\varphi t\varphi} = \frac{A_{\text{RN}}c^2(r_s r - 2\rho Q^2) \sin^2 \vartheta}{2r^2}, \quad R_{r\vartheta r\vartheta} = -\frac{r_s r - 2\rho Q^2}{2r^2 A_{\text{RN}}}, \quad (2.20.5b)$$

$$R_{r\varphi r\varphi} = -\frac{(r_s r - 2\rho Q^2) \sin^2 \vartheta}{2r^2 A_{\text{RN}}}, \quad R_{\vartheta\varphi\vartheta\varphi} = (r_s r - \rho Q^2) \sin^2 \vartheta. \quad (2.20.5c)$$

Ricci-Tensor:

$$R_{tt} = \frac{c^2 \rho Q^2 A_{\text{RN}}}{r^4}, \quad R_{rr} = -\frac{\rho Q^2}{r^4 A_{\text{RN}}}, \quad R_{\vartheta\vartheta} = \frac{\rho Q^2}{r^2}, \quad R_{\varphi\varphi} = \frac{\rho Q^2 \sin^2 \vartheta}{r^2}. \quad (2.20.6)$$

While the Ricci scalar vanishes identically, the Kretschmann scalar reads

$$\mathcal{K} = 4 \frac{3r_s^2 r^2 - 12r_s r \rho Q^2 + 14\rho^2 Q^4}{r^8}. \quad (2.20.7)$$

Weyl-Tensor:

$$C_{ttrr} = -\frac{c^2(r_s r - 2\rho Q^2)}{r^4}, \quad C_{t\vartheta t\vartheta} = -\frac{A_{\text{RN}}c^2(r_s r - 2\rho Q^2)}{2r^2}, \quad (2.20.8a)$$

$$C_{t\varphi t\varphi} = \frac{A_{\text{RN}}c^2(r_s r - 2\rho Q^2) \sin^2 \vartheta}{2r^2}, \quad C_{r\vartheta r\vartheta} = -\frac{r_s r - 2\rho Q^2}{2r^2 A_{\text{RN}}}, \quad (2.20.8b)$$

$$C_{r\varphi r\varphi} = -\frac{(r_s r - 2\rho Q^2) \sin^2 \vartheta}{2r^2 A_{\text{RN}}}, \quad C_{\vartheta\varphi\vartheta\varphi} = (r_s r - 2\rho Q^2) \sin^2 \vartheta. \quad (2.20.8c)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{A_{\text{RN}}}}\partial_t, \quad \mathbf{e}_{(r)} = \sqrt{A_{\text{RN}}}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_{\varphi}. \quad (2.20.9)$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{A_{\text{RN}}}\,dt, \quad \theta^{(r)} = \frac{dr}{\sqrt{A_{\text{RN}}}}, \quad \theta^{(\vartheta)} = r\,d\vartheta, \quad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \quad (2.20.10)$$

Ricci rotation coefficients:

$$\gamma_{(t)(t)(t)} = \frac{rr_s - 2\rho Q^2}{2r^3\sqrt{A_{\text{RN}}}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{\sqrt{A_{\text{RN}}}}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \quad (2.20.11)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r^2 - 3rr_s + 2\rho Q^2}{2r^3\sqrt{A_{\text{RN}}}}, \quad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \quad (2.20.12)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(r)(r)(t)(r)} = -\frac{r_s r - 3\rho Q^2}{r^4}, \quad R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{r_s r - \rho Q^2}{r^4}, \quad (2.20.13a)$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s r - 2\rho Q^2}{2r^4}. \quad (2.20.13b)$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{\rho Q^2}{r^4}. \quad (2.20.14)$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s r - 2\rho Q^2}{r^4}, \quad (2.20.15a)$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{r_s r - 2\rho Q^2}{2r^4}. \quad (2.20.15b)$$

Embedding:

The embedding function follows from the numerical integration of

$$\frac{dz}{dr} = \sqrt{\frac{1}{1 - r_s/r + \rho Q^2/r^2} - 1}. \quad (2.20.16)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \frac{1}{2}\frac{k^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r} + \frac{\rho Q^2}{r^2}\right)\left(\frac{h^2}{r^2} - \kappa c^2\right) \quad (2.20.17)$$

with constants of motion $k = A_{\text{RN}}c^2i$ and $h = r^2\dot{\varphi}$. For null geodesics, $\kappa = 0$, there are two extremal points

$$r_{\pm} = \frac{3}{4}r_s \left(1 \pm \sqrt{1 - \frac{32\rho Q^2}{9r_s^2}}\right), \quad (2.20.18)$$

where r_+ is a maximum and r_- a minimum.

Further reading:

Eiroa[ERT02]

2.21 de Sitter spacetime

The de Sitter spacetime with $\Lambda > 0$ is a solution of the Einstein field equations with constant curvature. A detailed discussion can be found for example in Hawking and Ellis[HE99]. Here, we use the coordinate transformations given by Bičák[BK01].

2.21.1 Standard coordinates

The de Sitter metric in standard coordinates $\{\tau \in \mathbb{R}, \chi \in [-\pi, \pi], \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ reads

$$ds^2 = -d\tau^2 + \alpha^2 \cosh^2 \frac{\tau}{\alpha} [d\chi^2 + \sin^2 \chi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)], \quad (2.21.1)$$

where $\alpha^2 = 3/\Lambda$.

Christoffel symbols:

$$\Gamma_{\tau\chi}^{\chi} = \frac{1}{\alpha} \tanh \frac{\tau}{\alpha}, \quad \Gamma_{\tau\vartheta}^{\vartheta} = \frac{1}{\alpha} \tanh \frac{\tau}{\alpha}, \quad \Gamma_{\tau\varphi}^{\varphi} = \frac{1}{\alpha} \tanh \frac{\tau}{\alpha}, \quad (2.21.2a)$$

$$\Gamma_{\chi\chi}^{\tau} = \alpha \sinh \frac{\tau}{\alpha} \cosh \frac{\tau}{\alpha}, \quad \Gamma_{\chi\vartheta}^{\vartheta} = \cot \chi, \quad \Gamma_{\chi\varphi}^{\varphi} = \cot \chi, \quad (2.21.2b)$$

$$\Gamma_{\vartheta\vartheta}^{\tau} = \alpha \sin^2 \chi \sinh \frac{\tau}{\alpha} \cosh \frac{\tau}{\alpha}, \quad \Gamma_{\vartheta\vartheta}^{\chi} = -\sin \chi \cos \chi, \quad \Gamma_{\vartheta\varphi}^{\varphi} = \cot \vartheta, \quad (2.21.2c)$$

$$\Gamma_{\varphi\varphi}^{\tau} = \alpha \sin^2 \chi \sin^2 \vartheta \sinh \frac{\tau}{\alpha} \cosh \frac{\tau}{\alpha}, \quad \Gamma_{\varphi\varphi}^{\chi} = -\sin^2 \vartheta \sin \chi \cos \chi, \quad \Gamma_{\varphi\varphi}^{\vartheta} = -\sin \vartheta \cos \vartheta. \quad (2.21.2d)$$

Riemann-Tensor:

$$R_{\tau\chi\tau\chi} = -\cosh^2 \frac{\tau}{\alpha}, \quad R_{\tau\vartheta\tau\vartheta} = -\cosh^2 \frac{\tau}{\alpha} \sin^2 \chi, \quad (2.21.3a)$$

$$R_{\tau\varphi\tau\varphi} = -\cosh^2 \frac{\tau}{\alpha} \sin^2 \chi \sin^2 \vartheta, \quad R_{\chi\vartheta\chi\vartheta} = \alpha^2 \left(1 + \sinh^2 \frac{\tau}{\alpha}\right)^2 \sin^2 \chi, \quad (2.21.3b)$$

$$R_{\chi\varphi\chi\varphi} = \alpha^2 \left(1 + \sinh^2 \frac{\tau}{\alpha}\right)^2 \sin^2 \chi \sin^2 \vartheta, \quad R_{\vartheta\varphi\vartheta\varphi} = \alpha^2 \left(1 + \sinh^2 \frac{\tau}{\alpha}\right)^2 \sin^4 \chi \sin^2 \vartheta. \quad (2.21.3c)$$

Ricci-Tensor:

$$R_{\tau\tau} = -\frac{3}{\alpha^2}, \quad R_{\chi\chi} = 3 \cosh^2 \frac{\tau}{\alpha}, \quad R_{\vartheta\vartheta} = 3 \cosh^2 \frac{\tau}{\alpha} \sin^2 \chi, \quad R_{\varphi\varphi} = 3 \cosh^2 \frac{\tau}{\alpha} \sin^2 \chi \sin^2 \vartheta. \quad (2.21.4)$$

Ricci and Kretschmann scalars:

$$\mathcal{R} = \frac{12}{\alpha^2}, \quad \mathcal{K} = \frac{24}{\alpha^4}. \quad (2.21.5)$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_{\tau}, \quad \mathbf{e}_{(\chi)} = \frac{1}{\alpha \cosh \frac{\tau}{\alpha}} \partial_{\chi}, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{\alpha \cosh \frac{\tau}{\alpha} \sin \chi} \partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{1}{\alpha \cosh \frac{\tau}{\alpha} \sin \chi \sin \vartheta} \partial_{\varphi}. \quad (2.21.6)$$

Dual tetrad:

$$\theta^{(\tau)} = d\tau, \quad \theta^{(\chi)} = \alpha \cosh \frac{\tau}{\alpha} d\chi, \quad \theta^{(\vartheta)} = \alpha \cosh \frac{\tau}{\alpha} \sin \chi d\vartheta, \quad \theta^{(\varphi)} = \alpha \cosh \frac{\tau}{\alpha} \sin \chi \sin \vartheta d\varphi. \quad (2.21.7)$$

2.21.2 Conformally Einstein coordinates

In conformally Einstein coordinates $\{\eta \in [0, \pi], \chi \in [-\pi, \pi], \vartheta \in [0, \pi], \varphi \in [0, 2\pi)\}$, the de Sitter metric reads

$$ds^2 = \frac{\alpha^2}{\sin^2 \eta} [-d\eta^2 + d\chi^2 + \sin^2 \chi (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)]. \quad (2.21.8)$$

It follows from the standard form (2.21.1) by the transformation

$$\eta = 2 \arctan \left(e^{\tau/\alpha} \right). \quad (2.21.9)$$

2.21.3 Conformally flat coordinates

Conformally flat coordinates $\{T \in \mathbb{R}, r \in \mathbb{R}, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ follow from conformally Einstein coordinates by means of the transformations

$$T = \frac{\alpha \sin \eta}{\cos \chi + \cos \eta}, \quad r = \frac{\alpha \sin \chi}{\cos \chi + \cos \eta}, \quad \text{or} \quad \eta = \arctan \frac{2T\alpha}{\alpha^2 - T^2 + r^2}, \quad \chi = \arctan \frac{2r\alpha}{\alpha^2 + T^2 - r^2}. \quad (2.21.10)$$

For the transformation $(T, R) \rightarrow (\eta, \chi)$, we have to take care of the coordinate domains. In that case, if $\kappa^2 - T^2 + r^2 < 0$, we have to map $\eta \rightarrow \eta + \pi$. On the other hand, if $\kappa^2 + T^2 - r^2 < 0$, we have to consider the sign of r . If $r > 0$, then $\chi \rightarrow \chi + \pi$, otherwise $\chi \rightarrow \chi - \pi$.

The resulting metric reads

$$ds^2 = \frac{\alpha^2}{T^2} [-dT^2 + dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)]. \quad (2.21.11)$$

Note that we identify points $(r < 0, \vartheta, \varphi)$ with $(r > 0, \pi - \vartheta, \varphi - \pi)$.

Christoffel symbols:

$$\Gamma_{TT}^T = \Gamma_{Tr}^r = \Gamma_{T\vartheta}^\vartheta = \Gamma_{T\varphi}^\varphi = \Gamma_{rr}^r = -\frac{1}{T}, \quad \Gamma_{r\vartheta}^\vartheta = \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^T = -\frac{r^2}{T}, \quad \Gamma_{\vartheta\vartheta}^r = -r, \quad (2.21.12a)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^T = -\frac{r^2 \sin^2 \vartheta}{T}, \quad \Gamma_{\varphi\varphi}^r = -r \sin^2 \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta. \quad (2.21.12b)$$

Riemann-Tensor:

$$R_{TrTr} = -\frac{\alpha^2}{T^4}, \quad R_{T\vartheta T\vartheta} = -\frac{\alpha^2 r^2}{T^4}, \quad R_{T\varphi T\varphi} = -\frac{\alpha^2 r^2 \sin^2 \vartheta}{T^4}, \quad (2.21.13a)$$

$$R_{r\vartheta r\vartheta} = \frac{\alpha^2 r^2}{T^4}, \quad R_{r\varphi r\varphi} = \frac{\alpha^2 r^2 \sin^2 \vartheta}{T^4}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{\alpha^2 r^4 \sin^2 \vartheta}{T^4}. \quad (2.21.13b)$$

Ricci-Tensor:

$$R_{TT} = -\frac{3}{T^2}, \quad R_{rr} = \frac{3}{T^2}, \quad R_{\vartheta\vartheta} = \frac{3r^2}{T^2}, \quad R_{\varphi\varphi} = \frac{3r^2 \sin^2 \vartheta}{T^2}. \quad (2.21.14)$$

The Ricci and Kretschmann scalar read:

$$\mathcal{R} = \frac{12}{\alpha^2}, \quad \mathcal{K} = \frac{24}{\alpha^4}. \quad (2.21.15)$$

Local tetrad:

$$\mathbf{e}_{(T)} = \frac{T}{\alpha} \partial_T, \quad \mathbf{e}_{(r)} = \frac{T}{\alpha} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{T}{\alpha r} \partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{T}{\alpha r \sin \vartheta} \partial_\varphi. \quad (2.21.16)$$

2.21.4 Static coordinates

The de Sitter metric in static spherical coordinates $\{t \in \mathbb{R}, r \in \mathbb{R}^+, \vartheta \in (0, \pi), \varphi \in [0, 2\pi)\}$ reads

$$ds^2 = - \left(1 - \frac{\Lambda}{3} r^2 \right) c^2 dt^2 + \left(1 - \frac{\Lambda}{3} r^2 \right)^{-1} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (2.21.17)$$

It follows from the conformally Einstein form (2.21.8) by the transformations

$$t = \frac{\alpha}{2} \ln \frac{\cos \chi - \cos \eta}{\cos \chi + \cos \eta}, \quad r = \alpha \frac{\sin \chi}{\sin \eta}. \quad (2.21.18)$$

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{(\Lambda r^2 - 3)}{9} c^2 \Lambda r, \quad \Gamma_{tr}^r = \frac{\Lambda r}{\Lambda r^2 - 3}, \quad \Gamma_{rr}^r = \frac{\Lambda r}{3 - \Lambda r^2}, \quad (2.21.19a)$$

$$\Gamma_{r\vartheta}^{\vartheta} = \frac{1}{r}, \quad \Gamma_{r\phi}^{\phi} = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^r = \frac{(\Lambda r^2 - 3)r}{3}, \quad (2.21.19b)$$

$$\Gamma_{\vartheta\phi}^{\phi} = \cot(\vartheta), \quad \Gamma_{\phi\phi}^r = \frac{\Lambda r^2 - 3}{3} r \sin^2(\vartheta), \quad \Gamma_{\phi\phi}^{\vartheta} = -\sin(\vartheta) \cos(\vartheta). \quad (2.21.19c)$$

Riemann-Tensor:

$$R_{trtr} = -\frac{\Lambda}{3} c^2, \quad R_{t\vartheta t\vartheta} = -\frac{3 - \Lambda r^2}{9} c^2 \Lambda r^2, \quad R_{t\phi t\phi} = -\frac{3 - \Lambda r^2}{9} c^2 \Lambda r^2 \sin^2(\vartheta)^2, \quad (2.21.20a)$$

$$R_{r\vartheta r\vartheta} = \frac{\Lambda r^2}{-\Lambda r^2 + 3}, \quad R_{r\phi r\phi} = \frac{\Lambda r^2 \sin^2(\vartheta)^2}{-\Lambda r^2 + 3}, \quad R_{\vartheta\phi\vartheta\phi} = \frac{r^4 \sin^2(\vartheta)\Lambda}{3}. \quad (2.21.20b)$$

Ricci-Tensor:

$$R_{tt} = \frac{\Lambda r^2 - 3}{3} c^2 \Lambda, \quad R_{rr} = \frac{3\Lambda}{3 - \Lambda r^2}, \quad R_{\vartheta\vartheta} = \Lambda r^2, \quad R_{\phi\phi} = r^2 \sin^2(\vartheta)\Lambda. \quad (2.21.21)$$

The Ricci scalar and Kretschmann scalar read:

$$\mathcal{R} = 4\Lambda, \quad \mathcal{K} = \frac{8}{3}\Lambda^2. \quad (2.21.22)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \sqrt{\frac{3}{3 - \Lambda r^2}} \frac{\partial}{\partial t}, \quad \mathbf{e}_{(r)} = \sqrt{1 - \frac{\Lambda r^2}{3}} \frac{\partial}{\partial r}, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r} \frac{\partial}{\partial \vartheta}, \quad \mathbf{e}_{(\phi)} = \frac{1}{r \sin(\vartheta)} \frac{\partial}{\partial \phi}. \quad (2.21.23)$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(t)} = -\frac{\Lambda r}{\sqrt{9 - 3\Lambda r^2}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\phi)(r)(\phi)} = \frac{\sqrt{9 - 3\Lambda r^2}}{3r}, \quad \gamma_{(\phi)(\vartheta)(\phi)} = \frac{\cot \vartheta}{r}. \quad (2.21.24)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{\sqrt{9 - 3\Lambda r^2}(\Lambda r^2 - 2)}{(\Lambda r^2 - 3)r}, \quad \gamma_{(\vartheta)} = \frac{\cot \vartheta}{r}. \quad (2.21.25)$$

Riemann-Tensor with respect to local tetrad:

$$-R_{(t)(r)(t)(r)} = -R_{(t)(\vartheta)(t)(\vartheta)} = -R_{(t)(\phi)(t)(\phi)} = R_{(r)(\vartheta)(r)(\vartheta)} = R_{(r)(\phi)(r)(\phi)} = R_{(\vartheta)(\phi)(\vartheta)(\phi)} = \frac{1}{3}\Lambda. \quad (2.21.26)$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\phi)(\phi)} = \Lambda. \quad (2.21.27)$$

2.21.5 Lemaître-Robertson form

The de Sitter universe in the Lemaître-Robertson form reads

$$ds^2 = -c^2 dt^2 + e^{2Ht} [dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2)], \quad (2.21.28)$$

with Hubble's Parameter $H = \sqrt{\frac{\Lambda c^2}{3}} = \frac{c}{\alpha}$, which is assumed here to be time-independent.

This a special case of the first and second form of the Friedman-Robertson-Walker metric defined in Eqs. (2.9.2) and (2.9.12) with $R(t) = e^{Ht}$ and $k = 0$.

Christoffel symbols:

$$\Gamma_{tr}^r = H, \quad \Gamma_{t\vartheta}^{\vartheta} = H, \quad \Gamma_{t\varphi}^{\varphi} = H, \quad (2.21.29a)$$

$$\Gamma_{rr}^r = \frac{e^{2Ht} H}{c^2}, \quad \Gamma_{r\vartheta}^{\vartheta} = \frac{1}{r}, \quad \Gamma_{r\varphi}^{\varphi} = \frac{1}{r}, \quad (2.21.29b)$$

$$\Gamma_{\vartheta\vartheta}^{\vartheta} = \frac{e^{2Ht} r^2 H}{c^2}, \quad \Gamma_{\vartheta\vartheta}^r = -r, \quad \Gamma_{\vartheta\varphi}^{\varphi} = \cot(\vartheta), \quad (2.21.29c)$$

$$\Gamma_{\varphi\varphi}^{\varphi} = \frac{e^{2Ht} r^2 \sin^2(\vartheta) H}{c^2}, \quad \Gamma_{\varphi\varphi}^r = -r \sin^2(\vartheta), \quad \Gamma_{\varphi\varphi}^{\vartheta} = -\sin(\vartheta) \cos(\vartheta). \quad (2.21.29d)$$

Riemann-Tensor:

$$R_{trtr} = -e^{2Ht} H^2, \quad R_{t\vartheta t\vartheta} = -e^{2Ht} r^2 H^2, \quad (2.21.30a)$$

$$R_{t\varphi t\varphi} = -e^{2Ht} r^2 \sin^2(\vartheta) H^2, \quad R_{r\vartheta r\vartheta} = \frac{e^{4Ht} r^2 H^2}{c^2}, \quad (2.21.30b)$$

$$R_{r\varphi r\varphi} = \frac{e^{4Ht} r^2 \sin^2(\vartheta) H^2}{c^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = \frac{e^{4Ht} r^4 \sin^2(\vartheta) H^2}{c^2}. \quad (2.21.30c)$$

Ricci-Tensor:

$$R_{tt} = -3H^2, \quad R_{rr} = 3 \frac{e^{2Ht} H^2}{c^2}, \quad R_{\vartheta\vartheta} = 3 \frac{e^{2Ht} r^2 H^2}{c^2}, \quad R_{\varphi\varphi} = 3 \frac{e^{2Ht} r^2 \sin^2(\vartheta) H^2}{c^2}. \quad (2.21.31)$$

Ricci and Kretschmann scalars:

$$\mathcal{R} = \frac{12H^2}{c^2}, \quad \mathcal{K} = \frac{24H^4}{c^4}. \quad (2.21.32)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c} \partial_t, \quad \mathbf{e}_{(r)} = e^{-Ht} \partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{e^{-Ht}}{r} \partial_{\vartheta}, \quad \mathbf{e}_{(\varphi)} = \frac{e^{-Ht}}{r \sin \vartheta} \partial_{\varphi}. \quad (2.21.33)$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(r)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{H}{c} \quad (2.21.34a)$$

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{e^{Ht} r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot(\vartheta)}{e^{Ht} r}. \quad (2.21.34b)$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = 3 \frac{H}{c}, \quad \gamma_{(r)} = \frac{2}{e^{Ht} r}, \quad \gamma_{(\vartheta)} = \frac{\cot(\vartheta)}{e^{Ht} r}. \quad (2.21.35)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{H^2}{c^2} \quad (2.21.36a)$$

$$R_{(r)(\vartheta)(r)(\vartheta)} = R_{(r)(\varphi)(r)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{H^2}{c^2}. \quad (2.21.36b)$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = 3\frac{H^2}{c^2}. \quad (2.21.37)$$

2.21.6 Cartesian coordinates

The de Sitter universe in Lemaître-Robertson form can also be expressed in Cartesian coordinates:

$$ds^2 = -c^2 dt^2 + e^{2Ht} [dx^2 + dy^2 + dz^2]. \quad (2.21.38)$$

Christoffel symbols:

$$\Gamma_{tx}^x = H, \quad \Gamma_{ty}^y = H, \quad \Gamma_{tz}^z = H, \quad (2.21.39a)$$

$$\Gamma_{xx}^t = \frac{e^{2Ht} H}{c^2}, \quad \Gamma_{yy}^t = \frac{e^{2Ht} H}{c^2}, \quad \Gamma_{zz}^t = \frac{e^{2Ht} H}{c^2}. \quad (2.21.39b)$$

$$(2.21.39c)$$

Partial derivatives

$$\Gamma_{xx,t}^t = \Gamma_{yy,t}^t = \Gamma_{zz,t}^t = \frac{2H^2 e^{2Ht}}{c^2}. \quad (2.21.40)$$

Riemann-Tensor:

$$R_{txtx} = R_{txxt} = R_{tztz} = -e^{2Ht} H^2, \quad R_{xyxy} = R_{xzyz} = R_{yzyz} = \frac{e^{4Ht} H^2}{c^2}. \quad (2.21.41)$$

Ricci-Tensor:

$$R_{tt} = -3H^2, \quad R_{xx} = R_{yy} = R_{zz} = 3\frac{e^{2Ht} H^2}{c^2}. \quad (2.21.42)$$

The Ricci and Kretschmann scalar read:

$$\mathcal{R} = 12\frac{H^2}{c^2}, \quad \mathcal{K} = 24\frac{H^4}{c^4}. \quad (2.21.43)$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \quad \mathbf{e}_{(x)} = e^{-Ht}\partial_x, \quad \mathbf{e}_{(y)} = e^{-Ht}\partial_y, \quad \mathbf{e}_{(z)} = e^{-Ht}\partial_z. \quad (2.21.44)$$

Ricci rotation coefficients:

$$\gamma_{(x)(t)(x)} = \gamma_{(y)(t)(y)} = \gamma_{(z)(t)(z)} = \frac{H}{c}. \quad (2.21.45)$$

The only non-vanishing contraction of the Ricci rotation coefficients read

$$\gamma_{(t)} = 3\frac{H}{c}. \quad (2.21.46)$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(x)(t)(x)} = R_{(t)(y)(t)(y)} = R_{(t)(z)(t)(z)} = -\frac{H^2}{c^2}, \quad (2.21.47a)$$

$$R_{(x)(y)(x)(y)} = R_{(x)(z)(x)(z)} = R_{(y)(z)(y)(z)} = \frac{H^2}{c^2}. \quad (2.21.47b)$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(x)(x)} = R_{(y)(y)} = R_{(z)(z)} = 3\frac{H^2}{c^2}. \quad (2.21.48)$$

Further reading:

Tolman[[Tol34](#), sec. 142], Bičák[[BK01](#)]

2.22 Straight spinning string

The metric of a straight spinning string in cylindrical coordinates (t, ρ, φ, z) reads

$$ds^2 = -(cdt - ad\varphi)^2 + d\rho^2 + k^2\rho^2 d\varphi^2 + dz^2, \quad (2.22.1)$$

where $a \in \mathbb{R}$ and $k > 0$ are two parameters, see Perlick[Per04].

Metric-Tensor:

$$g_{tt} = -c^2, \quad g_{t\varphi} = ac, \quad g_{\rho\rho} = g_{zz} = 1, \quad g_{\varphi\varphi} = k^2\rho^2 - a^2. \quad (2.22.2)$$

Christoffel symbols:

$$\Gamma_{\rho\varphi}^t = \frac{a}{c\rho}, \quad \Gamma_{\rho\varphi}^\varphi = \frac{1}{\rho}, \quad \Gamma_{\varphi\varphi}^\rho = -k^2\rho. \quad (2.22.3)$$

Partial derivatives

$$\Gamma_{\rho\varphi,\rho}^t = -\frac{\alpha}{c\rho^2}, \quad \Gamma_{\rho\varphi,\rho}^\varphi = -\frac{1}{\rho^2}, \quad \Gamma_{\varphi\varphi,\rho}^\rho = -k^2. \quad (2.22.4)$$

The Riemann-, Ricci-, and Weyl-tensors as well as the Ricci- and Kretschmann-scalar vanish identically.

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{c}\partial_t, \quad \mathbf{e}_{(1)} = \partial_\rho, \quad \mathbf{e}_{(2)} = \frac{1}{k\rho} \left(\frac{a}{c}\partial_t + \partial_\varphi \right), \quad \mathbf{e}_{(3)} = \partial_z. \quad (2.22.5)$$

Dual tetrad:

$$\theta^{(0)} = cdt - ad\varphi, \quad \theta^{(1)} = d\rho, \quad \theta^{(2)} = k\rho d\varphi, \quad \theta^{(3)} = dz. \quad (2.22.6)$$

Ricci rotation coefficients and their contractions read

$$\gamma_{(2)(1)(2)} = \frac{1}{\rho}, \quad \gamma_{(0)} = \gamma_{(2)} = \gamma_{(3)} = 0, \quad \gamma_{(1)} = \frac{1}{\rho}. \quad (2.22.7)$$

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{\sqrt{k^2\rho^2 - a^2}}{k\rho} \left(\frac{1}{c}\partial_t - \frac{a}{k^2\rho^2 - a^2}\partial_\varphi \right), \quad \mathbf{e}_{(1)} = \partial_\rho, \quad (2.22.8a)$$

$$\mathbf{e}_{(2)} = \frac{1}{\sqrt{k^2\rho^2 - a^2}}\partial_\varphi, \quad \mathbf{e}_{(3)} = \partial_z. \quad (2.22.8b)$$

Dual tetrad:

$$\theta^{(0)} = \frac{k\rho}{\sqrt{k^2\rho^2 - a^2}}cdt, \quad \theta^{(1)} = d\rho, \quad \theta^{(2)} = \frac{acdt}{\sqrt{k^2\rho^2 - a^2}} + \sqrt{k^2\rho^2 - a^2}d\varphi, \quad \theta^{(3)} = dz. \quad (2.22.9)$$

Ricci rotation coefficients and their contractions read

$$\gamma_{(0)(1)(0)} = \frac{a^2}{\rho(k^2\rho^2 - a^2)}, \quad \gamma_{(2)(1)(0)} = \gamma_{(0)(2)(1)} = \gamma_{(0)(1)(2)} = \frac{ak}{k^2\rho^2 - a^2}, \quad (2.22.10a)$$

$$\gamma_{(2)(1)(2)} = \frac{k^2\rho}{k^2\rho^2 - a^2}, \quad (2.22.10b)$$

$$\gamma_{(1)} = \frac{1}{\rho}. \quad (2.22.10c)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\dot{\rho}^2 + \frac{1}{k^2 \rho^2} \left(h_2 - \frac{a h_1}{c} \right)^2 - \kappa c^2 = \frac{h_1^2}{c^2}, \quad (2.22.11)$$

with the constants of motion $h_1 = c(ct - a\phi)$ and $h_2 = a(ct - a\phi) + k^2 \rho^2 \dot{\phi}$.

The point of closest approach ρ_{pca} for a null geodesic that starts at $\rho = \rho_i$ with $\mathbf{y} = \pm \mathbf{e}_{(0)} + \cos \xi \mathbf{e}_{(1)} + \sin \xi \mathbf{e}_{(2)}$ with respect to the static tetrad is given by $\rho = \rho_i \sin \xi$. Hence, the ρ_{pca} is independent of a and k . The same is also true for timelike geodesics.

2.23 Sultana-Dyer spacetime

The Sultana-Dyer metric represents a black hole in the Einstein-de Sitter universe. In spherical coordinates $(t, r, \vartheta, \varphi)$, the metric reads[SD05] ($G = c = 1$)

$$ds^2 = t^4 \left[\left(1 - \frac{2M}{r}\right) dt^2 - \frac{4M}{r} dt dr - \left(1 + \frac{2M}{r}\right) dr^2 - r^2 d\Omega^2 \right], \quad (2.23.1)$$

where M is the mass of the black hole and $\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2$ is the spherical surface element. Note that here, the signature of the metric is $\text{sign}(\mathbf{g}) = -2$.

Christoffel symbols:

$$\Gamma_{tt}^t = \frac{2(r^3 + 4M^2r + M^2t)}{tr^3}, \quad \Gamma_{tt}^r = \frac{M(r-2M)(4r+t)}{tr^3}, \quad \Gamma_{tr}^t = \frac{M(r+2M)(4r+t)}{tr^3}, \quad (2.23.2a)$$

$$\Gamma_{tr}^r = \frac{2(r^3 - 4M^2r - M^2t)}{tr^3}, \quad \Gamma_{t\vartheta}^\vartheta = \frac{2}{t}, \quad \Gamma_{t\varphi}^\varphi = \frac{2}{t}, \quad (2.23.2b)$$

$$\Gamma_{r\vartheta}^\vartheta = \frac{1}{r}, \quad \Gamma_{r\varphi}^\varphi = \frac{1}{r}, \quad \Gamma_{\vartheta\vartheta}^\vartheta = \frac{2(r^2 + 2Mr - Mt)}{t}, \quad (2.23.2c)$$

$$\Gamma_{\vartheta\vartheta}^r = -\frac{4Mr + tr - 2Mt}{t}, \quad \Gamma_{\vartheta\varphi}^\varphi = \cot \vartheta, \quad \Gamma_{\varphi\varphi}^\vartheta = -\sin \vartheta \cos \vartheta, \quad (2.23.2d)$$

$$\Gamma_{rr}^r = \frac{2(r^3 + 4Mr^2 + 4M^2r + M^2t + Mtr)}{tr^3}, \quad \Gamma_{rr}^t = -\frac{M(4r^2 + 8Mr + 2Mt + tr)}{tr^3}, \quad (2.23.2e)$$

$$\Gamma_{\varphi\varphi}^r = \frac{2(r^2 + 2Mr - Mt) \sin^2 \vartheta}{t}, \quad \Gamma_{\varphi\varphi}^t = -\frac{(4Mr + tr - 2Mt) \sin^2 \vartheta}{t}. \quad (2.23.2f)$$

Riemann-Tensor:

$$R_{trtr} = \frac{2t^2(-2Mr^2 - r^3 + Mt^2 + 2Mtr)}{r^3}, \quad (2.23.3a)$$

$$R_{r\vartheta t\vartheta} = -\frac{t^2(2r^4 + 16M^2r^2 + 4Mtr^2 - 4M^2r^2t + Mt^2r - 2M^2t^2)}{r^2}, \quad (2.23.3b)$$

$$R_{t\vartheta r\vartheta} = -\frac{2Mt^2(4r+t)(r^2 + 2Mr - Mt)}{r^2}, \quad (2.23.3c)$$

$$R_{r\varphi t\varphi} = -\frac{t^2 \sin^2 \vartheta (2r^4 + 16M^2r^2 + 4Mtr^2 - 4M^2r^2t + Mt^2r - 2M^2t^2)}{r^2}, \quad (2.23.3d)$$

$$R_{t\varphi r\varphi} = -\frac{2Mt^2 \sin^2 \vartheta (4r+t)(r^2 + 2Mr - Mt)}{r^2}, \quad (2.23.3e)$$

$$R_{r\vartheta r\vartheta} = -\frac{t^2(4r^4 + 16Mr^4 - 4M^2tr + 16M^2r^2 - 2M^2t^2 - Mt^2r)}{r^2}, \quad (2.23.3f)$$

$$R_{r\varphi r\varphi} = -\frac{t^2 \sin^2 \vartheta (4r^4 + 16Mr^4 - 4M^2tr + 16M^2r^2 - 2M^2t^2 - Mt^2r)}{r^2}, \quad (2.23.3g)$$

$$R_{\vartheta\varphi\vartheta\varphi} = -2t^2 r \sin^2 \vartheta (2r^3 + 4Mr^2 - 4Mtr + Mt^2). \quad (2.23.3h)$$

Ricci-Tensor:

$$R_{tt} = \frac{2(3r^2 + 12M^2 + 2Mt)}{t^2 r^2}, \quad R_{tr} = \frac{4M(3r+t+6M)}{t^2 r^2}, \quad (2.23.4a)$$

$$R_{rr} = \frac{2(3r^2 + 12Mr + 2Mt + 12M^2)}{t^2 r^2}, \quad R_{\vartheta\vartheta} = \frac{6(r^2 + 2Mr - 2Mt)}{t^2}, \quad (2.23.4b)$$

$$R_{\varphi\varphi} = \frac{6(r^2 + 2Mr - 2Mt) \sin^2 \vartheta}{t^2}. \quad (2.23.4c)$$

Ricci and Kretschmann scalars:

$$R = -\frac{12(r^2 + 2Mr - 2Mt)}{t^6 r^2}, \quad (2.23.5a)$$

$$\mathcal{K} = \frac{48(M^2 t^4 + 20M^2 r^4 + 20Mr^5 + 8M^2 r^2 t^2 - 4Mr^4 t - 16M^2 r^3 t + 5t^6)}{t^{12} r^6}. \quad (2.23.5b)$$

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{\sqrt{1+2M/r}}{t^2} \partial_t - \frac{2M/r}{t^2 \sqrt{1+2M/r}} \partial_r, \quad \mathbf{e}_{(1)} = \frac{1}{t^2 \sqrt{1+2M/r}} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{t^2 r} \partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{1}{t^2 r \sin \vartheta} \partial_\varphi. \quad (2.23.6)$$

Static local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{t^2 \sqrt{1-2M/r}} \partial_t, \quad \mathbf{e}_{(1)} = \frac{2M/r}{t^2 \sqrt{1-2M/r}} \partial_t + \frac{\sqrt{1-2M/r}}{t^2} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{t^2 r} \partial_\vartheta, \quad \mathbf{e}_{(3)} = \frac{1}{t^2 r \sin \vartheta} \partial_\varphi. \quad (2.23.7)$$

Further reading:

Sultana and Dyer[SD05].

2.24 TaubNUT

The TaubNUT metric in Boyer-Lindquist like spherical coordinates $(t, r, \vartheta, \varphi)$ reads[BCJ02] ($G = c = 1$)

$$ds^2 = -\frac{\Delta}{\Sigma} (dt + 2\ell \cos \vartheta d\varphi)^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right), \quad (2.24.1)$$

where $\Sigma = r^2 + \ell^2$ and $\Delta = r^2 - 2Mr - \ell^2$. Here, M is the mass of the black hole and ℓ the magnetic monopole strength.

Christoffel symbols:

$$\Gamma_{tt}^r = \frac{\Delta \rho}{\Sigma^3}, \quad \Gamma_{tr}^r = \frac{\rho}{\Delta \Sigma}, \quad \Gamma_{t\vartheta}^r = -2\ell^2 \cos \vartheta \frac{\Delta}{\Sigma^2}, \quad (2.24.2a)$$

$$\Gamma_{t\vartheta}^\varphi = \frac{\ell \Delta}{\Sigma^2 \sin \vartheta}, \quad \Gamma_{t\varphi}^r = \frac{2\ell \rho \Delta \cos \vartheta}{\Sigma^3}, \quad \Gamma_{t\varphi}^\varphi = -\frac{\ell \Delta \sin \vartheta}{\Sigma^2}, \quad (2.24.2b)$$

$$\Gamma_{rr}^r = -\frac{\rho}{\Sigma \Delta}, \quad \Gamma_{r\vartheta}^\varphi = \frac{r}{\Sigma}, \quad \Gamma_{r\varphi}^\varphi = \frac{r}{\Sigma}, \quad \Gamma_{\vartheta\vartheta}^r = -\frac{r\Delta}{\Sigma}, \quad (2.24.2c)$$

$$\Gamma_{r\varphi}^r = \frac{-2\ell(r^3 - 3Mr^2 - 3r\ell^2 + M\ell^2) \cos \vartheta}{\Sigma \Delta}, \quad (2.24.2d)$$

$$\Gamma_{\vartheta\varphi}^r = -\frac{\ell [\cos^2 \vartheta (6r^2 \ell^2 - 8\ell^2 Mr - 3\ell^4 + r^4) + \Sigma^2]}{\Sigma^2 \sin \vartheta}, \quad (2.24.2e)$$

$$\Gamma_{\varphi\varphi}^r = \frac{\Delta}{\Sigma^3} [\cos^2 \vartheta (9r\ell^4 + 4\ell^2 Mr^2 - 4\ell^4 M + r^5 + 2r^3 \ell^2) - r\Sigma^2], \quad (2.24.2f)$$

$$\Gamma_{\vartheta\varphi}^\varphi = \frac{(4r^2 \ell^2 - 4Mr\ell^2 - \ell^4 + r^4) \cot \vartheta}{\Sigma^2}, \quad (2.24.2g)$$

$$\Gamma_{\varphi\varphi}^\varphi = -\frac{(6r^2 \ell^2 - 8Mr\ell^2 - 3\ell^4 + r^4) \sin \vartheta \cos \vartheta}{\Sigma^2}, \quad (2.24.2h)$$

where $\rho = 2r\ell^2 + Mr^2 - M\ell^2$.

Static local tetrad:

$$\mathbf{e}_{(0)} = \sqrt{\frac{\Sigma}{\Delta}} \partial_t, \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}} \partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}} \partial_\vartheta, \quad \mathbf{e}_{(3)} = -\frac{2\ell \cot \vartheta}{\sqrt{\Sigma}} \partial_t + \frac{1}{\sqrt{\Sigma} \sin \vartheta} \partial_\varphi. \quad (2.24.3)$$

Dual tetrad:

$$\theta^{(0)} = \sqrt{\frac{\Delta}{\Sigma}} (dt + 2\ell \cos \vartheta d\varphi), \quad \theta^{(1)} = \sqrt{\frac{\Sigma}{\Delta}} dr, \quad \theta^{(2)} = \sqrt{\Sigma} d\vartheta, \quad \theta^{(3)} = \sqrt{\Sigma} \sin \vartheta d\varphi. \quad (2.24.4)$$

Euler-Lagrange:

The Euler-Lagrangian formalism, Sec. 1.8.4, for geodesics in the $\vartheta = \pi/2$ hyperplane yields

$$\frac{1}{2} \dot{r}^2 + V_{\text{eff}} = \frac{1}{2} \frac{k^2}{c^2}, \quad V_{\text{eff}} = \frac{1}{2} \frac{\Delta}{\Sigma} \left(\frac{h^2}{\Sigma} - \kappa \right) \quad (2.24.5)$$

with the constants of motion $k = (\Delta/\Sigma)t$ and $h = \Sigma\dot{\varphi}$. For null geodesics, we obtain a photon orbit at $r = r_{\text{po}}$ with

$$r_{\text{po}} = M + 2\sqrt{M^2 + \ell^2} \cos \left(\frac{1}{3} \arccos \frac{M}{\sqrt{M^2 + \ell^2}} \right) \quad (2.24.6)$$

Further reading:

Bini et al.[BCdMJ03].

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