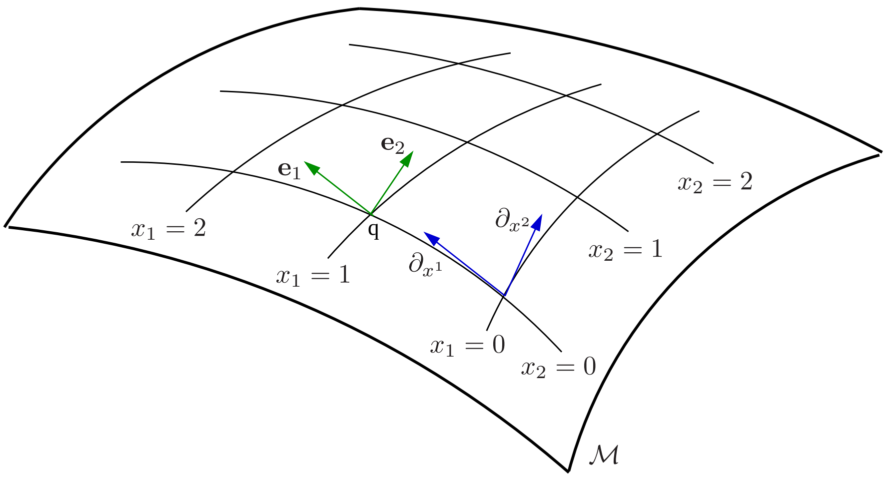
Catalogue of Spacetimes

Thomas Müller and Frank Grave



Contact: Visualisierungsinstitut der Universität Stuttgart
Nobelstrasse 15
70569 Stuttgart, Germany
Thomas.Mueller@vis.uni-stuttgart.de

1. Institut für Theoretische Physik
Pfaffenwaldring 57 // IV
70550 Stuttgart, Germany
Frank.Grave@vis.uni-stuttgart.de

URL: http://www.vis.uni-stuttgart.de/~muelleta/CoS

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Co-authors

Andreas Lemmer, Institut für Theoretische Physik, Universität Stuttgart
Alcubierre Warp

Sebastian Boblest, Institut für Theoretische Physik, Universität Stuttgart
de-Sitter, Friedmann-Robertson-Walker

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Chapter 1

Introduction and Notation

The *Catalogue of Spacetimes* is a collection of four-dimensional Lorentzian spacetimes in the context of the General Theory of Relativity (GR). The aim of the catalogue is to give a quick reference for students who need some basic facts of the most well-known spacetimes in GR. For a detailed discussion of a metric, the reader is referred to the standard literature or the original articles. An important resource for exact solutions is the book by Stephani et al[SKM+03].
Some of the metrics in this catalogue are implemented in the Motion4D-library[MG09] and can be visualized using the GeodesicViewer[MG].

1.1 General remarks

The Einstein field equation in the most general form reads[MTW73]

$$G_{\mu\nu} = \varkappa T_{\mu\nu} - \Lambda g_{\mu\nu}, \qquad \varkappa = \frac{8\pi G}{c^4}, \tag{1.1.1}$$

with the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R, the metric tensor $g_{\mu\nu}$, the energy-momentum tensor $T_{\mu\nu}$, the cosmological constant Λ, Newton's gravitational constant G, and the speed of light c.

A solution to the field equation is given by the line element

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{1.1.2}$$

with the symmetric, covariant metric tensor $g_{\mu\nu}$. The contravariant metric tensor $g^{\mu\nu}$ is related to the covariant tensor via $g_{\mu\nu}g^{\nu\lambda} = \delta_\mu^\lambda$ with the Kronecker-δ. Even though $g_{\mu\nu}$ is only a component of the metric tensor $\mathbf{g} = g_{\mu\nu}dx^\mu \otimes dx^\nu$, we will call $g_{\mu\nu}$ as the metric tensor.

Note that, in this catalogue, we use the convention that the signature of the metric is $+2$. In general, we will also keep the physical constants c and G within the metrics.

1.2 Basic objects of a metric

The basic objects of a metric are the Christoffel symbols, the Riemann and Ricci tensors as well as the Ricci and Kretschman scalars which are defined as follows:

Christoffel symbols of the first kind:[1]

$$\Gamma_{\nu\lambda\mu} = \frac{1}{2}\left(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}\right) \tag{1.2.1}$$

with the relation

$$g_{\nu\lambda,\mu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\mu\lambda\nu} \tag{1.2.2}$$

[1]The notation of the Christoffel symbols of the first kind differs from the notation used by Rindler[Rin01].

Christoffel symbols of the second kind:

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho} \right) \tag{1.2.3}$$

which are related to the Christoffel symbols of the first kind via

$$\Gamma^{\mu}_{\nu\lambda} = g^{\mu\rho} \Gamma_{\nu\lambda\rho} \tag{1.2.4}$$

Riemann tensor:

$$R^{\mu}{}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} + \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\rho} \tag{1.2.5}$$

or

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^{\lambda}{}_{\nu\rho\sigma} = \Gamma_{\nu\sigma\mu,\rho} - \Gamma_{\nu\rho\mu,\sigma} + \Gamma^{\lambda}_{\nu\rho}\Gamma_{\mu\sigma\lambda} - \Gamma^{\lambda}_{\nu\sigma}\Gamma_{\mu\sigma\lambda} \tag{1.2.6}$$

with symmetries

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}, \qquad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}, \qquad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \tag{1.2.7}$$

and

$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 \tag{1.2.8}$$

Ricci tensor:

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} \tag{1.2.9}$$

Ricci and Kretschman scalar:

$$\mathscr{R} = R^{\mu}{}_{\mu}, \qquad \mathscr{K} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = R^{\gamma\delta}{}_{\alpha\beta} R^{\alpha\beta}{}_{\gamma\delta} \tag{1.2.10}$$

Weyl tensor:

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2} \left(g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu} \right) + \frac{1}{3} R g_{\mu[\rho} g_{\sigma]\nu} \tag{1.2.11}$$

Symmetrization and Antisymmetrization brackets:

$$a_{(\mu} b_{\nu)} = \frac{1}{2} \left(a_\mu b_\nu + a_\nu b_\mu \right), \qquad a_{[\mu} b_{\nu]} = \frac{1}{2} \left(a_\mu b_\nu - a_\nu b_\mu \right) \tag{1.2.12}$$

Covariant derivative

$$\nabla_\lambda g_{\mu\nu} = g_{\mu\nu;\lambda} = 0. \tag{1.2.13}$$

Covariant derivative of the vector field ψ^μ:

$$\nabla_\nu \psi^\mu = \psi^{\mu}_{;\nu} = \partial_\nu \psi^\mu + \Gamma^{\mu}_{\nu\lambda} \psi^\lambda \tag{1.2.14}$$

Covariant derivative of a r-s-tensor field:

$$\begin{aligned} \nabla_c T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} = \partial_c T^{a_1 \ldots a_r}{}_{b_1 \ldots b_s} + \Gamma^{a_1}_{dc} T^{d \ldots a_r}{}_{b_1 \ldots b_s} + \ldots + \Gamma^{a_r}_{dc} T^{a_1 \ldots a_{r-1} d}{}_{b_1 \ldots b_s} \\ - \Gamma^{d}_{b_1 c} T^{a_1 \ldots a_r}{}_{d \ldots b_s} - \ldots - \Gamma^{d}_{b_s c} T^{a_1 \ldots a_r}{}_{b_1 \ldots b_{s-1} d} \end{aligned} \tag{1.2.15}$$

1.3 Natural local tetrad and initial conditions for geodesics

We will call a local tetrad natural if it is adapted to the symmetries or the coordinates of the spacetime. The four base vectors $\mathbf{e}_{(i)} = e^{\mu}_{(i)} \partial_\mu$ are given with respect to coordinate directions $\partial/\partial x^\mu = \partial_\mu$, compare Nakahara[Nak90] or Chandrasekhar[Cha06] for an introduction to the tetrad formalism. The inverse tetrad is given by $\theta^{(i)} = \theta^{(i)}_{\mu} dx^\mu$ with

$$\theta^{(i)}_{\mu} e^{\mu}_{(j)} = \delta^{(i)}_{(j)} \qquad \text{and} \qquad \theta^{(i)}_{\mu} e^{\nu}_{(i)} = \delta^{\nu}_{\mu}. \tag{1.3.1}$$

Note that we us Latin indices in brackets for tetrads and Greek indices for coordinates.

1.3.1 Orthonormality condition

To be applicable as a local reference frame (Minkowski frame), a local tetrad $\mathbf{e}_{(i)}$ has to fulfill the orthonormality condition

$$\left\langle \mathbf{e}_{(i)}, \mathbf{e}_{(j)} \right\rangle_{\mathbf{g}} = \mathbf{g}\left(\mathbf{e}_{(i)}, \mathbf{e}_{(j)}\right) = g_{\mu\nu} e^{\mu}_{(i)} e^{\nu}_{(j)} = \eta_{(i)(j)}, \tag{1.3.2}$$

where $\eta_{(i)(j)} = \mathrm{diag}(-1,1,1,1)$. Thus, the line element of a metric can be written as

$$ds^2 = \eta_{(i)(j)} \theta^{(i)} \theta^{(j)} = \eta_{(i)(j)} \theta^{(i)}_{\mu} \theta^{(j)}_{\nu} dx^{\mu} dx^{\nu}. \tag{1.3.3}$$

To obtain a local tetrad $\mathbf{e}_{(i)}$, we could first determine the dual tetrad $\theta^{(i)}$ via Eq. (1.3.3). If we combine all four dual tetrad vectors into one matrix Θ, we only have to determine its inverse Θ^{-1} to find the tetrad vectors,

$$\Theta = \begin{pmatrix} \theta^{(0)}_0 & \theta^{(0)}_1 & \theta^{(0)}_2 & \theta^{(0)}_3 \\ \theta^{(1)}_0 & \theta^{(1)}_1 & \theta^{(1)}_2 & \theta^{(1)}_3 \\ \theta^{(2)}_0 & \theta^{(2)}_1 & \theta^{(2)}_2 & \theta^{(2)}_3 \\ \theta^{(3)}_0 & \theta^{(3)}_1 & \theta^{(3)}_2 & \theta^{(3)}_3 \end{pmatrix} \quad \Rightarrow \quad \Theta^{-1} = \begin{pmatrix} e^0_{(0)} & e^0_{(1)} & e^0_{(2)} & e^0_{(3)} \\ e^1_{(0)} & e^1_{(1)} & e^1_{(2)} & e^1_{(3)} \\ e^2_{(0)} & e^2_{(1)} & e^2_{(2)} & e^2_{(3)} \\ e^3_{(0)} & e^3_{(1)} & e^3_{(2)} & e^3_{(3)} \end{pmatrix}. \tag{1.3.4}$$

There are also several useful relations:

$$e_{(a)\mu} = g_{\mu\nu} e^{\nu}_{(a)}, \qquad \eta_{(a)(b)} = e^{\mu}_{(a)} e_{(b)\mu}, \qquad e_{(b)\mu} = \eta_{(a)(b)} \theta^{(a)}_{\mu}, \tag{1.3.5a}$$

$$\theta^{(b)}_{\mu} = \eta^{(a)(b)} e_{(a)\mu}, \qquad g_{\mu\nu} = e_{(a)\mu} \theta^{(a)}_{\nu}, \qquad \eta^{(a)(b)} = \theta^{(a)}_{\mu} \theta^{(b)}_{\nu} g^{\mu\nu}. \tag{1.3.5b}$$

1.3.2 Ricci rotation-, connection-, and structure coefficients

The Ricci rotation coefficients $\gamma_{(i)(j)(k)}$ with respect to the local tetrad $\mathbf{e}_{(i)}$ are defined by

$$\gamma_{(i)(j)(k)} := g_{\mu\lambda} e^{\mu}_{(i)} \nabla_{\mathbf{e}_{(k)}} e^{\lambda}_{(j)} = g_{\mu\lambda} e^{\mu}_{(i)} e^{\nu}_{(k)} \nabla_{\nu} e^{\lambda}_{(j)} = g_{\mu\lambda} e^{\mu}_{(i)} e^{\nu}_{(k)} \left(\partial_{\nu} e^{\lambda}_{(j)} + \Gamma^{\lambda}_{\nu\beta} e^{\beta}_{(j)} \right). \tag{1.3.6}$$

They are antisymmetric in the first two indices, $\gamma_{(i)(j)(k)} = -\gamma_{(j)(i)(k)}$, which follows from the definition, Eq. (1.3.6), and the relation

$$0 = \partial_{\mu} \eta_{(i)(j)} = \nabla_{\mu} \left(g_{\beta\nu} e^{\beta}_{(i)} e^{\nu}_{(j)} \right), \tag{1.3.7}$$

where $\nabla_{\mu} g_{\beta\nu} = 0$, compare [Cha06]. Otherwise, we have

$$\gamma^{(i)}{}_{(j)(k)} = \theta^{(i)}_{\lambda} e^{\nu}_{(k)} \nabla_{\nu} e^{\lambda}_{(j)} = -e^{\lambda}_{(j)} e^{\nu}_{(k)} \nabla_{\nu} \theta^{(i)}_{\lambda}. \tag{1.3.8}$$

The contraction of the first and the last index is given by

$$\gamma_{(j)} = \gamma^{(k)}{}_{(j)(k)} = \eta^{(k)(i)} \gamma_{(i)(j)(k)} = -\gamma_{(0)(j)(0)} + \gamma_{(1)(j)(1)} + \gamma_{(2)(j)(2)} + \gamma_{(3)(j)(3)} = \nabla_{\nu} e^{\nu}_{(j)}. \tag{1.3.9}$$

The connection coefficients $\omega^{(m)}_{(j)(n)}$ with respect to the local tetrad $\mathbf{e}_{(i)}$ are defined by

$$\omega^{(m)}_{(j)(n)} := \theta^{(m)}_{\mu} \nabla_{\mathbf{e}_{(j)}} e^{\mu}_{(n)} = \theta^{(m)}_{\mu} e^{\alpha}_{(j)} \nabla_{\alpha} e^{\mu}_{(n)} = \theta^{(m)}_{\mu} e^{\alpha}_{(j)} \left(\partial_{\alpha} e^{\mu}_{(n)} + \Gamma^{\mu}_{\alpha\beta} e^{\beta}_{(n)} \right), \tag{1.3.10}$$

compare Nakahara[Nak90]. They are related to the Ricci rotation coefficients via

$$\gamma_{(i)(j)(k)} = \eta_{(i)(m)} \omega^{(m)}_{(k)(j)}. \tag{1.3.11}$$

Furthermore, the local tetrad has a non-vanishing Lie-bracket $[X,Y]^{\nu} = X^{\mu} \partial_{\mu} Y^{\nu} - Y^{\mu} \partial_{\mu} X^{\nu}$. Thus,

$$\left[\mathbf{e}_{(i)}, \mathbf{e}_{(j)}\right] = c^{(k)}_{(i)(j)} \mathbf{e}_{(k)} \qquad \text{or} \qquad c^{(k)}_{(i)(j)} = \theta^{(k)} \left[\mathbf{e}_{(i)}, \mathbf{e}_{(j)}\right]. \tag{1.3.12}$$

These structure coefficients are related to the connection coefficients or the Ricci rotation coefficients via

$$c^{(k)}_{(i)(j)} = \omega^{(k)}_{(i)(j)} - \omega^{(k)}_{(j)(i)} = \eta^{(k)(m)} \left(\gamma_{(m)(j)(i)} - \gamma_{(m)(i)(j)} \right) = \gamma^{(k)}{}_{(j)(i)} - \gamma^{(k)}{}_{(i)(j)}. \tag{1.3.13}$$

1.3.3 Riemann-, Ricci-, and Weyl-tensor with respect to a local tetrad

The transformation between the coordinate representations of the Riemann-, Ricci-, and Weyl-tensors and their representation with respect to a local tetrad $\mathbf{e}_{(i)}$ is given by

$$R_{(a)(b)(c)(d)} = R_{\mu\nu\rho\sigma}e^{\mu}_{(a)}e^{\nu}_{(b)}e^{\rho}_{(c)}e^{\sigma}_{(d)}, \tag{1.3.14a}$$

$$R_{(a)(b)} = R_{\mu\nu}e^{\mu}_{(a)}e^{\nu}_{(b)}, \tag{1.3.14b}$$

$$C_{(a)(b)(c)(d)} = C_{\mu\nu\rho\sigma}e^{\mu}_{(a)}e^{\nu}_{(b)}e^{\rho}_{(c)}e^{\sigma}_{(d)} \tag{1.3.14c}$$

$$= R_{(a)(b)(c)(d)} - \frac{1}{2}\left(\eta_{(a)[(c)}R_{(d)](b)} - \eta_{(b)[(c)}R_{(d)](a)}\right) + \frac{R}{3}\eta_{(a)[(c)}\eta_{(d)](b)}. \tag{1.3.14d}$$

1.3.4 Null or timelike directions

A null or timelike direction $\upsilon = \upsilon^{(i)}\mathbf{e}_{(i)}$ with respect to a local tetrad $\mathbf{e}_{(i)}$ can be written as

$$\upsilon = \upsilon^{(0)}\mathbf{e}_{(0)} + \psi\left(\sin\chi\cos\xi\,\mathbf{e}_{(1)} + \sin\chi\sin\xi\,\mathbf{e}_{(2)} + \cos\chi\,\mathbf{e}_{(3)}\right) = \upsilon^{(0)}\mathbf{e}_{(0)} + \mathbf{n}. \tag{1.3.15}$$

In the case of a null direction we have $\psi = 1$ and $\upsilon^{(0)} = \pm 1$. A timelike direction can be identified with an initial four-velocity $\mathbf{u} = c\gamma(\mathbf{e}_0 + \beta\mathbf{n})$, where

$$\mathbf{u}^2 = \langle\mathbf{u},\mathbf{u}\rangle_{\mathbf{g}} = c^2\gamma^2\left\langle\mathbf{e}_{(0)} + \beta\mathbf{n}, \mathbf{e}_{(0)} + \beta\mathbf{n}\right\rangle = c^2\gamma^2\left(-1+\beta^2\right) = -c^2. \tag{1.3.16}$$

Thus, $\psi = c\beta\gamma$ and $\upsilon^0 = \pm c\gamma$. The sign of $\upsilon^{(0)}$ determines the time direction.

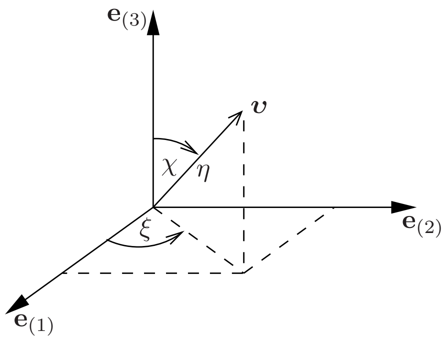

Figure 1.1: Null or timelike direction υ with respect to the local tetrad $\mathbf{e}_{(i)}$.

The transformations between a local direction $\upsilon^{(i)}$ and its coordinate representation υ^{μ} read

$$\upsilon^{\mu} = \upsilon^{(i)}e^{\mu}_{(i)} \qquad \text{and} \qquad \upsilon^{(i)} = \theta^{(i)}_{\mu}\upsilon^{\mu}. \tag{1.3.17}$$

1.3.5 Local tetrad for stationary axisymmetric spacetimes

The line element of a stationary axisymmetric spacetime is given by

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}\,dt\,d\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\vartheta\vartheta}d\vartheta^2, \tag{1.3.18}$$

where the metric components are functions of r and ϑ only.
The local tetrad for an observer on a stationary circular orbit, ($r = \text{const}, \vartheta = \text{const}$), with four velocity $\mathbf{u} = c\Gamma\left(\partial_t + \zeta\partial_\varphi\right)$ can be defined as, compare Bini[BJ00],

$$\mathbf{e}_{(0)} = \Gamma\left(\partial_t + \zeta\partial_\varphi\right), \qquad \mathbf{e}_{(1)} = \frac{1}{\sqrt{g_{rr}}}\partial_r, \qquad \mathbf{e}_{(2)} = \frac{1}{\sqrt{g_{\vartheta\vartheta}}}\partial_\vartheta, \tag{1.3.19a}$$

$$\mathbf{e}_{(3)} = \Delta\Gamma\left(\pm(g_{t\varphi} + \zeta g_{\varphi\varphi})\partial_t \mp (g_{tt} + \zeta g_{t\varphi})\partial_\varphi\right), \tag{1.3.19b}$$

where

$$\Gamma = \frac{1}{\sqrt{-\left(g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi}\right)}}, \qquad \Delta = \frac{1}{\sqrt{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}}}. \tag{1.3.20}$$

The angular velocity ζ is limited due to $g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi} < 0$

$$\zeta_{\text{min}} = \omega - \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \qquad \text{and} \qquad \zeta_{\text{max}} = \omega + \sqrt{\omega^2 - \frac{g_{tt}}{g_{\varphi\varphi}}} \tag{1.3.21}$$

with $\omega = -g_{t\varphi}/g_{\varphi\varphi}$.
For $\zeta = 0$, the observer is static with respect to spatial infinity. The locally nonrotating frame (LNRF) has angular velocity $\zeta = \omega$, see also MTW[MTW73], exercise 33.3.
Static limit: $\zeta_{\text{min}} = 0 \quad \Rightarrow \quad g_{tt} = 0$.
The transformation between the local direction $\upsilon^{(i)}$ and the coordinate direction υ^μ reads

$$\upsilon^0 = \Gamma\left(\upsilon^{(0)} \pm \upsilon^{(3)}\Delta w_1\right), \qquad \upsilon^1 = \frac{\upsilon^{(1)}}{\sqrt{g_{rr}}}, \qquad \upsilon^2 = \frac{\upsilon^{(2)}}{\sqrt{g_{\vartheta\vartheta}}}, \qquad \upsilon^3 = \Gamma\left(\upsilon^{(0)}\zeta \mp \upsilon^{(3)}\Delta w_2\right), \tag{1.3.22}$$

with

$$w_1 = g_{t\varphi} + \zeta g_{\varphi\varphi} \qquad \text{and} \qquad w_2 = g_{tt} + \zeta g_{t\varphi}. \tag{1.3.23}$$

The back transformation reads

$$\upsilon^{(0)} = \frac{1}{\Gamma}\frac{\upsilon^0 w_2 + \upsilon^3 w_1}{\zeta w_1 + w_2}, \qquad \upsilon^{(1)} = \sqrt{g_{rr}}\,\upsilon^1, \quad \upsilon^{(2)} = \sqrt{g_{\vartheta\vartheta}}\,\upsilon^2, \qquad \upsilon^{(3)} = \pm\frac{1}{\Delta\Gamma}\frac{\zeta\upsilon^0 - \upsilon^3}{\zeta w_1 + w_2}. \tag{1.3.24}$$

Note, to obtain a right-handed local tetrad, $\det\left(e^\mu_{(i)}\right) > 0$, the upper sign has to be used.

1.4 Coordinate relations

1.4.1 Spherical and cartesian coordinates

The well-known relation between the spherical coordinates (r, ϑ, φ) and the cartesian coordinates (x, y, z), compare Fig. 1.2, are

$$x = r\sin\vartheta\cos\varphi, \qquad r = \sqrt{x^2 + y^2 + z^2}, \tag{1.4.1a}$$

$$y = r\sin\vartheta\sin\varphi, \qquad \text{and} \qquad \vartheta = \text{arctan}2(\sqrt{x^2 + y^2}, z), \tag{1.4.1b}$$

$$z = r\cos\vartheta, \qquad \varphi = \text{arctan}2(y, x), \tag{1.4.1c}$$

where arctan2() ensures that $\varphi \in [0, 2\pi)$ and $\vartheta \in (0, \pi)$.
The total differentials of the spherical coordinates read

$$dr = \frac{x\,dx + y\,dy + z\,dz}{r}, \tag{1.4.2a}$$

$$d\vartheta = \frac{xz\,dx + yz\,dy - (x^2 + y^2)dz}{r^2\sqrt{x^2 + y^2}}, \tag{1.4.2b}$$

$$d\varphi = \frac{-y\,dx + x\,dy}{x^2 + y^2}, \tag{1.4.2c}$$

whereas the coordinate derivatives read

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y + \frac{\partial z}{\partial r}\partial_z \quad = \sin\vartheta\cos\varphi\,\partial_x + \sin\vartheta\sin\varphi\,\partial_y + \cos\vartheta\,\partial_z, \tag{1.4.3a}$$

$$\partial_\vartheta = \frac{\partial x}{\partial\vartheta}\partial_x + \frac{\partial y}{\partial\vartheta}\partial_y + \frac{\partial z}{\partial\vartheta}\partial_z = r\cos\vartheta\cos\varphi\,\partial_x + r\cos\vartheta\sin\varphi\,\partial_y - r\sin\vartheta\,\partial_z, \tag{1.4.3b}$$

$$\partial_\varphi = \frac{\partial x}{\partial\varphi}\partial_x + \frac{\partial y}{\partial\varphi}\partial_y + \frac{\partial z}{\partial\varphi}\partial_z \quad = -r\sin\vartheta\sin\varphi\,\partial_x + r\sin\vartheta\cos\varphi\,\partial_y, \tag{1.4.3c}$$

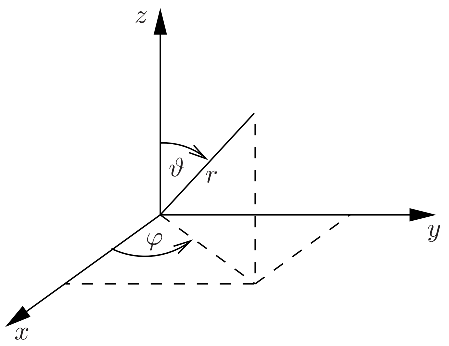

Figure 1.2: Relation between spherical and cartesian coordinates.

and

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial \vartheta}{\partial x}\partial_\vartheta + \frac{\partial \varphi}{\partial x}\partial_\varphi = \sin\vartheta\cos\varphi\,\partial_r + \frac{\cos\vartheta\cos\varphi}{r}\,\partial_\vartheta - \frac{\sin\varphi}{r\sin\vartheta}\partial_\varphi, \tag{1.4.4a}$$

$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial \vartheta}{\partial y}\partial_\vartheta + \frac{\partial \varphi}{\partial y}\partial_\varphi = \sin\vartheta\sin\varphi\,\partial_r + \frac{\cos\vartheta\sin\varphi}{r}\,\partial_\vartheta + \frac{\cos\varphi}{r\sin\vartheta}\partial_\varphi, \tag{1.4.4b}$$

$$\partial_z = \frac{\partial r}{\partial z}\partial_r + \frac{\partial \vartheta}{\partial z}\partial_\vartheta + \frac{\partial \varphi}{\partial z}\partial_\varphi = \cos\vartheta\,\partial_r - \frac{\sin\vartheta}{r}\,\partial_\vartheta. \tag{1.4.4c}$$

1.4.2 Cylindrical and cartesian coordinates

The relation between cylindrical coordinates (r,φ,z) and cartesian coordinates (x,y,z) is given by

$$x = r\cos\varphi, \qquad r = \sqrt{x^2+y^2}, \tag{1.4.5a}$$

$$y = r\sin\varphi, \qquad \varphi = \operatorname{arctan2}(y,x), \tag{1.4.5b}$$

where $\operatorname{arctan2}()$ again ensures that the angle $\varphi \in [0,2\pi)$.

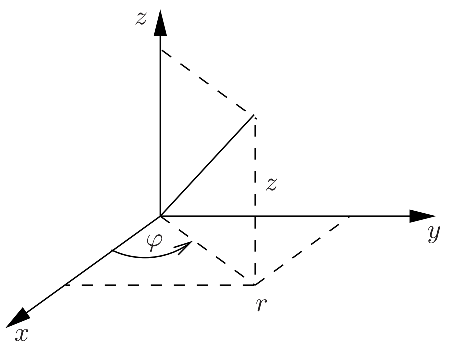

Figure 1.3: Relation between cylindrical and cartesian coordinates.

The total differentials of the spherical coordinates are given by

$$dr = \frac{x\,dx + y\,dy}{r}, \tag{1.4.6a}$$

$$d\varphi = \frac{-y\,dx + x\,dy}{r^2}, \tag{1.4.6b}$$

and

$$dx = \cos\varphi\, dr - r\sin\varphi\, d\varphi, \quad (1.4.7a)$$
$$dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi. \quad (1.4.7b)$$

The coordinate derivatives are

$$\partial_r = \frac{\partial x}{\partial r}\partial_x + \frac{\partial y}{\partial r}\partial_y \quad = \cos\varphi\,\partial_x + \sin\varphi\,\partial_y, \quad (1.4.8a)$$
$$\partial_\varphi = \frac{\partial x}{\partial \varphi}\partial_x + \frac{\partial y}{\partial \varphi}\partial_y = -r\sin\varphi\,\partial_x + r\cos\varphi\,\partial_y m \quad (1.4.8b)$$

and

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial \varphi}{\partial x}\partial_\varphi = \cos\varphi\,\partial_r - \frac{\sin\varphi}{r}\partial_y, \quad (1.4.9a)$$
$$\partial_y = \frac{\partial r}{\partial y}\partial_r + \frac{\partial \varphi}{\partial y}\partial_\varphi = \sin\varphi\,\partial_r + \frac{\cos\varphi}{r}\partial_y. \quad (1.4.9b)$$

1.5 Embedding diagram

A two-dimensional hypersurface with line segment

$$d\sigma^2 = g_{rr}(r)dr^2 + g_{\varphi\varphi}(r)d\varphi^2 \quad (1.5.1)$$

can be embedded in a three-dimensional Euclidean space with cylindrical coordinates,

$$d\sigma^2 = \left[1 + \left(\frac{dz}{d\rho}\right)^2\right] d\rho^2 + \rho^2 d\varphi^2. \quad (1.5.2)$$

With $\rho(r)^2 = g_{\varphi\varphi}(r)$ and $dr = (dr/d\rho)d\rho$, we obtain for the embedding function $z = z(r)$,

$$\frac{dz}{dr} = \pm\sqrt{g_{rr} - \left(\frac{d\sqrt{g_{\varphi\varphi}}}{dr}\right)^2}. \quad (1.5.3)$$

If $g_{\varphi\varphi}(r) = r^2$, then $d\sqrt{g_{\varphi\varphi}}/dr = 1$.

1.6 Equations of motion

1.6.1 Geodesic equation

The geodesic equation reads

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \quad (1.6.1)$$

with the affine parameter λ. For timelike geodesics, we replace the affine parameter by the proper time τ.

The geodesic equation (1.6.1) is a system of ordinary differential equations of second order. Hence, to solve these differential equations, we need an initial position $x^\mu(\lambda = 0)$ as well as an initial direction $(dx^\mu/d\lambda)(\lambda = 0)$. This initial direction has to fulfill the constraint equation

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \kappa c^2, \quad (1.6.2)$$

where $\kappa = 0$ for lightlike and $\kappa = -1$ for timelike geodesics.

1.6.2 Fermi-Walker transport

The Fermi-Walker transport of a vector $\mathbf{X} = X^\mu \partial_\mu$ along the worldline $x^\mu(\tau)$ with four-velocity $u^\mu(\tau)$ is given by[SS90]

$$\frac{dX^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} u^\rho X^\sigma + \frac{1}{c^2}\left(u^\sigma a^\mu - a^\sigma u^\mu\right) g_{\rho\sigma} X^\rho = 0. \tag{1.6.3}$$

The four-acceleration follows from the four-velocity via

$$a^\mu = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma. \tag{1.6.4}$$

1.6.3 Parallel transport

If the four-acceleration vanishes, the Fermi-Walker transport simplifies to the parallel transport

$$\frac{dX^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma} u^\rho X^\sigma = 0. \tag{1.6.5}$$

1.7 Tools

1.7.1 Maple/GRTensorII

The Christoffel symbols, the Riemann- and Ricci-tensors as well as the Ricci and Kretschman scalars in this catalogue were determined by means of the software Maple together with the package by Musgrave, Pollney, and Lake.[2]

A typical worksheet to enter a new metric may look like this:

```
> grtw();
> makeg(Schwarzschild);

Makeg 2.0: GRTensor metric/basis entry utility
To quit makeg, type 'exit' at any prompt.
Do you wish to enter a 1) metric [g(dn,dn)],
                       2) line element [ds],
                       3) non-holonomic basis [e(1)...e(n)], or
                       4) NP tetrad [l,n,m,mbar]?
> 2:

Enter coordinates as a LIST (eg. [t,r,theta,phi]):
> [t,r,theta,phi]:

Enter the line element using d[coord] to indicate differentials.
(for example,  r^2*(d[theta]^2 + sin(theta)^2*d[phi]^2)
[Type 'exit' to quit makeg]
ds^2 =

If there are any complex valued coordinates, constants or functions
for this spacetime, please enter them as a SET ( eg. { z, psi } ).

Complex quantities [default={}]:
> {}:
```

[2] The commercial software Maple can be found here: http://www.maplesoft.com. The GRTensorII-package is free: http://grtensor.phy.queensu.ca.

```
You may choose to 0) Use the metric WITHOUT saving it,
                  1) Save the metric as it is,
                  2) Correct an element of the metric,
                  3) Re-enter the metric,
                  4) Add/change constraint equations,
                  5) Add a text description, or
                  6) Abandon this metric and return to Maple.
> 0:
```

The worksheets for some of the metrics in this catalogue can be found on the authors homepage.

To determine the objects that are defined with respect to a local tetrad, the metric must be given as non-holonomic basis.

1.7.2 Mathematica

The calculation of the Christoffel-symbols, the Riemann- or Ricci-tensor within *Mathematica* could read like this:

```
Clearing the values of symbols:
    In[1]:=  Clear[coord, metric, inversemetric, affine,
                    t, r, Theta, Phi]

Setting the dimension:
    In[2]:=  n := 4

Defining a list of coordinates:
    In[3]:=  coord := {t, r, Theta, Phi}

Defining the metric:
    In[4]:=  metric := {{-(1 - rs/r) c^2, 0, 0, 0},
                        {0, 1/(1 - rs/r), 0, 0},
                        {0, 0, r^2, 0},
                        {0, 0, 0, r ^2 Sin[Theta]^2}}
    In[5]:=  metric // MatrixForm

Calculating the inverse metric:
    In[6]:=  inversemetric := Simplify[Inverse[metric]]

    In[7]:=  inversemetric // MatrixForm

Calculating the Christoffel symbols of the second kind:
    In[8]:=  affine :=  affine = Simplify[
      Table[(1/2) Sum[inversemetric[[Mu, Rho]] (
         D[metric[[Rho, Nu]], coord[[Lambda]]] +
         D[metric[[Rho, Lambda]], coord[[Nu]]] -
         D[metric[[Nu, Lambda]], coord[[Mu]]]),
      {Rho, 1, n}], {Nu, 1, n}, {Lambda, 1, n}, {Mu, 1, n}]]

Displaying the Christoffel symbols of the second kind:
    In[9]:=  listaffine :=
      Table[If[UnsameQ[affine[[Nu, Lambda, Mu]], 0],
        {Style[ Subsuperscript[\[CapitalGamma],
           Row[{coord[[Nu]], coord[[Lambda]]}], coord[[Mu]]], 18],
           "=",
           Style[affine[[Nu, Lambda, Mu]], 14]}],
        {Lambda, 1, n}, {Nu, 1, Lambda}, {Mu, 1, n}]
```

```
   In[10]:=  TableForm[Partition[DeleteCases[Flatten[listaffine],
                                                Null], 3],
                    TableSpacing -> {1, 2}]
```

Defining the Riemann tensor:

```
   In[11]:=  riemann :=  riemann =
    Table[D[affine[[Nu, Sigma, Mu]], coord[[Rho]]] -
          D[affine[[Nu, Rho, Mu]], coord[[Sigma]]] +
          Sum[affine[[Rho, Lambda, Mu]]
              affine[[Nu, Sigma, Lambda]] -
              affine[[Sigma, Lambda, Mu]]
              affine[[Nu, Rho, Lambda]],
            {Lambda, 1, n}],
    {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]
```

Defining the Riemann tensor with lower indices:

```
   In[12]:= riemannDn :=  riemannDn =
     Table[Simplify[
        Sum[metric[[Mu, Kappa]] riemann[[Kappa, Nu, Rho, Sigma]],
        {Kappa, 1, n}]],
     {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]

  In[13]:= listRiemann :=
     Table[If[UnsameQ[riemannDn[[Mu, Nu, Rho, Sigma]], 0],
    {Style[Subscript[R, Row[{coord[[Mu]], coord[[Nu]], coord[[Rho]],
      coord[[Sigma]]}]], 16], "=",
      riemannDn[[Mu, Nu, Rho, Sigma]]}],
    {Nu, 1, n}, {Mu, 1, Nu}, {Sigma, 1, n}, {Rho, 1, Sigma}]

  In[14]:= TableForm[Partition[DeleteCases[Flatten[listRiemann],
                                              Null], 3],
                  TableSpacing -> {2, 2}]
```

Defining the Ricci tensor:

```
  In[15]:= ricci := ricci =
    Table[Simplify[
     Sum[riemann[[Rho, Mu, Rho, Nu]], {Rho, 1, n}]],
    {Mu, 1, n}, {Nu, 1, n}]

  In[16]:= listRicci :=
   Table[If[UnsameQ[ricci[[Mu, Nu]], 0],
      {Style[Subscript[R, Row[{coord[[Mu]], coord[[Nu]]}]], 16],
      "=",
     Style[ricci[[Mu, Nu]], 16]}], {Nu, 1, 4}, {Mu, 1, Nu}]

  In[17]:= TableForm[Partition[DeleteCases[Flatten[listRicci],
                                              Null], 3],
                  TableSpacing -> {1, 2}]
```

Defining the Ricci scalar:

```
  In[18]:= ricciscalar :=  ricciscalar =
    Simplify[Sum[
      Sum[inversemetric[[Mu, Nu]] ricci[[Nu, Mu]],
      {Mu, 1, n}], {Nu, 1, n}]]
```

```
Defining the Kretschman scalar:
  In[19]:= riemannUp :=  riemannUp =
    Table[Simplify[
     Sum[inversemetric[[Nu, Kappa]]
           riemann[[Mu, Kappa, Rho, Sigma]], {Kappa, 1, n}]],
    {Mu, 1, n}, {Nu, 1, n}, {Rho, 1, n}, {Sigma, 1, n}]

  In[20]:= kretschman := kretschman =
     Simplify[Sum[ Sum[Sum[Sum[
        riemannUp[[Mu, Nu, Rho, Sigma]]
        riemannUp[[Rho, Sigma, Mu, Nu]],
       {Mu, 1, n}], {Nu, 1, n}], {Rho, 1, n}], {Sigma, 1, n}]]
```

Some example notebooks can be found on the authors homepage.

1.7.3 Maxima

Instead of using commercial software like *Maple* or *Mathematica,* Maxima also offers a tensor package that helps to calculate the Christoffel symbols etc. The above example for the Schwarzschild metric can be written as a maxima worksheet as follows:

```
/* load ctensor package */
load(ctensor);

/* define coordinates to use */
ct_coords:[t,r,theta,phi];

/* start with the identity metric */
lg:ident(4);
lg[1,1]:c^2*(1-rs/r);
lg[2,2]:-1/(1-rs/r);
lg[3,3]:-r^2;
lg[4,4]:-r^2*sin(theta)^2;
cmetric();

/* calculate the christoffel symbols of the second kind */
christof(mcs);

/* calculate the riemann tensor */
lriemann(mcs);

/* calculate the ricci tensor */
ricci(mcs);

/* calculate the ricci scalar */
scurvature();

/* calculate the Kretschman scalar */
uriemann(mcs);
rinvariant();
ratsimp(%);
```

As you may have noticed, the Schwarzschild metric must be given with negative signature.

Chapter 2

Spacetimes

2.1 Minkowski

2.1.1 Cartesian coordinates

The Minkowski metric in cartesian coordinates reads

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{2.1.1}$$

All Christoffel symbols as well as the Riemann- and Ricci-tensor vanish identically. The natural local tetrad is trivial,

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(x)} = \partial_x, \qquad \mathbf{e}_{(y)} = \partial_y, \qquad \mathbf{e}_{(z)} = \partial_z. \tag{2.1.2}$$

2.1.2 Cylindrical coordinates

The Minkowski metric in cylindrical coordinates

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\varphi^2 + dz^2 \tag{2.1.3}$$

has the natural local tetrad

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \partial_z. \tag{2.1.4}$$

Christoffel symbols:

$$\Gamma^r_{\varphi\varphi} = -r, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}. \tag{2.1.5}$$

Ricci rotation coefficients:

$$\gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r} \qquad \text{and} \qquad \gamma_{(r)} = \frac{1}{r}. \tag{2.1.6}$$

2.1.3 Spherical coordinates

In spherical coordinates, the Minkowski metric reads

$$ds^2 = -c^2dt^2 + dr^2 + r^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right). \tag{2.1.7}$$

Christoffel symbols:

$$\Gamma^r_{\vartheta\vartheta} = -r, \qquad \Gamma^r_{\varphi\varphi} = -r\sin^2\vartheta, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \tag{2.1.8a}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta. \tag{2.1.8b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.1.9}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.1.10}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2}{r}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.1.11}$$

2.1.4 Conform-compactified coordinates

The Minkowski metric in conform-compactified coordinates $(\psi, \xi, \vartheta, \varphi)$ reads[HE99]

$$ds^2 = -d\psi^2 + d\xi^2 + \sin^2\xi\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{2.1.12}$$

where $\psi \in [-\pi, \pi]$ and $\xi \in [0, \pi]$. This form follows from the spherical Minkowski metric (2.1.7) by means of the coordinate transformation

$$ct + r = \tan\frac{\psi+\xi}{2}, \qquad ct - r = \tan\frac{\psi-\xi}{2}, \tag{2.1.13}$$

resulting in the metric

$$d\tilde{s}^2 = \frac{-d\psi^2 + d\xi^2}{4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}} + \frac{\sin^2\xi}{4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}}\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{2.1.14}$$

and by the conformal transformation $ds^2 = \Omega^2 d\tilde{s}^2$ with $\Omega^2 = 4\cos^2\frac{\psi+\xi}{2}\cos^2\frac{\psi-\xi}{2}$.

Christoffel symbols:

$$\Gamma^{\vartheta}_{\xi\vartheta} = \cot\xi, \qquad \Gamma^{\varphi}_{\xi\varphi} = \cot\xi, \qquad \Gamma^{\xi}_{\vartheta\vartheta} = -\sin\xi\cos\xi, \tag{2.1.15a}$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^{\xi}_{\varphi\varphi} = -\sin\xi\cos\xi\sin^2\vartheta, \qquad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.1.15b}$$

Riemann-Tensor:

$$R_{\xi\vartheta\xi\vartheta} = \sin^2\xi, \qquad R_{\xi\varphi\xi\varphi} = \sin^2\xi\sin^2\vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = \sin^4\xi\sin^2\vartheta. \tag{2.1.16}$$

Ricci-Tensor:

$$R_{\xi\xi} = 2, \qquad R_{\vartheta\vartheta} = 2\sin^2\xi, \qquad R_{\varphi\varphi} = 2\sin^2\xi\sin^2\vartheta. \tag{2.1.17}$$

The *Ricci*- and *Kretschman*-scalars:

$$\mathscr{R} = 6, \qquad \mathscr{K} = 12. \tag{2.1.18}$$

The Weyl tensor vanishs identically.

Local tetrad:

$$\mathbf{e}_{(\psi)} = \partial_\psi, \qquad \mathbf{e}_{(\xi)} = \partial_\xi, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{\sin\xi}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\sin\xi\sin\vartheta}\partial_\varphi. \tag{2.1.19}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(\xi)(\vartheta)} = \gamma_{(\varphi)(\xi)(\varphi)} = \cot\xi, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{\sin\xi}. \tag{2.1.20}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\xi)} = 2\cot\xi, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{\sin\xi}. \tag{2.1.21}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\xi)(\vartheta)(\xi)(\vartheta)} = R_{(\xi)(\varphi)(\xi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = 1. \tag{2.1.22}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\xi)(\xi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = 2. \tag{2.1.23}$$

2.1.5 Rotating coordinates

The transformation $d\varphi \mapsto d\varphi + \omega\, dt$ brings the Minkowski metric (2.1.3) into the rotating form[Rin01] with coordinates (t,r,φ,z),

$$ds^2 = -\left(1-\frac{\omega^2 r^2}{c^2}\right)\left[c\,dt - \Omega(r)d\varphi\right]^2 + dr^2 + \frac{r^2}{1-\omega^2 r^2/c^2}d\varphi^2 + dz^2 \tag{2.1.24}$$

with $\Omega(r) = (r^2\omega/c)/(1-\omega^2 r^2/c^2)$.
The local tetrad of the comoving observer is

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t - \frac{\omega}{c}\partial_\varphi, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r}\partial_\varphi, \qquad \mathbf{e}_{(z)} = \partial_z, \tag{2.1.25}$$

whereas the static observer has the local tetrad

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1-\omega^2 r^2/c^2}}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(z)} = \partial_z, \tag{2.1.26a}$$

$$\mathbf{e}_{(\varphi)} = \frac{\omega r}{c^2\sqrt{1-\omega^2 r^2/c^2}}\partial_t + \frac{\sqrt{1-\omega^2 r^2/c^2}}{r}\partial_\varphi. \tag{2.1.26b}$$

Christoffel symbols:

$$\Gamma^r_{tt} = -\omega^2 r, \qquad \Gamma^\varphi_{tr} = \frac{\omega}{r}, \qquad \Gamma^r_{t\varphi} = -\omega r, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\varphi\varphi} = -r. \tag{2.1.27}$$

2.1.6 Rindler coordinates

The worldline of an observer in the Minkowski spacetime who moves with constant proper acceleration α along the x direction reads

$$x = \frac{c^2}{\alpha}\cosh\frac{\alpha t'}{c}, \qquad ct = \frac{c^2}{\alpha}\sinh\frac{\alpha t'}{c}, \tag{2.1.28}$$

where t' is the observer's proper time. The observer starts at $x = 1$ with zero velocity.
However, such an observer could also be described with Rindler coordinates. With the coordinate transformation

$$(ct,x) \mapsto (\tau,\rho): \qquad ct = \frac{1}{\rho}\sinh\tau, \qquad x = \frac{1}{\rho}\cosh\tau, \tag{2.1.29}$$

the Rindler metric reads

$$ds^2 = -\frac{1}{\rho^2}d\tau^2 + \frac{1}{\rho^4}d\rho^2 + dy^2 + dz^2. \tag{2.1.30}$$

Christoffel symbols:

$$\Gamma^{\rho}_{\tau\tau} = -\rho, \qquad \Gamma^{\tau}_{\tau\rho} = -\frac{1}{\rho}, \qquad \Gamma^{\rho}_{\rho\rho} = -\frac{2}{\rho}. \tag{2.1.31}$$

The Riemann and Ricci tensors as well as the Ricci and Kretschman scalar vanish identically.
Local tetrad:

$$\mathbf{e}_{(\tau)} = \rho\partial_\tau, \qquad \mathbf{e}_{(\rho)} = \rho^2\partial_\rho, \qquad \mathbf{e}_{(y)} = \partial_y, \qquad \mathbf{e}_{(z)} = \partial_z. \tag{2.1.32}$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(\rho)(\tau)} = \rho, \qquad \text{and} \qquad \gamma_{(\rho)} = -\rho. \tag{2.1.33}$$

2.2 Schwarzschild spacetime

2.2.1 Schwarzschild coordinates

The Schwarzschild metric represented by Schwarzschild coordinates $(t, r, \vartheta, \varphi)$ reads

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dt^2 + \frac{1}{1 - r_s/r} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \tag{2.2.1}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, and M is the mass of the black hole. The critical point $r = 0$ is a real curvature singularity while the event horizon, $r = r_s$, is only a coordinate singularity, compare e.g. the Kretschman scalar.

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2 r_s (r - r_s)}{2r^3}, \qquad \Gamma^t_{tr} = \frac{r_s}{2r(r - r_s)}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2r(r - r_s)}, \tag{2.2.2a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -(r - r_s), \tag{2.2.2b}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -(r - r_s)\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.2c}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2 r_s}{r^3}, \qquad R_{t\vartheta t\vartheta} = \frac{1}{2}\frac{c^2 (r - r_s) r_s}{r^2}, \qquad R_{t\varphi t\varphi} = \frac{1}{2}\frac{c^2 (r - r_s) r_s \sin^2\vartheta}{r^2}, \tag{2.2.3a}$$

$$R_{r\vartheta r\vartheta} = -\frac{1}{2}\frac{r_s}{r - r_s}, \qquad R_{r\varphi r\varphi} = -\frac{1}{2}\frac{r_s \sin^2\vartheta}{r - r_s}, \qquad R_{\vartheta\varphi\vartheta\varphi} = r r_s \sin^2\vartheta. \tag{2.2.3b}$$

As aspected, the Ricci tensor as well as the Ricci scalar vanish identically because the Schwarzschild spacetime is a vacuum solution of the field equations. Hence, the Weyl tensor is identical to the Riemann tensor. The Kretschman scalar reads

$$\mathscr{K} = 12\frac{r_s^2}{r^6}. \tag{2.2.4}$$

Here, it becomes clear that at $r = r_s$ there is no real singularity.

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{1 - r_s/r}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{1 - \frac{r_s}{r}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.2.5}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{1 - \frac{r_s}{r}}dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{1 - r_s/r}}, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.2.6}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2\sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}\sqrt{1 - \frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.2.7}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s}{2r^2\sqrt{1 - r_s/r}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.2.8}$$

Structure coefficients:

$$c^{(t)}_{(t)(r)} = \frac{r_s}{2r^2\sqrt{1 - r_s/r}}, \qquad c^{(\vartheta)}_{(r)(\vartheta)} = c^{(\varphi)}_{(r)(\varphi)} = -\frac{1}{r}\sqrt{1 - \frac{r_s}{r}}, \qquad c^{(\varphi)}_{(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.2.9}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.2.10a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.2.10b}$$

Embedding:
The embedding function reads

$$z = 2\sqrt{r_s}\sqrt{r-r_s}. \tag{2.2.11}$$

Effective potential:
The Euler-Lagrangian formalism[Rin01] yields the effective potential

$$V_{\text{eff}} = \frac{1}{2}\left(1-\frac{r_s}{r}\right)\left(\frac{h^2}{r^2}-\kappa c^2\right) \tag{2.2.12}$$

with the constants of motion $k = (1-r_s/r)c^2\dot{t}$, $h = r^2\dot{\varphi}$, and κ as in Eq. (1.6.2). For timelike geodesics, the effective potential has the extremal points

$$r_\pm = \frac{h^2 \pm h\sqrt{h^2-3c^2r_s}}{c^2 r_s}, \tag{2.2.13}$$

where r_+ is a maximum and r_- is a minimum. Null geodesics, however, have only a maximum at $r = \frac{3}{2}r_s$. The corresponding circular orbit is called photon orbit.
Further reading:
Schwarzschild[Sch16, Sch03]

2.2.2 Schwarzschild in pseudo-cartesian coordinates

The Schwarzschild spacetime in pseudo-cartesian coordinates

$$ds^2 = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \left(\frac{x^2}{1-r_s/r}+y^2+z^2\right)\frac{dx^2}{r^2} + \left(x^2+\frac{y^2}{1-r_s/r}+z^2\right)\frac{dy^2}{r^2} \tag{2.2.14a}$$

$$+\left(x^2+y^2+\frac{z^2}{1-r_s/r}\right)\frac{dz^2}{r^2} + \frac{2r_s}{r^2(r-r_s)}\left(xy\,dxdy + xz\,dxdz + yz\,dydz\right). \tag{2.2.14b}$$

ansatz for a local tetrad

$$\mathbf{e}_{(0)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_t, \qquad \mathbf{e}_{(1)} = A\partial_x, \qquad \mathbf{e}_{(2)} = B\partial_x + C\partial_y, \qquad \mathbf{e}_{(3)} = D\partial_x + E\partial_y + F\partial_z. \tag{2.2.15}$$

$$A = \frac{1}{\sqrt{g_{xx}}}, \qquad B = \frac{-g_{xy}}{g_{xx}\sqrt{-g_{xy}^2/g_{xx}+g_{yy}}}, \qquad C = \frac{1}{\sqrt{-g_{xy}^2/g_{xx}+g_{yy}}}, \tag{2.2.16a}$$

$$D = \frac{g_{xy}g_{yz}-g_{xz}g_{yy}}{\sqrt{NW}}, \qquad E = \frac{g_{xz}g_{xy}-g_{xx}g_{yz}}{\sqrt{NW}}, \qquad F = \frac{\sqrt{N}}{\sqrt{W}}. \tag{2.2.16b}$$

with

$$N = g_{xx}g_{yy} - g_{xy}^2, \tag{2.2.17a}$$

$$W = g_{xx}g_{yy}g_{zz} - g_{xz}^2 g_{yy} + 2g_{xz}g_{xy}g_{yz} - g_{xy}^2 g_{zz} - g_{xx}g_{yz}^2. \tag{2.2.17b}$$

2.2.3 Eddington-Finkelstein

The transformation of the Schwarzschild metric (2.2.1) from the usual Schwarzschild time coordinate t to the advanced null coordinate v with

$$cv = ct + r + r_s \ln(r - r_s) \tag{2.2.18}$$

leads to the ingoing Eddington-Finkelstein[Edd24, Fin58] metric with coordinates $(v, r, \vartheta, \varphi)$,

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) c^2 dv^2 + 2c\,dv\,dr + r^2 d\Omega^2. \tag{2.2.19}$$

Christoffel symbols:

$$\Gamma^v_{vv} = \frac{cr_s}{2r^2}, \qquad \Gamma^r_{vv} = \frac{c^2 r_s(r - r_s)}{2r^3}, \qquad \Gamma^r_{vr} = -\frac{cr_s}{2r^2}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \tag{2.2.20a}$$

$$\Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^v_{\vartheta\vartheta} = -\frac{r}{c}, \qquad \Gamma^r_{\vartheta\vartheta} = -(r - r_s), \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \tag{2.2.20b}$$

$$\Gamma^v_{\varphi\varphi} = -\frac{r\sin^2\vartheta}{c}, \quad \Gamma^r_{\varphi\varphi} = -(r - r_s)\sin^2\vartheta, \quad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.20c}$$

Riemann-Tensor:

$$R_{vrvr} = -\frac{c^2 r_s}{r^3}, \qquad R_{v\vartheta v\varphi} = \frac{c^2 r_s(r - r_s)}{2r^2}, \qquad R_{v\vartheta r\vartheta} = -\frac{cr_s}{2r}, \tag{2.2.21a}$$

$$R_{v\varphi v\varphi} = \frac{c^2 r_s(r - r_s)\sin^2\vartheta}{2r^2}, \qquad R_{v\varphi r\varphi} = -\frac{cr_s\sin^2\vartheta}{2r}, \qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s\sin^2\vartheta. \tag{2.2.21b}$$

While the Ricci tensor and the Ricci scalar vanish identically, the Kretschman scalar is $\mathscr{K} = 12r_s^2/r^6$.

Static Local tetrad:

$$\mathbf{e}_{(v)} = \frac{1}{c\sqrt{1 - r_s/r}}\partial_v, \quad \mathbf{e}_{(r)} = \frac{1}{c\sqrt{1 - r_s/r}}\partial_v + \sqrt{1 - \frac{r_s}{r}}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.2.22}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2\sqrt{1 - r_s/r}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}\sqrt{1 - \frac{r_s}{r}}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.2.23}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s}{2r^2\sqrt{1 - r_s/r}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.2.24}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.2.25a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.2.25b}$$

2.2.4 Isotropic coordinates

The Schwarzschild metric (2.2.1) in isotropic coordinates $(t, \rho, \vartheta, \varphi)$ reads, compare MTW[MTW73] page 840,

$$ds^2 = -\left(\frac{1 - r_s/(4\rho)}{1 + r_s/(4\rho)}\right)^2 c^2 dt^2 + \left(1 + \frac{r_s}{4\rho}\right)^4 \left[d\rho^2 + \rho^2\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right)\right], \tag{2.2.26}$$

where

$$r = \rho\left(1+\frac{r_s}{4\rho}\right)^2 \tag{2.2.27}$$

is the coordinate transformation between the Schwarzschild radial coordinate r and the isotropic radial coordinate ρ.

Christoffel symbols:

$$\Gamma^{\rho}_{tt} = 2048\frac{(4\rho-r_s)\rho^4 r_s c^2}{(4\rho+r_s)^7},\quad \Gamma^{t}_{t\rho} = \frac{8r_s}{16\rho^2-r_s^2},\quad \Gamma^{\rho}_{\rho\rho} = -\frac{2r_s}{(4\rho+r_s)\rho}, \tag{2.2.28a}$$

$$\Gamma^{\vartheta}_{\rho\vartheta} = \frac{4\rho-r_s}{(4\rho+r_s)\rho},\quad \Gamma^{\varphi}_{\rho\varphi} = \frac{4\rho-r_s}{(4\rho+r_s)\rho},\quad \Gamma^{\rho}_{\vartheta\vartheta} = -\rho\frac{4\rho-r_s}{4\rho+r_s}, \tag{2.2.28b}$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta,\quad \Gamma^{\rho}_{\varphi\varphi} = -\frac{(4\rho-r_s)\rho\sin^2\vartheta}{4\rho+r_s},\quad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.28c}$$

Riemann-Tensor:

$$R_{t\rho t\rho} = -16\frac{(4\rho-r_s)^2 r_s c^2}{(4\rho+r_s)^4\rho},\quad R_{t\vartheta t\vartheta} = 8\frac{(4\rho-r_s)^2\rho r_s c^2}{(4\rho+r_s)^4}, \tag{2.2.29a}$$

$$R_{t\varphi t\varphi} = 8\frac{(4\rho-r_s)^2\rho c^2 r_s\sin^2\vartheta}{(4\rho+r_s)^4},\quad R_{\rho\vartheta\rho\vartheta} = -\frac{(4\rho+r_s)^2 r_s}{32\rho^3}, \tag{2.2.29b}$$

$$R_{\rho\varphi\rho\varphi} = -\frac{(4\rho+r_s)^2 r_s\sin^2\vartheta}{32\rho^3},\quad R_{\vartheta\varphi\vartheta\varphi} = \frac{(4\rho+r_s)^2 r_s\sin^2\vartheta}{16\rho}. \tag{2.2.29c}$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschman scalar:

$$\mathscr{K} = 3\cdot 4^{13}\frac{\rho^6 r_s^2}{(4\rho+r_s)^{12}}. \tag{2.2.30}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1+r_s/(4\rho)}{1-r_s/(4\rho)}\frac{\partial_t}{c},\quad \mathbf{e}_{(r)} = \frac{1}{[1+r_s/(4\rho)]^2}\partial_\rho, \tag{2.2.31a}$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{\rho\,[1+r_s/(4\rho)]^2}\partial_\vartheta,\quad \mathbf{e}_{(\varphi)} = \frac{1}{\rho\,[1+r_s/(4\rho)]^2\sin^2\vartheta}\partial_\varphi. \tag{2.2.31b}$$

Ricci rotation coefficients:

$$\gamma_{(\rho)(t)(t)} = \frac{128r_s\rho^2}{(4\rho+r_s)^3(4\rho-r_s)},\quad \gamma_{(\vartheta)(\rho)(\vartheta)} = \gamma_{(\varphi)(\rho)(\varphi)} = \frac{16\rho(4\rho-r_s)}{(4\rho+r_s)^3}, \tag{2.2.32a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{16\rho\cot\vartheta}{(4\rho+r_s)^2}. \tag{2.2.32b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\rho)} = \frac{32\rho(16\rho^2-4\rho r_s+r_s^2)}{(4\rho+r_s)^3(4\rho-r_s)},\quad \gamma_{(\vartheta)} = \frac{16\rho\cot\vartheta}{(4\rho+r_s)^2}. \tag{2.2.33}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\rho)(t)(\rho)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{4096\rho^3 r_s}{(4\rho+r_s)^6}, \tag{2.2.34a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(\rho)(\vartheta)(\rho)(\vartheta)} = -R_{(\rho)(\varphi)(\rho)(\varphi)} = \frac{2048\rho^3 r_s}{(4\rho+r_s)^6}. \tag{2.2.34b}$$

2.2.5 Kruskal-Szekeres

The Schwarzschild metric in Kruskal-Szekeres[Kru60] coordinates $(T, X, \vartheta, \varphi)$ reads

$$ds^2 = \frac{4r_s^3}{r} e^{-r/r_s} \left(-dT^2 + dX^2\right) + r^2 d\Omega^2, \tag{2.2.35}$$

where $r \in \mathbb{R}_+ \setminus \{0\}$ is given by means of the LambertW-function $\mathscr{W}$,

$$\left(\frac{r}{r_s} - 1\right) e^{r/r_s} = X^2 - T^2 \qquad \text{or} \qquad r = r_s \left[\mathscr{W}\left(\frac{X^2 - T^2}{e}\right) + 1\right]. \tag{2.2.36}$$

The Schwarzschild coordinate time t in terms of the Kruskal coordinates T and X reads

$$t = 2r_s \text{arctanh} \frac{T}{X}, \qquad r > r_s, \tag{2.2.37a}$$

$$t = 2r_s \text{arctanh} \frac{X}{T}, \qquad r < r_s, \tag{2.2.37b}$$

$$t = \infty, \qquad r = r_s. \tag{2.2.37c}$$

The transformations between Kruskal- and Schwarzschild coordinates read

$$X = \sqrt{1 - \frac{r}{r_s}} e^{r/(2r_s)} \sinh \frac{ct}{2r_s}, \quad T = \sqrt{1 - \frac{r}{r_s}} e^{r/(2r_s)} \cosh \frac{ct}{2r_s}, \qquad 0 < r < r_2, \tag{2.2.38a}$$

$$X = \sqrt{\frac{r}{r_s} - 1} e^{r/(2r_s)} \cosh \frac{ct}{2r_s}, \quad T = \sqrt{\frac{r}{r_s} - 1} e^{r/(2r_s)} \sinh \frac{ct}{2r_s}, \qquad r \geq r_s. \tag{2.2.38b}$$

Christoffel symbols:

$$\Gamma^T_{TT} = \Gamma^X_{TX} = \Gamma^T_{XX} = \frac{Tr_s(r + r_s)}{r^2} e^{-r/r_s}, \tag{2.2.39a}$$

$$\Gamma^X_{TT} = \Gamma^T_{TX} = \Gamma^X_{XX} = -\frac{Xr_s(r + r_s)}{r^2} e^{-r/r_s}, \tag{2.2.39b}$$

$$\Gamma^\vartheta_{T\vartheta} = -\frac{2r_s^2 T}{r^2} e^{-r/r_s}, \qquad \Gamma^\vartheta_{X\vartheta} = \frac{2r_s^2 X}{r^2} e^{-r/r_s}, \tag{2.2.39c}$$

$$\Gamma^T_{\vartheta\vartheta} = -\frac{r}{2r_s} T, \qquad \Gamma^X_{\vartheta\vartheta} = \frac{r}{2r_s} X, \tag{2.2.39d}$$

$$\Gamma^T_{\vartheta\vartheta} = -\frac{r}{2r_s} T \sin^2\vartheta, \qquad \Gamma^X_{\vartheta\vartheta} = \frac{r}{2r_s} X \sin^2\vartheta, \tag{2.2.39e}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.39f}$$

Riemann-Tensor:

$$R_{TXTX} = -16\frac{r_s^7}{r^5} e^{-2r/r_s}, \qquad R_{T\vartheta T\vartheta} = \frac{2r_s^4}{r^2} e^{-r/r_s}, \tag{2.2.40a}$$

$$R_{T\varphi T\varphi} = \frac{2r_s^4}{r^2} e^{-r/r_s} \sin^2\vartheta, \qquad R_{X\vartheta X\vartheta} = -\frac{2r_s^4}{r^2} e^{-r/r_s}, \tag{2.2.40b}$$

$$R_{X\varphi X\varphi} = -\frac{2r_s^4}{r^2} e^{-r/r_s} \sin^2\vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s \sin^2\vartheta. \tag{2.2.40c}$$

The *Ricci-Tensor* as well as the *Ricci-scalar* vanish identically.
Kretschman scalar:

$$\mathscr{K} = \frac{12r_s^2}{r^6}. \tag{2.2.41}$$

Local tetrad:

$$\mathbf{e}_{(T)} = \frac{\sqrt{r}}{2r_s\sqrt{r_s}} e^{r/(2r_s)} \partial_T, \quad \mathbf{e}_{(X)} = \frac{\sqrt{r}}{2r_s\sqrt{r_s}} e^{r/(2r_s)} \partial_X, \quad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi \tag{2.2.42}$$

2.2.6 Painlevé-Gullstrand

The Schwarzschild metric expressed in Painlevé-Gullstrand coordinates[MP01] reads

$$ds^2 = -c^2 dT^2 + \left(dr + \sqrt{\frac{r_s}{r}} c\, dT\right)^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{2.2.43}$$

where the new time coordinate T follows from the Schwarzschild time t in the following way:

$$cT = ct + 2r_s \left(\sqrt{\frac{r}{r_s}} + \frac{1}{2}\ln\left|\frac{\sqrt{r/r_s}-1}{\sqrt{r/r_s}+1}\right|\right). \tag{2.2.44}$$

Christoffel symbols:

$$\Gamma^T_{TT} = \frac{cr_s}{2r^2}\sqrt{\frac{r_s}{r}}, \qquad \Gamma^r_{TT} = \frac{c^2 r_s(r-r_s)}{2r^3}, \qquad \Gamma^T_{Tr} = \frac{r_s}{2r^2}, \tag{2.2.45a}$$

$$\Gamma^r_{Tr} = -\frac{cr_s}{2r^2}\sqrt{\frac{r_s}{r}}, \qquad \Gamma^T_{rr} = \frac{r_s}{2cr^2}\sqrt{\frac{r}{r_s}}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2r^2}, \tag{2.2.45b}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^T_{\vartheta\vartheta} = -\frac{r}{c}\sqrt{\frac{r_s}{r}}, \tag{2.2.45c}$$

$$\Gamma^r_{\vartheta\vartheta} = -(r-r_s), \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^T_{\varphi\varphi} = -\frac{r}{c}\sqrt{\frac{r_s}{r}}\sin^2\vartheta, \tag{2.2.45d}$$

$$\Gamma^r_{\varphi\varphi} = -(r-r_s)\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.2.45e}$$

Riemann-Tensor:

$$R_{TrTr} = -\frac{c^2 r_s}{r^3}, \qquad R_{T\vartheta T\vartheta} = \frac{c^2 r_s(r-r_s)}{2r^2}, \qquad R_{T\vartheta r\vartheta} = -\frac{cr_s}{2r}\sqrt{\frac{r_s}{r}}, \tag{2.2.46a}$$

$$R_{T\varphi T\varphi} = \frac{c^2 r_s(r-r_s)\sin^2\vartheta}{2r^2}, \qquad R_{T\varphi r\varphi} = -\frac{cr_s}{2r}\sqrt{\frac{r_s}{r}}\sin^2\vartheta, \qquad R_{r\vartheta r\vartheta} = -\frac{r_s}{2r}, \tag{2.2.46b}$$

$$R_{r\varphi r\varphi} = -\frac{r_s\sin^2\vartheta}{2r}, \qquad R_{\vartheta\varphi\vartheta\varphi} = rr_s\sin^2\vartheta. \tag{2.2.46c}$$

The Ricci tensor and the Ricci scalar vanish identically while the Kretschman scalar reads $\mathscr{K} = 12r_s^2/r^6$.
For the Painlevé-Gullstrand coordinates, we can define two natural local tetrads. The static tetrads reads

$$\hat{\mathbf{e}}_{(T)} = \frac{1}{c\sqrt{1-r_s/r}}\partial_T, \quad \hat{\mathbf{e}}_{(r)} = \frac{\sqrt{r_s}}{c\sqrt{r-r_s}}\partial_T + \sqrt{1-\frac{r_s}{r}}\partial_r, \quad \hat{\mathbf{e}}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \quad \hat{\mathbf{e}}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi, \tag{2.2.47}$$

whereas the tetrad for the freely falling observer reads

$$\mathbf{e}_{(T)} = \frac{1}{c}\partial_T - \sqrt{\frac{r_s}{r}}\partial_r, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.2.48}$$

2.3 Reissner-Nordstrøm

The metric of the Reissner-Nordstrøm black hole in spherical coordinates $(t, r, \vartheta, \varphi)$ reads[MTW73]

$$ds^2 = -A_{\mathrm{RN}} c^2 dt^2 + A_{\mathrm{RN}}^{-1} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{2.3.1}$$

where

$$A_{\mathrm{RN}} = 1 - \frac{r_s}{r} + \frac{\rho Q^2}{r^2} \tag{2.3.2}$$

with $r_s = 2GM/c^2$, the charge Q, and $\rho = G/(\varepsilon_0 c^4) \approx 9.33 \cdot 10^{-34}$. As in the Schwarzschild case, there is a true curvature singularity at $r = 0$. However, for $Q^2 < r_s^2/(4\rho)$ there are also two critical points at

$$r = \frac{r_s}{2} \pm \frac{r_s}{2}\sqrt{1 - \frac{4\rho Q^2}{r_s^2}}. \tag{2.3.3}$$

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{A_{\mathrm{RN}} c^2 (r_s r - 2\rho Q^2)}{2r^3}, \qquad \Gamma^t_{tr} = \frac{r_s r - 2\rho Q^2}{2r^3 A_{\mathrm{RN}}}, \qquad \Gamma^r_{rr} = -\frac{r_s r - 2\rho Q^2}{2r^3 A_{\mathrm{RN}}}, \tag{2.3.4a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -r A_{\mathrm{RN}}, \tag{2.3.4b}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -r A_{\mathrm{RN}} \sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.3.4c}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2(r_s r - 3\rho Q^2)}{r^4}, \qquad R_{t\vartheta t\vartheta} = \frac{A_{\mathrm{RN}} c^2 (rs_r - 2\rho Q^2)}{2r^2}, \tag{2.3.5a}$$

$$R_{t\varphi t\varphi} = \frac{A_{\mathrm{RN}} c^2 (r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2}, \qquad R_{r\vartheta r\vartheta} = -\frac{r_s r - 2\rho Q^2}{2r^2 A_{\mathrm{RN}}}, \tag{2.3.5b}$$

$$R_{r\varphi r\varphi} = -\frac{(r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2 A_{\mathrm{RN}}}, \qquad R_{\vartheta\varphi\vartheta\varphi} = (r_s r - \rho Q^2)\sin^2\vartheta. \tag{2.3.5c}$$

Ricci-Tensor:

$$R_{tt} = \frac{c^2 \rho Q^2 A_{\mathrm{RN}}}{r^4}, \qquad R_{rr} = -\frac{\rho Q^2}{r^4 A_{\mathrm{RN}}}, \qquad R_{\vartheta\vartheta} = \frac{\rho Q^2}{r^2}, \qquad R_{\varphi\varphi} = \frac{\rho Q^2 \sin^2\vartheta}{r^2}. \tag{2.3.6}$$

While the Ricci scalar vanishes identically, the Kretschman scalar reads

$$\mathcal{K} = 4\frac{3r_s^2 r^2 - 12 r_s r \rho Q^2 + 14\rho^2 Q^4}{r^8}. \tag{2.3.7}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2(r_s r - 2\rho Q^2)}{r^4}, \qquad C_{t\vartheta t\vartheta} = -\frac{A_{\mathrm{RN}} c^2 (rs_r - 2\rho Q^2)}{2r^2}, \tag{2.3.8a}$$

$$C_{t\varphi t\varphi} = \frac{A_{\mathrm{RN}} c^2 (r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2}, \qquad C_{r\vartheta r\vartheta} = -\frac{r_s r - 2\rho Q^2}{2r^2 A_{\mathrm{RN}}}, \tag{2.3.8b}$$

$$C_{r\varphi r\varphi} = -\frac{(r_s r - 2\rho Q^2)\sin^2\vartheta}{2r^2 A_{\mathrm{RN}}}, \qquad C_{\vartheta\varphi\vartheta\varphi} = (r_s r - 2\rho Q^2)\sin^2\vartheta. \tag{2.3.8c}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{A_{\mathrm{RN}}}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{A_{\mathrm{RN}}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.3.9}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{A_{\mathrm{RN}}}\,dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{A_{\mathrm{RN}}}}, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.3.10}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{rr_s - 2\rho Q^2}{2r^3\sqrt{A_{\mathrm{RN}}}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{\sqrt{A_{\mathrm{RN}}}}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.3.11}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r^2 - 3rr_s + 2\rho Q^2}{2r^3\sqrt{A_{\mathrm{RN}}}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.3.12}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -\frac{r_s r - 3\rho Q^2}{r^4}, \qquad R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{r_s r - \rho Q^2}{r^4}, \tag{2.3.13a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{r_s r - 2\rho Q^2}{2r^4}. \tag{2.3.13b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{\rho Q^2}{r^4}. \tag{2.3.14}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s r - 2\rho Q^2}{r^4}, \tag{2.3.15a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{r_s r - 2\rho Q^2}{2r^4}. \tag{2.3.15b}$$

Effective potential:
The Euler-Lagrangian formalism[Rin01] yields the effective potential

$$V_{\mathrm{eff}} = \frac{1}{2}\left(1 - \frac{r_s}{r} + \frac{\rho Q^2}{r^2}\right)\left(\frac{h^2}{r^2} - \kappa c^2\right). \tag{2.3.16}$$

For null geodesics, $\kappa = 0$, there are two extremal points

$$r_\pm = \frac{3}{4}r_s\left(1 \pm \sqrt{1 - \frac{32\rho Q^2}{9r_s^2}}\right), \tag{2.3.17}$$

where r_+ is a maximum and r_- a minimum.
Further reading:
Eiroa[ERT02]

2.4 Janis-Newman-Winicour

The Janis-Newman-Winicour[JNW68] spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ is represented by the line element

$$ds^2 = -\alpha^{\gamma} c^2 dt^2 + \alpha^{-\gamma} dr^2 + r^2 \alpha^{-\gamma+1} \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right), \tag{2.4.1}$$

where $\alpha = 1 - r_s/(\gamma r)$. The Schwarzschild radius $r_s = 2GM/c^2$ is defined by Newton's constant G, the speed of light c, the mass parameter M, and the constant γ.

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{r_s c^2}{2r^2}\alpha^{2\gamma-1}, \qquad \Gamma^t_{tr} = \frac{r_s}{2\gamma r^2 \alpha}, \qquad \Gamma^r_{rr} = -\frac{r_s}{2\gamma r^2\alpha}, \tag{2.4.2a}$$

$$\Gamma^{\vartheta}_{r\vartheta} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2\alpha}, \qquad \Gamma^{\varphi}_{r\varphi} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2\alpha}, \qquad \Gamma^r_{\vartheta\vartheta} = -\frac{2\gamma r - r_s(\gamma+1)}{2\gamma}, \tag{2.4.2b}$$

$$\Gamma^r_{\varphi\varphi} = \Gamma^r_{\vartheta\vartheta}\sin^2\vartheta, \qquad \Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.4.2c}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{r_s c^2 \left[2\gamma r - r_s(\gamma+1)\right]\alpha^{\gamma-2}}{2\gamma r^4}, \qquad R_{t\vartheta t\vartheta} = \frac{r_s c^2\left[2\gamma r - r_s(\gamma+1)\right]\alpha^{\gamma-1}}{4\gamma r^2}, \tag{2.4.3a}$$

$$R_{t\varphi t\varphi} = \frac{r_s c^2\left[2\gamma r - r_s(\gamma+1)\right]\alpha^{\gamma-1}\sin^2\vartheta}{4\gamma r^2}, \quad R_{r\vartheta r\vartheta} = -\frac{r_s\left[2\gamma^2 r - r_s(\gamma+1)\right]}{4\gamma^2 r^2\alpha^{\gamma-1}}, \tag{2.4.3b}$$

$$R_{r\varphi r\varphi} = -\frac{r_s\left[2\gamma^2 r - r_s(\gamma+1)\right]\sin^2\vartheta}{4\gamma^2 r^2 \alpha^{\gamma-1}}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{r_s\left[4\gamma^2 r - r_s(\gamma+1)^2\right]\sin^2\vartheta}{4\gamma^2\alpha^{\gamma}}. \tag{2.4.3c}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{r_s c^2\alpha^{\gamma-2}\beta}{6\gamma^2 r^4}, \qquad C_{t\vartheta t\vartheta} = \frac{r_s c^2\alpha^{\gamma-1}\beta}{12\gamma^2 r^2}, \tag{2.4.4a}$$

$$C_{t\varphi t\varphi} = \frac{r_s c^2\alpha^{\gamma-1}\beta\sin^2\vartheta}{12\gamma^2 r^2}, \qquad C_{r\vartheta r\vartheta} = -\frac{r_s\beta}{12\gamma^2 r^2\alpha^{\gamma-1}}, \tag{2.4.4b}$$

$$C_{r\varphi r\varphi} = -\frac{r_s\beta\sin^2\vartheta}{12\gamma^2 r^2\alpha^{\gamma-1}}, \qquad C_{\vartheta\varphi\vartheta\varphi} = \frac{r_s\beta\sin^2\vartheta}{6\gamma^2\alpha^{\gamma}}, \tag{2.4.4c}$$

where $\beta = 6\gamma^2 r - r_s(\gamma+1)(2\gamma+1)$.

Ricci-Tensor:

$$R_{rr} = \frac{r_s^2(1-\gamma^2)}{2\gamma^2 r^4\alpha^2}. \tag{2.4.5}$$

The *Ricci scalar* reads

$$\mathscr{R} = \frac{r_s^2(1-\gamma^2)\alpha^{\gamma-2}}{2\gamma^2 r^4}, \tag{2.4.6}$$

whereas the *Kretschman scalar* is given by

$$\mathscr{K} = \frac{r_s^2\alpha^{2\gamma-4}}{4\gamma^4 r^8}\left[7\gamma^2 r_s^2(2+\gamma^2) + 48\gamma^4 r^2\alpha + 8\gamma r_s(2\gamma^2+1)(r_s - 2\gamma r) + 3r_s^2\right]. \tag{2.4.7}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\alpha^{\gamma/2}}\partial_t, \qquad \mathbf{e}_{(r)} = \alpha^{\gamma/2}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{\alpha^{(\gamma-1)/2}}{r}\partial_{\vartheta}, \qquad \mathbf{e}_{(\varphi)} = \frac{\alpha^{(\gamma-1)/2}}{r\sin\vartheta}\partial_{\varphi}. \tag{2.4.8}$$

Dual tetrad:

$$\theta^{(t)} = c\alpha^{\gamma/2}dt, \qquad \theta^{(r)} = \frac{dr}{\alpha^{\gamma/2}}, \qquad \theta^{(\vartheta)} = \frac{r}{\alpha^{(\gamma-1)/2}}d\vartheta, \qquad \theta^{(\varphi)} = \frac{r\sin\vartheta}{\alpha^{(\gamma-1)/2}}d\varphi. \tag{2.4.9}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s}{2r^2}\alpha^{(\gamma-2)/2}, \qquad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2}\alpha^{(\gamma-2)/2}, \tag{2.4.10a}$$

$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \tag{2.4.10b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4\gamma r - r_s(2+\gamma)}{2\gamma r^2}\alpha^{(\gamma-1)/2}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \tag{2.4.11}$$

Structure coefficients:

$$c_{(t)(r)}^{(t)} = \frac{r_s}{2r^2}\alpha^{(\gamma-2)/2}, \qquad c_{(r)(\vartheta)}^{(\vartheta)} = c_{(r)(\varphi)}^{(\varphi)} = -\frac{2\gamma r - r_s(\gamma+1)}{2\gamma r^2}\alpha^{(\gamma-2)/2}, \tag{2.4.12a}$$

$$c_{(\vartheta)(\varphi)}^{(\varphi)} = -\frac{\cot\vartheta}{r}\alpha^{(\gamma-1)/2}. \tag{2.4.12b}$$

Effective potential:
The Euler-Lagrangian formalism[Rin01] yields the effective potential

$$V_{\text{eff}} = \frac{1}{2}\alpha^{\gamma}\left(\frac{h^2\alpha^{\gamma-1}}{r^2} - \kappa c^2\right). \tag{2.4.13}$$

For null geodesics ($\kappa = 0$) and $\gamma > \frac{1}{2}$, there is an extremum at

$$r = r_s\frac{1+2\gamma}{2\gamma}. \tag{2.4.14}$$

2.5 Kerr

The Kerr spacetime, found by Roy Kerr in 1963[Ker63], describes a rotating black hole. Further reading: Boyer and Lindquist[BL67], Wilkins[Wil72], Brill[BC66].

2.5.1 Boyer-Lindquist coordinates

The Kerr metric in Boyer-Lindquist coordinates

$$ds^2 = -\left(1-\frac{r_s r}{\Sigma}\right)c^2dt^2 - \frac{2r_s a r\sin^2\vartheta}{\Sigma}c\,dt\,d\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\vartheta^2 \tag{2.5.1a}$$

$$+\left(r^2+a^2+\frac{r_s a^2 r\sin^2\vartheta}{\Sigma}\right)\sin^2\vartheta d\varphi^2, \tag{2.5.1b}$$

with $\Sigma = r^2+a^2\cos^2\vartheta$, $\Delta = r^2 - r_s r + a^2$, and $r_s = 2GM/c^2$, is taken from Bardeen[BPT72]. M is the mass and a is the angular momentum per unit mass of the black hole. The contravariant form of the metric reads

$$\partial_s^2 = -\frac{A}{c^2\Sigma\Delta}\partial_t^2 - \frac{2r_s a r}{c\Sigma\Delta}\partial_t\partial_\varphi + \frac{\Delta}{\Sigma}\partial_r^2 + \frac{1}{\Sigma}\partial_\vartheta^2 + \frac{\Delta - a^2\sin^2\vartheta}{\Sigma\Delta\sin^2\vartheta}\partial_\varphi^2, \tag{2.5.2}$$

where $A = \left(r^2+a^2\right)^2 - a^2\Delta\sin^2\vartheta = \left(r^2+a^2\right)\Sigma + r_s a^2 r\sin^2\vartheta$.
The event horizon r_+ is defined by the outer root of Δ,

$$r_+ = \frac{r_s}{2} + \sqrt{\frac{r_s^2}{4} - a^2}, \tag{2.5.3}$$

whereas the outer boundary r_0 of the ergosphere follows from the outer root of $\Sigma - r_s r$,

$$r_0 = \frac{r_s}{2} + \sqrt{\frac{r_s^2}{4} - a^2\cos^2\vartheta}, \tag{2.5.4}$$

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2 r_s\Delta(r^2-a^2\cos^2\vartheta)}{2\Sigma^3}, \qquad \Gamma^\vartheta_{tt} = -\frac{c^2 r_s a^2 r\sin\vartheta\cos\vartheta}{\Sigma^3}, \tag{2.5.5a}$$

$$\Gamma^t_{tr} = \frac{r_s(r^2+a^2)(r^2-a^2\cos^2\vartheta)}{2\Sigma^2\Delta}, \qquad \Gamma^\varphi_{tr} = \frac{c r_s a(r^2-a^2\cos^2\vartheta)}{2\Sigma^2\Delta}, \tag{2.5.5b}$$

$$\Gamma^t_{t\vartheta} = -\frac{r_s a^2 r\sin\vartheta\cos\vartheta}{\Sigma^2}, \qquad \Gamma^\varphi_{t\vartheta} = -\frac{c r_s a r\cot\vartheta}{\Sigma^2}, \tag{2.5.5c}$$

$$\Gamma^r_{t\varphi} = -\frac{c\Delta r_s a\sin^2\vartheta(r^2-a^2\cos^2\vartheta)}{2\Sigma^3}, \qquad \Gamma^\vartheta_{t\varphi} = \frac{c r_s a r(r^2+a^2)\sin\vartheta\cos\vartheta}{\Sigma^3}, \tag{2.5.5d}$$

$$\Gamma^r_{rr} = \frac{2ra^2\sin^2\vartheta - r_s(r^2-a^2\cos^2\vartheta)}{2\Sigma\Delta}, \qquad \Gamma^\vartheta_{rr} = \frac{a^2\sin\vartheta\cos\vartheta}{\Sigma\Delta}, \tag{2.5.5e}$$

$$\Gamma^r_{r\vartheta} = -\frac{a^2\sin\vartheta\cos\vartheta}{\Sigma}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{r}{\Sigma}, \tag{2.5.5f}$$

$$\Gamma^t_{r\varphi} = \frac{r_s a\sin^2\vartheta\left[a^2\cos^2\vartheta(a^2-r^2) - r^2(a^2+3r^2)\right]}{2\Sigma^2\Delta}, \qquad \Gamma^r_{\vartheta\vartheta} = -\frac{r\Delta}{\Sigma}, \tag{2.5.5g}$$

$$\Gamma^\varphi_{r\varphi} = \frac{2r\Sigma^2 + r_s\left[a^4\sin^2\vartheta\cos^2\vartheta - r^2(\Sigma+r^2+a^2)\right]}{2\Sigma^2\Delta}, \qquad \Gamma^\vartheta_{\vartheta\vartheta} = -\frac{a^2\sin\vartheta\cos\vartheta}{\Sigma}, \tag{2.5.5h}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \frac{\cot\vartheta}{\Sigma^2}\left[\Sigma^2 + r_s a^2 r\sin^2\vartheta\right], \qquad \Gamma^t_{\vartheta\varphi} = \frac{r_s a^3 r\sin^3\vartheta\cos\vartheta}{\Sigma^2}, \tag{2.5.5i}$$

$$\Gamma^r_{\varphi\varphi} = \frac{\Delta\sin^2\vartheta}{2\Sigma^3}\left[-2r\Sigma^2 + r_s a^2\sin^2\vartheta(r^2-a^2\cos^2\vartheta\right], \tag{2.5.5j}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\frac{\sin\vartheta\cos\vartheta}{\Sigma^3}\left[A\Sigma + \left(r^2+a^2\right)r_s a^2 r\sin^2\vartheta\right], \tag{2.5.5k}$$

General local tetrad:

$$\mathbf{e}_{(0)} = \Gamma\left(\partial_t + \zeta\partial_\varphi\right), \qquad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \tag{2.5.6a}$$

$$\mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}}\partial_\vartheta, \qquad \mathbf{e}_{(3)} = \frac{\Gamma}{c}\left(\mp\frac{g_{t\varphi}+\zeta g_{\varphi\varphi}}{\sqrt{\Delta}\sin\vartheta}\partial_t \pm \frac{g_{tt}+\zeta g_{t\varphi}}{\sqrt{\Delta}\sin\vartheta}\partial_\varphi\right), \tag{2.5.6b}$$

where $-\Gamma^{-2} = g_{tt} + 2\zeta g_{t\varphi} + \zeta^2 g_{\varphi\varphi}$,

$$\Gamma^{-2} = \left(1-\frac{r_s r}{\Sigma}\right) + \frac{2r_s a r\sin^2\vartheta}{\Sigma}\frac{\zeta}{c} - \left(r^2+a^2+\frac{r_s a^2 r\sin^2\vartheta}{\Sigma}\right)\frac{\zeta^2}{c^2}\sin^2\vartheta \tag{2.5.7}$$

Non-rotating local tetrad ($\zeta = \omega$)**:**

$$\mathbf{e}_{(0)} = \sqrt{\frac{A}{\Sigma\Delta}}\left(\frac{1}{c}\partial_t + \omega\partial_\varphi\right), \quad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \quad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}}\partial_\vartheta, \quad \mathbf{e}_{(3)} = \sqrt{\frac{\Sigma}{A}}\frac{1}{\sin\vartheta}\partial_\varphi, \tag{2.5.8}$$

where $\omega = -g_{t\varphi}/g_{\varphi\varphi} = r_s a r/A$.
The relation between the constants of motion E, L, Q, and μ (defined in Bardeen[BPT72]) and the initial direction υ, compare Sec. (1.3.4), with respect to the LNRF reads ($c = 1$)

$$\upsilon^{(0)} = \sqrt{\frac{A}{\Sigma\Delta}}E - \frac{r_s r a}{\sqrt{A\Sigma\Delta}}L, \qquad \upsilon^{(1)} = \sqrt{\frac{\Delta}{\Sigma}}p_r, \tag{2.5.9a}$$

$$\upsilon^{(2)} = \frac{1}{\sqrt{\Sigma}}\sqrt{Q-\cos^2\vartheta\left[a^2\left(\mu^2-E^2\right)+\frac{L^2}{\sin^2\vartheta}\right]}, \qquad \upsilon^{(3)} = \sqrt{\frac{\Sigma}{A}}\frac{L}{\sin\vartheta}. \tag{2.5.9b}$$

Static local tetrad ($\zeta = 0$)**:**

$$\mathbf{e}_{(0)} = \frac{1}{c\sqrt{1-r_s r/\Sigma}}\partial_t, \qquad \mathbf{e}_{(1)} = \sqrt{\frac{\Delta}{\Sigma}}\partial_r, \qquad \mathbf{e}_{(2)} = \frac{1}{\sqrt{\Sigma}}\partial_\vartheta, \tag{2.5.10a}$$

$$\mathbf{e}_{(3)} = \pm\frac{r_s a r\sin\vartheta}{c\sqrt{1-r_s r/\Sigma}\sqrt{\Delta\Sigma}}\partial_t \mp \frac{\sqrt{1-r_s r/\Sigma}}{\sqrt{\Delta}\sin\vartheta}\partial_\varphi. \tag{2.5.10b}$$

2.6 Oppenheimer-Snyder collapse

2.6.1 Outer metric

The metric of the outer spacetime ($R > R_b$) in comoving coordinates $(\tau, R, \vartheta, \varphi)$ with $(c = 1)$ is given by

$$ds^2 = -d\tau^2 + \frac{R}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}} dR^2 + \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{4/3} \left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right). \tag{2.6.1}$$

Christoffel symbols:

$$\Gamma^R_{\tau R} = \frac{1}{2}\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \qquad \Gamma^\vartheta_{\tau\vartheta} = -\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \tag{2.6.2a}$$

$$\Gamma^\varphi_{\tau\varphi} = -\frac{\sqrt{r_s}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \qquad \Gamma^\tau_{RR} = \frac{R\sqrt{r_s}}{2\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{5/3}}, \tag{2.6.2b}$$

$$\Gamma^R_{RR} = -\frac{3\sqrt{r_s}\tau}{4\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)R}, \qquad \Gamma^\vartheta_{R\vartheta} = \frac{\sqrt{R}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \tag{2.6.2c}$$

$$\Gamma^\varphi_{R\varphi} = \frac{\sqrt{R}}{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}, \qquad \Gamma^\tau_{\vartheta\vartheta} = -\sqrt{r_s}\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}, \tag{2.6.2d}$$

$$\Gamma^R_{\vartheta\vartheta} = -\frac{R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau}{\sqrt{R}}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \tag{2.6.2e}$$

$$\Gamma^\tau_{\varphi\varphi} = -\sqrt{r_s}\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \tag{2.6.2f}$$

$$\Gamma^R_{\varphi\varphi} = -\frac{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)\sin^2\vartheta}{\sqrt{R}}. \tag{2.6.2g}$$

Riemann-Tensor:

$$R_{\tau R\tau R} = -\frac{Rr_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{8/3}}, \qquad R_{\tau\vartheta\tau\vartheta} = \frac{1}{2}\frac{r_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{2/3}}, \tag{2.6.3a}$$

$$R_{\tau\varphi\tau\varphi} = \frac{1}{2}\frac{r_s\sin^2\vartheta}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{2/3}}, \qquad R_{R\vartheta R\vartheta} = -\frac{1}{2}\frac{Rr_s}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{4/3}}, \tag{2.6.3b}$$

$$R_{R\varphi R\varphi} = -\frac{1}{2}\frac{Rr_s\sin^2\vartheta}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{4/3}}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^{2/3} r_s\sin^2\vartheta. \tag{2.6.3c}$$

The Ricci tensor and the Ricci scalar vanish identically.

Kretschman scalar:

$$\mathscr{K} = 12\frac{r_s^2}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\,\tau\right)^4}. \tag{2.6.4}$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \qquad \mathbf{e}_{(R)} = \frac{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{1/3}}{\sqrt{R}}\partial_R, \tag{2.6.5a}$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\left(R^{3/2} - \frac{3}{2}\sqrt{r_s}\tau\right)^{2/3}\sin\vartheta}\partial_\varphi. \tag{2.6.5b}$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(R)(R)} = -\frac{\sqrt{r_s}}{2R^{3/2}-3\sqrt{r_s}\tau}, \quad \gamma_{(\tau)(\vartheta)(\vartheta)} = \gamma_{(\tau)(\varphi)(\varphi)} = \frac{2\sqrt{r_s}}{2R^{3/2}-3\sqrt{r_s}\tau}, \tag{2.6.6a}$$

$$\gamma_{(R)(\varphi)(\varphi)} = \gamma_{(R)(\vartheta)(\vartheta)} = -\left(R^{3/2}-\frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}. \tag{2.6.6b}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\tau)} = -\frac{3\sqrt{r_s}}{2R^{3/2}-3\sqrt{r_s}\tau}, \quad \gamma_{(R)} = 2\left(R^{3/2}-\frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}, \quad \gamma_{(\vartheta)} = \cot\vartheta\left(R^{3/2}-\frac{3}{2}\sqrt{r_s}\tau\right)^{-2/3}. \tag{2.6.7}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\tau)(R)(\tau)(R)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{4r_s}{\left(2R^{3/2}-3\sqrt{r_s}\tau\right)^2}, \tag{2.6.8a}$$

$$R_{(\tau)(\vartheta)(\tau)(\vartheta)} = R_{(\tau)(\varphi)(\tau)(\varphi)} = -R_{(R)(\vartheta)(R)(\vartheta)} = -R_{(R)(\varphi)(R)(\varphi)} = \frac{2r_s}{\left(2R^{3/2}-3\sqrt{r_s}\tau\right)^2}. \tag{2.6.8b}$$

The Ricci tensor with respect to the local tetrad vanishes identically.

2.6.2 Inner metric

The metric of the inside ($R \le R_b$) reads

$$ds^2 = -d\tau^2 + \left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{4/3}\left[dR^2 + R^2\left(d\vartheta^2+\sin^2\vartheta\, d\varphi^2\right)\right]. \tag{2.6.9}$$

Christoffel symbols:

$$\Gamma^R_{\tau R} = -\frac{\sqrt{r_s}R_b^{3/2}}{1-\frac{3}{2}\sqrt{r_s}R_b^{3/2}\tau}, \qquad \Gamma^\vartheta_{\tau\vartheta} = -\frac{\sqrt{r_s}R_b^{3/2}}{1-\frac{3}{2}\sqrt{r_s}R_b^{3/2}\tau}, \tag{2.6.10a}$$

$$\Gamma^\varphi_{\tau\varphi} = -\frac{\sqrt{r_s}R_b^{3/2}}{1-\frac{3}{2}\sqrt{r_s}R_b^{3/2}\tau}, \qquad \Gamma^\tau_{RR} = -\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{1/3}\sqrt{r_s}R_b^{-3/2}, \tag{2.6.10b}$$

$$\Gamma^\vartheta_{R\vartheta} = \frac{1}{R}, \qquad \Gamma^\varphi_{R\varphi} = \frac{1}{R}, \tag{2.6.10c}$$

$$\Gamma^R_{\vartheta\vartheta} = -R, \qquad \Gamma^\tau_{\vartheta\vartheta} = -\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{1/3}\sqrt{r_s}R_b^{-3/2}R^2, \tag{2.6.10d}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^\tau_{\varphi\varphi} = -\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{1/3}\sqrt{r_s}R_b^{-3/2}R^2\sin^2\vartheta, \tag{2.6.10e}$$

$$\Gamma^R_{\varphi\varphi} = -R\sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.6.10f}$$

Riemann-Tensor:

$$R_{\tau R\tau R} = -\frac{1}{2}\frac{r_s}{R_b^3\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{2/3}}, \qquad R_{\tau\vartheta\tau\vartheta} = \frac{1}{2}\frac{r_sR^2}{R_b^3\left(1-\frac{3}{2}\sqrt{r_s}R^{-3/2}\tau\right)^{2/3}}, \tag{2.6.11a}$$

$$R_{\tau\varphi\tau\varphi} = \frac{1}{2}\frac{r_sR^2\sin^2\vartheta}{R_b^3\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{2/3}}, \qquad R_{R\varphi R\varphi} = \frac{r_sR^2\sin^2\vartheta}{R_b^3}\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{2/3}, \tag{2.6.11b}$$

$$R_{R\vartheta R\vartheta} = \frac{r_sR^2}{R_b^3}\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{2/3}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{r_sR^4\sin^2\vartheta}{R_b^3}\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{2/3}. \tag{2.6.11c}$$

Ricci-Tensor:

$$R_{\tau\tau} = \frac{3}{2}\frac{r_s}{R_b^3\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\,\tau\right)^2}, \qquad R_{RR} = \frac{3}{2}\frac{r_s}{R_b^3\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\,\tau\right)^{2/3}}, \tag{2.6.12a}$$

$$R_{\vartheta\vartheta} = \frac{3}{2}\frac{r_s R^2}{R_b^3\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\,\tau\right)^{2/3}}, \qquad R_{\varphi\varphi} = \frac{3}{2}\frac{r_s R^2\sin^2\vartheta}{R_b^3\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\,\tau\right)^{2/3}}, \tag{2.6.12b}$$

The *Ricci*- and *Kretschman*- scalars read:

$$\mathscr{R} = \frac{3r_s}{R_b^3\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\,\tau\right)^2}, \qquad \mathscr{K} = 15\frac{r_s^2}{R_b^6\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\,\tau\right)^4}. \tag{2.6.13}$$

Local tetrad:

$$\mathbf{e}_{(\tau)} = \partial_\tau, \qquad \mathbf{e}_{(R)} = \frac{1}{\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{2/3}}\partial_R, \tag{2.6.14a}$$

$$\mathbf{e}_{(\vartheta)} = \frac{1}{R\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{2/3}}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\left(1-\frac{3}{2}\sqrt{r_s}R_b^{-3/2}\tau\right)^{2/3}R\sin\vartheta}\partial_\varphi. \tag{2.6.14b}$$

Ricci rotation coefficients:

$$\gamma_{(\tau)(R)(R)} = \gamma_{(\tau)(\vartheta)(\vartheta)} = \gamma_{(\tau)(\varphi)(\varphi)} = \frac{2\sqrt{r_s}}{2R_b^{3/2}-3\sqrt{r_s}\tau}, \tag{2.6.15a}$$

$$\gamma_{(R)(\vartheta)(\vartheta)} = \gamma_{(R)(\varphi)(\varphi)} = -\frac{2^{2/3}R_b}{R\left(2R_b^{3/2}-3\sqrt{r_s}\tau\right)^{2/3}}, \tag{2.6.15b}$$

$$\gamma_{(\vartheta)(\varphi)(\varphi)} = -\frac{2^{2/3}R_b\cot\vartheta}{R\left(2R_b^{3/2}-3\sqrt{r_s}\tau\right)^{2/3}}. \tag{2.6.15c}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(\tau)} = -\frac{6\sqrt{r_s}}{2R_b^{3/2}-3\sqrt{r_s}\tau}, \quad \gamma_{(R)} = \frac{2^{5/3}R_b}{R\left(2R_b^{3/2}-3\sqrt{r_s}\tau\right)^{2/3}}, \quad \gamma_{(\vartheta)} = \frac{2^{2/3}R_b\cot\vartheta}{R\left(2R_b^{3/2}-3\sqrt{r_s}\tau\right)^{2/3}}. \tag{2.6.16}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\tau)(R)(\tau)(R)} = R_{(\tau)(\vartheta)(\tau)(\vartheta)} = R_{(\tau)(\varphi)(\tau)(\varphi)} = \frac{2r_s}{\left(2R_b^{3/2}-3\sqrt{r_s}\tau\right)^2}, \tag{2.6.17a}$$

$$R_{(R)(\vartheta)(R)(\vartheta)} = R_{(R)(\varphi)(R)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{4r_s}{\left(2R_b^{3/2}-3\sqrt{r_s}\tau\right)^2}. \tag{2.6.17b}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\tau)(\tau)} = R_{(R)(R)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{6r_s}{\left(2R_b^{3/2}-3\sqrt{r_s}\tau\right)^2}. \tag{2.6.18}$$

2.7 Morris-Thorne

The most simple wormhole geometry is represented by the metric[MT88, Vis95]

$$ds^2 = -c^2dt^2 + dl^2 + (b_0^2 + l^2)\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{2.7.1}$$

where b_0 is the throat radius and l is the proper radial coordinate.
Christoffel symbols:

$$\Gamma^{\vartheta}_{l\vartheta} = \frac{l}{b_0^2+l^2}, \qquad \Gamma^{\varphi}_{l\varphi} = \frac{l}{b_0^2+l^2}, \qquad \Gamma^{l}_{\vartheta\vartheta} = -l, \tag{2.7.2a}$$

$$\Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^{l}_{\varphi\varphi} = -l\sin^2\vartheta, \qquad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\,\cos\vartheta. \tag{2.7.2b}$$

Riemann-Tensor:

$$R_{l\vartheta l\vartheta} = -\frac{b_0^2}{b_0^2+l^2}, \quad R_{l\varphi l\varphi} = -\frac{b_0^2\sin^2\vartheta}{b_0^2+l^2}, \quad R_{\vartheta\varphi\vartheta\varphi} = b_0^2\sin^2\vartheta. \tag{2.7.3}$$

Ricci tensor, Ricci and Kretschman scalar:

$$R_{ll} = -2\frac{b_0^2}{\left(b_0^2+l^2\right)^2}, \qquad \mathscr{R} = -2\frac{b_0^2}{\left(b_0^2+l^2\right)^2}, \qquad \mathscr{K} = \frac{12b_0^4}{\left(b_0^2+l^2\right)^4}. \tag{2.7.4}$$

Weyl-Tensor:

$$C_{tltl} = -\frac{2}{3}\frac{c^2b_0^2}{\left(b_0^2+l^2\right)^2}, \qquad C_{t\vartheta t\vartheta} = \frac{1}{3}\frac{c^2b_0^2}{b_0^2+l^2}, \qquad C_{t\varphi t\varphi} = \frac{1}{3}\frac{c^2b_0^2\sin^2\vartheta}{b_0^2+l^2}, \tag{2.7.5a}$$

$$C_{l\vartheta l\vartheta} = -\frac{1}{3}\frac{b_0^2}{b_0^2+l^2}, \qquad C_{l\varphi l\varphi} = -\frac{1}{3}\frac{b_0^2\sin^2\vartheta}{b_0^2+l^2}, \quad C_{\vartheta\varphi\vartheta\varphi} = \frac{2}{3}b_0^2\sin^2\vartheta. \tag{2.7.5b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(l)} = \partial_l, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{\sqrt{b_0^2+l^2}}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{\sqrt{b_0^2+l^2}\,\sin\vartheta}\partial_\varphi. \tag{2.7.6}$$

Dual tetrad

$$\theta^{(t)} = c\,dt, \qquad \theta^{(l)} = dl, \qquad \theta^{(\vartheta)} = \sqrt{b_0^2+l^2}\,d\vartheta, \qquad \theta^{(\varphi)} = \sqrt{b_0^2+l^2}\sin\vartheta\,d\varphi. \tag{2.7.7}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{l}{b_0^2+l^2}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{\sqrt{b_0^2+l^2}}. \tag{2.7.8}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2l}{b_0^2+l^2}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{\sqrt{b_0^2+l^2}}. \tag{2.7.9}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(l)(\vartheta)(l)(\vartheta)} = R_{(l)(\varphi)(l)(\varphi)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{b_0^2}{\left(b_0^2+l^2\right)^2}. \tag{2.7.10}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(l)(l)} = -\frac{2b_0^2}{\left(b_0^2 + l^2\right)^2}. \tag{2.7.11}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(l)(t)(l)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{2b_0^2}{3\left(b_0^2 + l^2\right)^2}, \tag{2.7.12a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(l)(\vartheta)(l)(\vartheta)} = -C_{(l)(\varphi)(l)(\varphi)} = \frac{b_0^2}{3\left(b_0^2 + l^2\right)^2}. \tag{2.7.12b}$$

Embedding:
The embedding function reads

$$z(r) = \pm b_0 \ln\left[\frac{r}{b_0} + \sqrt{\left(\frac{r}{b_0}\right)^2 - 1}\right] \tag{2.7.13}$$

with $r^2 = b_0^2 + l^2$.
Further reading:
Ellis[Ell73], Visser[Vis95].

2.8 Alcubierre Warp

The Warp metric given by Miguel Alcubierre[Alc94] reads

$$ds^2 = -c^2dt^2 + (dx - v_s f(r_s)dt)^2 + dy^2 + dz^2 \tag{2.8.1}$$

where

$$v_s = \frac{dx_s(t)}{dt}, \tag{2.8.2a}$$
$$r_s(t) = \sqrt{(x - x_s(t))^2 + y^2 + z^2}, \tag{2.8.2b}$$
$$f(r_s) = \frac{\tanh(\sigma(r_s + R)) - \tanh(\sigma(r_s - R))}{2\tanh(\sigma R)}. \tag{2.8.2c}$$

The parameter $R > 0$ defines the radius of the warp bubble and the parameter $\sigma > 0$ its thickness.
Christoffel symbols:

$$\Gamma^t_{tt} = \frac{f^2 f_x v_s^3}{c^2}, \quad \Gamma^z_{tt} = -f f_z v_s^2, \quad \Gamma^y_{tt} = -f f_y v_s^2, \tag{2.8.3a}$$
$$\Gamma^x_{tt} = \frac{f^3 f_x v_s^4 - c^2 f f_x v_s^2 - c^2 f_t v_s}{c^2}, \quad \Gamma^t_{tx} = -\frac{f f_x v_s^2}{c^2}, \quad \Gamma^x_{tx} = -\frac{f^2 f_x v_s^3}{c^2}, \tag{2.8.3b}$$
$$\Gamma^y_{tx} = \frac{f_y v_s}{2}, \quad \Gamma^z_{tx} = \frac{f_z v_s}{2}, \quad \Gamma^t_{ty} = -\frac{f f_y v_s^2}{2c^2}, \tag{2.8.3c}$$
$$\Gamma^x_{ty} = -\frac{f^2 f_y v_s^3 + c^2 f_y v_s}{2c^2}, \quad \Gamma^t_{tz} = -\frac{f f_z v_s^2}{2c^2}, \quad \Gamma^x_{tz} = -\frac{f^2 f_z v_s^3 + c^2 f_z v_s}{2c^2}, \tag{2.8.3d}$$
$$\Gamma^t_{xx} = \frac{f_x v_s}{c^2}, \quad \Gamma^x_{xx} = \frac{f f_x v_s^2}{c^2}, \quad \Gamma^t_{xy} = \frac{f_y v_s}{2c^2}, \tag{2.8.3e}$$
$$\Gamma^x_{xy} = \frac{f f_y v_s^2}{2c^2}, \quad \Gamma^t_{xz} = \frac{f_z v_s}{2c^2}, \quad \Gamma^x_{xz} = \frac{f f_z v_s^2}{2c^2}, \tag{2.8.3f}$$

with derivatives

$$f_t = \frac{df(r_s)}{dt} = \frac{-v_s\sigma(x - x_s(t))}{2r_s\tanh(\sigma R)}\left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R))\right] \tag{2.8.4a}$$
$$f_x = \frac{df(r_s)}{dx} = \frac{\sigma(x - x_s(t))}{2r_s\tanh(\sigma R)}\left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R))\right] \tag{2.8.4b}$$
$$f_y = \frac{df(r_s)}{dy} = \frac{\sigma y}{2r_s\tanh(\sigma R)}\left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R))\right] \tag{2.8.4c}$$
$$f_z = \frac{df(r_s)}{dz} = \frac{\sigma z}{2r_s\tanh(\sigma R)}\left[\operatorname{sech}^2(\sigma(r_s + R)) - \operatorname{sech}^2(\sigma(r_s - R))\right] \tag{2.8.4d}$$

Riemann- and Ricci-tensor as well as Ricci- and Kretschman-scalar are shown only in the Maple worksheet.
Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{c}(\partial_t + v_s f \partial_x), \quad \mathbf{e}_{(1)} = \partial_x, \quad \mathbf{e}_{(2)} = \partial_y, \quad \mathbf{e}_{(3)} = \partial_z. \tag{2.8.5}$$

Static Local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{\sqrt{c^2 - v_s^2 f^2}}\partial_t, \quad \mathbf{e}_{(1)} = \frac{v_s f}{c\sqrt{c^2 - v_s^2 f^2}}\partial_t + \frac{\sqrt{c^2 - v_s^2 f^2}}{c}\partial_x, \quad \mathbf{e}_{(2)} = \partial_y, \quad \mathbf{e}_{(3)} = \partial_z. \tag{2.8.6}$$

Further reading:
Pfenning[PF97], Clark[CHL99], Van Den Broeck[Bro99]

2.9 Barriola-Vilenkin monopol

The Barriola Vilenkin metric reads[BV89]

$$ds^2 = -c^2 dt^2 + dr^2 + k^2 r^2 \left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right), \tag{2.9.1}$$

where k is the scaling factor.

Christoffel symbols:

$$\Gamma^r_{\vartheta\vartheta} = -k^2 r, \qquad \Gamma^r_{\varphi\varphi} = -k^2 r \sin^2\vartheta, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \tag{2.9.2a}$$

$$\Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta. \tag{2.9.2b}$$

Riemann-Tensor:

$$R_{\vartheta\varphi\vartheta\varphi} = (1-k^2)k^2 r^2 \sin^2\vartheta. \tag{2.9.3}$$

Ricci tensor, Ricci and Kretschman scalar:

$$R_{\vartheta\vartheta} = (1-k^2), \qquad R_{\varphi\varphi} = (1-k^2)\sin^2\vartheta, \qquad \mathscr{R} = 2\frac{1-k^2}{k^2 r^2}, \qquad \mathscr{K} = 4\frac{(1-k^2)^2}{k^4 r^4}. \tag{2.9.4}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2(1-k^2)}{3k^2 r^2}, \quad C_{t\vartheta t\vartheta} = \frac{c^2}{6}(1-k^2), \qquad C_{t\varphi t\varphi} = \frac{c^2}{6}(1-k^2)\sin^2\vartheta, \tag{2.9.5a}$$

$$C_{r\vartheta r\vartheta} = -\frac{1}{6}(1-k^2), \quad C_{r\varphi r\varphi} = -\frac{1}{6}(1-k^2)\sin^2\vartheta, \; C_{\vartheta\varphi\vartheta\varphi} = \frac{k^2 r^2}{3}(1-k^2)\sin^2\vartheta. \tag{2.9.5b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = \partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{kr}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{kr\sin\vartheta}\partial_\varphi. \tag{2.9.6}$$

Dual tetrad:

$$\theta^{(t)} = c\,dt, \qquad \theta^{(r)} = dr, \qquad \theta^{(\vartheta)} = kr\,d\vartheta, \qquad \theta^{(\varphi)} = kr\sin\vartheta\,d\varphi. \tag{2.9.7}$$

Ricci rotation coefficients:

$$\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{r}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{kr}. \tag{2.9.8}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{2}{r}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{kr}. \tag{2.9.9}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{1-k^2}{k^2 r^2}. \tag{2.9.10}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{1-k^2}{k^2 r^2}. \tag{2.9.11}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{1-k^2}{3k^2 r^2}, \tag{2.9.12a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{1-k^2}{6k^2 r^2}. \tag{2.9.12b}$$

2.10 Gödel Universe

Gödel introduced a homogeneous and rotating universe model in [Göd49]. We follow the notation of [KWSD04]

2.10.1 Cylindrical coordinates

The Gödel metric in cylindrical coordinates is

$$ds^2 = -c^2dt^2 + \frac{dr^2}{1+[r/(2a)]^2} + r^2\left[1-\left(\frac{r}{2a}\right)^2\right]d\varphi^2 + dz^2 - 2r^2\frac{c}{\sqrt{2}a}dt\,d\varphi, \tag{2.10.1}$$

where $2a$ is the Gödel radius.

Christoffel symbols:

$$\Gamma^t_{tr} = \frac{r}{2a^2}\frac{1}{1+[r/(2a)]^2}, \qquad \Gamma^\varphi_{tr} = -\frac{c}{\sqrt{2}ar}\frac{1}{1+[r/(2a)]^2}, \tag{2.10.2a}$$

$$\Gamma^r_{t\varphi} = \frac{cr}{\sqrt{2}a}\left[1+\left(\frac{r}{2a}\right)\right]^2, \qquad \Gamma^r_{rr} = -\frac{r}{4a^2}\frac{1}{1+[r/(2a)]^2}, \tag{2.10.2b}$$

$$\Gamma^t_{r\varphi} = \frac{r^3}{4\sqrt{2}ca^3}\frac{1}{1+[r/(2a)]^2}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}\frac{1}{1+[r/(2a)]^2}, \tag{2.10.2c}$$

$$\Gamma^r_{\varphi\varphi} = r\left[1+\left(\frac{r}{2a}\right)^2\right]\left[1-\frac{1}{2}\left(\frac{r}{a}\right)^2\right]. \tag{2.10.2d}$$

Riemann-Tensor:

$$R_{trtr} = \frac{c^2}{2a^2}\frac{1}{1+[r/(2a)]^2}, \qquad R_{trr\varphi} = -\frac{cr^2}{2\sqrt{2}a^3}\frac{1}{1+[r/(2a)]^2}, \tag{2.10.3a}$$

$$R_{t\varphi t\varphi} = \frac{c^2r^2}{2a^2}\frac{1}{1+[r/(2a)]^2}, \qquad R_{r\varphi r\varphi} = \frac{r^2}{2a^2}\frac{1+3[r/(2a)]^2}{1+[r/(2a)]^2}. \tag{2.10.3b}$$

Ricci-Tensor:

$$R_{tt} = \frac{c^2}{a^2}, \qquad R_{t\varphi} = \frac{r^2c}{\sqrt{2}a^3}, \qquad R_{\varphi\varphi} = \frac{r^4}{2a^4}. \tag{2.10.4}$$

Ricci and Kretschman scalar

$$\mathscr{R} = -\frac{1}{a^2}, \qquad \mathscr{K} = \frac{3}{a^4}. \tag{2.10.5}$$

cosmological constant:

$$\Lambda = \frac{R}{2} \tag{2.10.6}$$

Killing vectors:
An infinitesimal isometric transformation $x'^\mu = x^\mu + \varepsilon\xi^\mu(x^\nu)$ leaves the metric unchanged, that is $g'_{\mu\nu}(x'^\sigma) = g_{\mu\nu}(x'^\sigma)$. A killing vector field ξ^μ is solution to the killing equation $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$. There exist five killing vector fields in Gödel's spacetime:

$$\underset{a}{\xi}^\mu = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \underset{b}{\xi}^\mu = \frac{1}{\sqrt{1+[r/(2a)]^2}}\begin{pmatrix}\frac{r}{\sqrt{2}c}\cos\varphi\\ a\left(1+[r/(2a)]^2\right)\sin\varphi\\ \frac{a}{r}\left(1+2[r/(2a)]^2\right)\cos\varphi\\ 0\end{pmatrix}, \quad \underset{c}{\xi}^\mu = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \tag{2.10.7a}$$

$$\underset{d}{\xi}^\mu = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \quad \underset{e}{\xi}^\mu = \frac{1}{\sqrt{1+[r/(2a)]^2}}\begin{pmatrix}\frac{r}{\sqrt{2}c}\sin\varphi\\ -a\left(1+[r/(2a)]^2\right)\cos\varphi\\ \frac{a}{r}\left(1+2[r/(2a)]^2\right)\sin\varphi\\ 0\end{pmatrix}. \tag{2.10.7b}$$

An arbitrary linear combination of killing vector fields is again a killing vector field.

Local tetrad:

For the local tetrad in Gödel's spacetime an ansatz similar to the local tetrad of a rotating spacetime in spherical coordinates (Sec. 1.3.5) can be used. After substituting $\vartheta \to z$ and swapping base vectors $\mathbf{e}_{(2)}$ and $\mathbf{e}_{(3)}$ an orthonormalized and right-handed local tetrad is obtained.

$$\mathbf{e}_{(0)} = \Gamma\left(\partial_t + \zeta\partial_\varphi\right), \quad \mathbf{e}_{(1)} = \sqrt{1+[r/(2a)]^2}\partial_r, \quad \mathbf{e}_{(2)} = \Delta\Gamma\left(A\partial_t + B\partial_\varphi\right), \quad \mathbf{e}_{(3)} = \partial_z, \tag{2.10.8a}$$

where

$$A = -\frac{r^2c}{\sqrt{2}a} + \zeta r^2\left(1-[r/(2a)]^2\right), \qquad B = c^2 + \frac{\zeta r^2 c}{\sqrt{2}a}, \tag{2.10.9a}$$

$$\Gamma = \frac{1}{\sqrt{c^2+\zeta r^2 c\sqrt{2}/a - \zeta^2 r^2\left(1-[r/(2a)]^2\right)}}, \qquad \Delta = \frac{1}{rc\sqrt{1+[r/(2a)]^2}}. \tag{2.10.9b}$$

Transformation between local direction $y^{(i)}$ and coordinate direction y^μ:

$$y^0 = y^{(0)}\Gamma + y^{(2)}\Delta\Gamma A, \quad y^1 = y^{(1)}\sqrt{1+[r/(2a)]^2}, \quad y^2 = y^{(0)}\Gamma\zeta + y^{(2)}\Delta\Gamma B, \quad y^3 = y^{(3)}. \tag{2.10.10}$$

with the above abbreviations.

2.10.2 Scaled cylindrical coordinates

If we apply the simple transformation

$$T = \frac{t}{r_G}, \qquad R = \frac{r}{r_G}, \qquad \phi = \varphi, \qquad Z = \frac{z}{r_G}, \tag{2.10.11}$$

with $r_G = 2a$, we find a formulation for the metric scaling with r_G, which is

$$ds^2 = r_G^2\left(-c^2dT^2 + \frac{dR^2}{1+R^2} + R^2(1-R^2)D\phi^2 + dZ^2 - 2\sqrt{2}cR^2\,dT\,d\phi\right). \tag{2.10.12}$$

Christoffel symbols:

$$\Gamma^T_{TR} = \frac{2R}{1+R^2}, \qquad \Gamma^\phi_{TR} = -\frac{\sqrt{2}c}{R(1+R^2)}, \tag{2.10.13a}$$

$$\Gamma^R_{T\phi} = \sqrt{2}cR(1+R^2), \qquad \Gamma^R_{RR} = -\frac{R}{1+R^2}, \tag{2.10.13b}$$

$$\Gamma^T_{R\phi} = \frac{\sqrt{2}R^3}{c(1+R^2)}, \qquad \Gamma^\phi_{R\phi} = \frac{1}{R(1+R^2)}, \tag{2.10.13c}$$

$$\Gamma^R_{\phi\phi} = R(1+R^2)(2R^2-1). \tag{2.10.13d}$$

Riemann-Tensor:

$$R_{TRTR} = \frac{2r_G^2c^2}{1+R^2}, \qquad R_{TRR\phi} = -\frac{2\sqrt{2}r_G^2cR^2}{1+R^2}, \tag{2.10.14a}$$

$$R_{T\phi T\phi} = 2c^2r_G^2R^2(1+R^2), \quad R_{R\phi R\phi} = \frac{2r_G^2R^2(1+3R^2)}{1+R^2}. \tag{2.10.14b}$$

Ricci-Tensor:

$$R_{TT} = 4c^2, \quad R_{T\phi} = 4\sqrt{2}cR^2, \quad R_{\phi\phi} = 8R^4. \tag{2.10.15}$$

Ricci and Kretschman scalar

$$\mathscr{R} = -\frac{4}{r_G^2}, \qquad \mathscr{K} = \frac{48}{r_G^4}. \tag{2.10.16}$$

cosmological constant:

$$\Lambda = \frac{R}{2} \tag{2.10.17}$$

Killing vectors:
The Killing vectors read

$$\underset{a}{\xi}^{\mu} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underset{b}{\xi}^{\mu} = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} \frac{R}{\sqrt{2}c}\cos\varphi \\ \frac{1}{2}(1+R^2)\sin\varphi \\ \frac{1}{2R}(1+2R^2)\cos\varphi \\ 0 \end{pmatrix}, \quad \underset{c}{\xi}^{\mu} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{2.10.18a}$$

$$\underset{d}{\xi}^{\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \underset{e}{\xi}^{\mu} = \frac{1}{\sqrt{1+R^2}} \begin{pmatrix} \frac{R}{\sqrt{2}c}\sin\varphi \\ -\frac{1}{2}(1+R^2)\cos\varphi \\ \frac{1}{2R}(1+2R^2)\sin\varphi \\ 0 \end{pmatrix}. \tag{2.10.18b}$$

Local tetrad:
After the transformation to scaled cylindrical coordinates, the local tetrad reads

$$\mathbf{e}_{(0)} = \frac{\Gamma}{r_G}\left(\partial_T + \zeta\partial_\phi\right), \quad \mathbf{e}_{(1)} = \frac{1}{r_G}\sqrt{1+R^2}\,\partial_R, \qquad \mathbf{e}_{(2)} = \frac{\Delta\Gamma}{r_G}\left(A\partial_T + B\partial_\phi\right), \quad \mathbf{e}_{(3)} = \frac{1}{r_G}\,\partial_Z, \tag{2.10.19a}$$

where

$$A = R^2\left[-\sqrt{2}c + (1-R^2)\zeta\right], \qquad B = c^2 + \sqrt{2}R^2 c\zeta, \tag{2.10.20a}$$

$$\Gamma = \frac{1}{\sqrt{c^2 + 2\sqrt{2}R^2 c\zeta - R^2(1-R^2)\zeta^2}}, \qquad \Delta = \frac{1}{Rc\sqrt{1+R^2}}. \tag{2.10.20b}$$

Transformation between local direction $y^{(i)}$ and coordinate direction y^{μ}:

$$y^0 = \frac{\Gamma}{r_G}y^{(0)} + \frac{\Delta\Gamma A}{r_G}y^{(2)}, \qquad y^1 = \frac{1}{r_G}\sqrt{1+R^2}y^{(1)}, \qquad y^2 = \frac{\Gamma\zeta}{r_G}y^{(0)} + \frac{\Delta\Gamma B}{r_G}y^{(2)}, \qquad y^3 = \frac{1}{r_G}y^{(3)}, \tag{2.10.21}$$

and the back transformation is given by

$$y^{(0)} = \frac{r_G}{\Gamma}\frac{By^0 - Ay^2}{B - \zeta A}, \qquad y^{(1)} = \frac{r_G}{\sqrt{1+R^2}}y^1, \qquad y^{(2)} = \frac{r_G}{\Delta\Gamma}\frac{y^2 - \zeta y^0}{B - \zeta A}, \qquad y^{(3)} = r_G y^3. \tag{2.10.22a}$$

2.11 De-Sitter spacetime

The de-Sitter spacetime describes a spherically symmetric spacetime that satisfies the Einstein field equations with $\Lambda \neq 0$. For further reading see also [Tol34, sec. 142]

2.11.1 Standard-Form

The de-Sitter spacetime in its original form reads:

$$ds^2 = -\left(1-\frac{\Lambda}{3}r^2\right)c^2dt^2 + \left(1-\frac{\Lambda}{3}r^2\right)^{-1}dr^2 + r^2\left(d\vartheta^2 + \sin(\vartheta)^2 d\varphi^2\right). \tag{2.11.1}$$

Christoffel symbols:

$$\begin{aligned}
&\Gamma^r_{tt} = \frac{(\Lambda r^2-3)}{9}c^2\Lambda r, && \Gamma^t_{tr} = \frac{\Lambda r}{\Lambda r^2-3}, && \Gamma^r_{rr} = \frac{\Lambda r}{3-\Lambda r^2},\\
&\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, && \Gamma^\phi_{r\phi} = \frac{1}{r}, && \Gamma^r_{\vartheta\vartheta} = \frac{(\Lambda r^2-3)r}{3},\\
&\Gamma^\phi_{\vartheta\phi} = \cot(\vartheta), && \Gamma^r_{\phi\phi} = \frac{\Lambda r^2-3}{3}r\sin^2(\vartheta), && \Gamma^\vartheta_{\phi\phi} = -\sin(\vartheta)\cos(\vartheta).
\end{aligned} \tag{2.11.2}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{\Lambda}{3}c^2, \qquad R_{t\vartheta t\vartheta} = -\frac{3-\Lambda r^2}{9}c^2\Lambda r^2, \qquad R_{t\varphi t\varphi} = -\frac{3-\Lambda r^2}{9}c^2\Lambda r^2\sin(\vartheta)^2, \tag{2.11.3a}$$

$$R_{r\vartheta r\vartheta} = \frac{\Lambda r^2}{-\Lambda r^2+3}, \qquad R_{r\varphi r\varphi} = \frac{\Lambda r^2\sin(\theta)^2}{-\Lambda r^2+3}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{r^4\sin^2(\theta)\Lambda}{3}. \tag{2.11.3b}$$

Ricci-Tensor:

$$R_{tt} = \frac{\Lambda r^2-3}{3}c^2\Lambda, \qquad R_{rr} = \frac{3\Lambda}{3-\Lambda r^2}, \qquad R_{\vartheta\vartheta} = \Lambda r^2, \qquad R_{\varphi\varphi} = r^2\sin^2(\vartheta)\Lambda. \tag{2.11.4}$$

The *Ricci scalar and Kretschman scalar read:*

$$\mathscr{R} = 4\Lambda, \qquad \mathscr{K} = \frac{8}{3}\Lambda^2. \tag{2.11.5}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{\sqrt{\frac{3}{3-\Lambda r^2}}}{c}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{1-\frac{\Lambda r^2}{3}}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin(\vartheta)}\partial_\varphi. \tag{2.11.6}$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(t)} = -\frac{\Lambda r}{\sqrt{9-3\Lambda r^2}}, \qquad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{\sqrt{9-3\Lambda r^2}}{3r}, \qquad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.11.7}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{\sqrt{9-3\Lambda r^2}(\Lambda r^2-2)}{(\Lambda r^2-3)r}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.11.8}$$

Riemann-Tensor with respect to local tetrad:

$$-R_{(t)(r)(t)(r)} = -R_{(t)(\vartheta)(t)(\vartheta)} = -R_{(t)(\varphi)(t)(\varphi)} = R_{(r)(\vartheta)(r)(\vartheta)} = R_{(r)(\varphi)(r)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{1}{3}\Lambda. \tag{2.11.9}$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \Lambda. \tag{2.11.10}$$

2.11.2 Lemaître-Robinson

The de-Sitter spacetime in the form found by Lemaître and Robinson reads:

$$ds^2 = -c^2dt^2 + e^{2Ht}\left[dr^2 + r^2\left(d\vartheta^2 + \sin(\vartheta)^2 d\varphi^2\right)\right]. \tag{2.11.11}$$

with Hubble's Parameter $H = \sqrt{\frac{\Lambda c^2}{3}}$, which is assumed here to be time-independent.
This a special case of the first and second form of the Friedman-Robertson-Walker metric defined in equations 2.13.2 and 2.13.12 with $R(t) = e^{Ht}$ and $k = 0$.

Christoffel symbols:

$$\begin{aligned} \Gamma^r_{tr} &= H, & \Gamma^\vartheta_{t\vartheta} &= H, & \Gamma^\varphi_{t\varphi} &= H,\\ \Gamma^t_{rr} &= \frac{e^{2Ht}H}{c^2}, & \Gamma^\vartheta_{r\vartheta} &= \frac{1}{r}, & \Gamma^\varphi_{r\varphi} &= \frac{1}{r},\\ \Gamma^t_{\vartheta\vartheta} &= \frac{e^{2Ht}r^2H}{c^2}, & \Gamma^r_{\vartheta\vartheta} &= -r, & \Gamma^\varphi_{\vartheta\varphi} &= \cot(\vartheta),\\ \Gamma^t_{\varphi\varphi} &= \frac{e^{2Ht}r^2\sin^2(\theta)H}{c^2}, & \Gamma^r_{\varphi\varphi} &= -r\sin(\vartheta)^2, & \Gamma^\vartheta_{\varphi\varphi} &= -\sin(\vartheta)\cos(\vartheta). \end{aligned} \tag{2.11.12}$$

Riemann-Tensor:

$$\begin{aligned} R_{trtr} &= -e^{2Ht}H^2, & R_{t\vartheta t\vartheta} &= -e^{2Ht}r^2H^2,\\ R_{t\varphi t\varphi} &= -e^{2Ht}r^2\sin^2(\vartheta)H^2, & R_{r\vartheta r\vartheta} &= \frac{e^{4Ht}r^2H^2}{c^2},\\ R_{r\varphi r\varphi} &= \frac{e^{4Ht}r^2\sin^2(\vartheta)H^2}{c^2}, & R_{\vartheta\varphi\vartheta\varphi} &= \frac{e^{4Ht}r^4\sin^2(\vartheta)H^2}{c^2}. \end{aligned} \tag{2.11.13}$$

Ricci-Tensor:

$$R_{tt} = -3H^2, \quad R_{rr} = 3\frac{e^{2Ht}H^2}{c^2}, \quad R_{\vartheta\vartheta} = 3\frac{e^{2Ht}r^2H^2}{c^2}, \quad R_{\varphi\varphi} = 3\frac{e^{2Ht}r^2\sin^2(\vartheta)H^2}{c^2}. \tag{2.11.14}$$

The *Ricci scalar and Kretschman scalar read:*

$$\mathscr{R} = \frac{12H^2}{c^2}, \qquad \mathscr{K} = \frac{24H^4}{c^4}. \tag{2.11.15}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \quad \mathbf{e}_{(r)} = e^{-Ht}\partial_r, \quad \mathbf{e}_{(\vartheta)} = \frac{e^{-Ht}}{r}\partial_\vartheta, \quad \mathbf{e}_{(\varphi)} = \frac{e^{-Ht}}{r\sin\vartheta}\partial_\varphi. \tag{2.11.16}$$

Ricci rotation coefficients:

$$\begin{aligned} &\gamma_{(r)(t)(r)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{H}{c}\\ &\gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{1}{e^{Ht}r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot(\theta)}{e^{Ht}r}. \end{aligned} \tag{2.11.17}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = 3\frac{H}{c}, \qquad \gamma_{(r)} = \frac{2}{e^{Ht}r}, \gamma_{(\vartheta)} = \frac{\cot(\theta)}{e^{Ht}r}. \tag{2.11.18}$$

Riemann-Tensor with respect to local tetrad:

$$\begin{aligned} &R_{(t)(r)(t)(r)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{H^2}{c^2}\\ &R_{(r)(\vartheta)(r)(\vartheta)} = R_{(r)(\varphi)(r)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{H^2}{c^2}. \end{aligned} \tag{2.11.19}$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = 3\frac{H^2}{c^2}. \tag{2.11.20}$$

Lemaître-Robinson in cartesian coordinates

The Lemaître and Robinson Form of de-Sitter space can also be expressed using cartesian coordinates:

$$ds^2 = -c^2dt^2 + e^{2Ht}\left\{dx^2 + dy^2 + dz^2\right\} \tag{2.11.21}$$

The *Christoffel symbols:*

$$\begin{aligned}
&\Gamma^x_{tx} = H, && \Gamma^y_{ty} = H, && \Gamma^z_{tz} = H,\\
&\Gamma^t_{xx} = \frac{e^{2Ht}H}{c^2}, && \Gamma^t_{yy} = \frac{e^{2Ht}H}{c^2}, && \Gamma^t_{zz} = \frac{e^{2Ht}H}{c^2}.
\end{aligned} \tag{2.11.22}$$

The *Riemann-Tensor:*

$$\begin{aligned}
R_{txtx} &= R_{txtx} = R_{tztz} = -e^{2Ht}H^2,\\
R_{xyxy} &= R_{xzxz} = R_{yzyz} = \frac{e^{4Ht}H^2}{c^2}.
\end{aligned} \tag{2.11.23}$$

The *Ricci-Tensor:*

$$R_{tt} = -3H^2, \qquad R_{xx} = R_{yy} = R_{zz} = 3\frac{e^{2Ht}H^2}{c^2}. \tag{2.11.24}$$

The *Ricci scalar and Kretschman scalar read:*

$$\mathscr{R} = 12\frac{H^2}{c^2}, \qquad \mathscr{K} = 24\frac{H^4}{c^4}. \tag{2.11.25}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(x)} = e^{-Ht}\partial_x, \qquad e_{(y)} = e^{-Ht}\partial_y, \qquad e_{(z)} = e^{-Ht}\partial_z. \tag{2.11.26}$$

Ricci rotation coefficients:

$$\gamma_{(x)(t)(x)} = \gamma_{(y)(t)(y)} = \gamma_{(z)(t)(z)} = \frac{H}{c}. \tag{2.11.27}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = 3\frac{H}{c}. \tag{2.11.28}$$

Riemann-Tensor with respect to local tetrad:

$$\begin{aligned}
R_{(t)(x)(t)(x)} &= R_{(t)(y)(t)(y)} = R_{(t)(z)(t)(z)} = -\frac{H^2}{c^2}\\
R_{(x)(y)(x)(y)} &= R_{(x)(z)(x)(z)} = R_{(y)(z)(y)(z)} = \frac{H^2}{c^2}.
\end{aligned} \tag{2.11.29}$$

Ricci-Tensor with respect to local tetrad:

$$-R_{(t)(t)} = R_{(x)(x)} = R_{(y)(y)} = R_{(z)(z)} = 3\frac{H^2}{c^2}. \tag{2.11.30}$$

2.12 Kottler spacetime

The Kottler (Schwarzschild-deSitter) spacetime is represented in spherical coordinates $(t, r, \vartheta, \varphi)$ by the line element

$$ds^2 = -\left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}\right) c^2 dt^2 + \frac{1}{1 - r_s/r - \Lambda r^2/3} dr^2 + r^2 d\Omega^2, \tag{2.12.1}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, G is Newton's constant, c is the speed of light, M is the mass of the black hole, and Λ is the cosmological constant.
For the following, we define the two abbreviations

$$\alpha = 1 - \frac{r_s}{r} - \frac{\Lambda r^2}{r} \qquad \text{and} \qquad \beta = \frac{r_s}{r} - \frac{2\Lambda}{3} r^2. \tag{2.12.2}$$

Christoffel symbols:

$$\Gamma^r_{tt} = \frac{c^2 \alpha\beta}{2r}, \qquad \Gamma^t_{tr} = \frac{\beta}{2r\alpha}, \qquad \Gamma^r_{rr} = -\frac{\beta}{2r\alpha)}, \tag{2.12.3a}$$

$$\Gamma^\vartheta_{r\vartheta} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \frac{1}{r}, \qquad \Gamma^r_{\vartheta\vartheta} = -\alpha r, \tag{2.12.3b}$$

$$\Gamma^\varphi_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^r_{\varphi\varphi} = -\alpha r \sin^2\vartheta, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.12.3c}$$

Riemann-Tensor:

$$R_{trtr} = -\frac{c^2\left(3r_s + \Lambda r^3\right)}{3r^3}, \qquad R_{t\vartheta t\vartheta} = \frac{1}{2} c^2 \alpha\beta, \tag{2.12.4a}$$

$$R_{t\varphi t\varphi} = \frac{1}{2} c^2 \alpha\beta \sin^2\vartheta, \qquad R_{r\vartheta r\vartheta} = -\frac{\beta}{2\alpha}, \tag{2.12.4b}$$

$$R_{r\varphi r\varphi} = -\frac{\beta}{2\alpha}\sin^2\vartheta, \qquad R_{\vartheta\varphi\vartheta\varphi} = r\left(r_s + \frac{\Lambda r^3}{3}\right)\sin^2\vartheta. \tag{2.12.4c}$$

Ricci-Tensor:

$$R_{tt} = -c^2\alpha\Lambda, \qquad R_{rr} = \frac{\Lambda}{\alpha}, \qquad R_{\vartheta\vartheta} = \Lambda r^2, \qquad R_{\varphi\varphi} = \Lambda r^2 \sin^2\vartheta. \tag{2.12.5}$$

The *Ricci scalar* and the *Kretschman scalar* read

$$\mathscr{R} = 4\Lambda, \qquad \mathscr{K} = 12\frac{r_s^2}{r^6} + \frac{8\Lambda^2}{3}. \tag{2.12.6}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{c^2 r_s}{r^3}, \qquad C_{t\vartheta t\vartheta} = \frac{c^2 \alpha r_s}{2r}, \qquad C_{t\varphi t\varphi} = \frac{c^2 \alpha r_s \sin^2\vartheta}{2r}, \tag{2.12.7a}$$

$$C_{r\vartheta r\vartheta} = -\frac{r_s}{2r\alpha}, \qquad C_{r\varphi r\varphi} = -\frac{r_s \sin^2\vartheta}{2r\alpha}, \qquad C_{\vartheta\varphi\vartheta\varphi} = r r_s \sin^2\vartheta. \tag{2.12.7b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c\sqrt{\alpha}}\partial_t, \qquad \mathbf{e}_{(r)} = \sqrt{\alpha}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \frac{1}{r}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{1}{r\sin\vartheta}\partial_\varphi. \tag{2.12.8}$$

Dual tetrad:

$$\theta^{(t)} = c\sqrt{\alpha}\,dt, \qquad \theta^{(r)} = \frac{dr}{\sqrt{\alpha}}, \qquad \theta^{(\vartheta)} = r\,d\vartheta, \qquad \theta^{(\varphi)} = r\sin\vartheta\,d\varphi. \tag{2.12.9}$$

Ricci rotation coefficients:

$$\gamma_{(r)(t)(t)} = \frac{r_s - \frac{2}{3}\Lambda r^3}{2r^2\sqrt{\alpha}}, \quad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = \frac{\sqrt{\alpha}}{r}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{r}. \tag{2.12.10}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(r)} = \frac{4r - 3r_s - 2\Lambda r^3}{2r^2\sqrt{\alpha}}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{r}. \tag{2.12.11}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{\Lambda r^3 + 3r_s}{3r^3}, \tag{2.12.12a}$$

$$R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -R_{(r)(\vartheta)(r)(\vartheta)} = -R_{(r)(\varphi)(r)(\varphi)} = \frac{3r_s - 2\Lambda r^3}{6r^3}. \tag{2.12.12b}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{r_s}{r^3}, \tag{2.12.13a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{r_s}{2r^3}. \tag{2.12.13b}$$

Further reading:
Kottler[Kot18], Weyl[Wey19], Hackmann[HL08]

2.13 Friedmann-Robertson-Walker

The Friedmann-Robertson-Walker metric describes a general homogeneous and isotropic universe. In a general form it reads:

$$ds^2 = -c^2dt^2 + R^2 d\sigma^2 \tag{2.13.1}$$

With $R = R(t)$ being an arbitrary function of time only and $d\sigma^2$ being a metric of a 3-space of constant curvature, for which three explicit forms will be described here.
In all formulas in this section a dot denotes differentiation with respect to t, e.g. $\dot{R} = dR(t)/dt$.

2.13.1 Form 1

$$ds^2 = -c^2dt^2 + R^2\left\{\frac{d\eta^2}{1-k\eta^2} + \eta^2\left(d\vartheta^2 + \sin(\vartheta)^2 d\varphi^2\right)\right\} \tag{2.13.2}$$

Christoffel symbols:

$$\begin{aligned}
&\Gamma^{\eta}_{t\eta} = \frac{\dot{R}}{R}, && \Gamma^{\vartheta}_{t\vartheta} = \frac{\dot{R}}{R}, && \Gamma^{\varphi}_{t\varphi} = \frac{\dot{R}}{R},\\
&\Gamma^{t}_{\eta\eta} = \frac{R\dot{R}}{c^2(1-k\eta^2)}, && \Gamma^{\eta}_{\eta\eta} = \frac{k\eta}{1-k\eta^2}, && \Gamma^{\vartheta}_{\eta\vartheta} = \frac{1}{\eta},\\
&\Gamma^{\varphi}_{\eta\varphi} = \frac{1}{\eta}, && \Gamma^{t}_{\vartheta\vartheta} = \frac{R\eta^2\dot{R}}{c^2}, && \Gamma^{\eta}_{\vartheta\vartheta} = (k\eta^2-1)\eta,\\
&\Gamma^{\varphi}_{\vartheta\varphi} = \cot(\vartheta), && \Gamma^{t}_{\varphi\varphi} = \frac{R\eta^2\sin(\vartheta)^2\dot{R}}{c^2}, && \Gamma^{\eta}_{\varphi\varphi} = (k\eta^2-1)\eta\sin(\vartheta)^2,\\
&\Gamma^{\vartheta}_{\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta).
\end{aligned} \tag{2.13.3}$$

Riemann-Tensor:

$$\begin{aligned}
&R_{t\eta t\eta} = \frac{R\ddot{R}}{k\eta^2-1}, && R_{t\vartheta t\vartheta} = -R\eta^2\ddot{R},\\
&R_{t\varphi t\varphi} = -R\eta^2\sin^2(\vartheta)\ddot{R}, && R_{\eta\vartheta\eta\vartheta} = -\frac{R^2\eta^2\left(\dot{R}^2+kc^2\right)}{c^2(k\eta^2-1)},\\
&R_{\eta\varphi\eta\varphi} = -\frac{R^2\eta^2\sin^2(\vartheta)\left(\dot{R}^2+kc^2\right)}{c^2(k\eta^2-1)}, && R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\eta^4\sin^2(\vartheta)\left(\dot{R}^2+kc^2\right)}{c^2}.
\end{aligned} \tag{2.13.4}$$

Ricci-Tensor:

$$\begin{aligned}
&R_{tt} = -3\frac{\ddot{R}}{R}, && R_{\eta\eta} = \frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2(1-k\eta^2)},\\
&R_{\vartheta\vartheta} = \eta^2\frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2}, && R_{\varphi\varphi} = \eta^2\sin(\vartheta)^2\frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2}.
\end{aligned} \tag{2.13.5}$$

The *Ricci scalar and Kretschman scalar read:*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2+kc^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4+2\dot{R}^2kc^2+k^2c^4}{R^4c^4}. \tag{2.13.6}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\eta)} = \frac{\sqrt{1-k\eta^2}}{R}\partial_\eta, \qquad e_\vartheta = \frac{1}{R\eta}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\eta\sin\vartheta}\partial_\varphi. \tag{2.13.7}$$

Ricci rotation coefficients:

$$\gamma_{(\eta)(t)(\eta)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\eta)(\vartheta)} = \gamma_{(\varphi)(\eta)(\varphi)} = \frac{\sqrt{1-k\eta^2}}{R\eta}, \quad \gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\vartheta}{R\eta}. \tag{2.13.8}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = \frac{2\sqrt{1-k\eta^2}}{R\eta}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\eta}. \tag{2.13.9}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\eta)(t)(\eta)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2} \quad R_{(\eta)(\vartheta)(\eta)(\vartheta)} = R_{(\eta)(\varphi)(\eta)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2+kc^2}{R^2c^2}. \tag{2.13.10}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2\dot{R}^2+2kc^2}{R^2c^2}. \tag{2.13.11}$$

2.13.2 Form 2

$$ds^2 = -c^2dt^2 + \frac{R^2}{(1+\frac{k}{4}r^2)^2}\left\{dr^2 + r^2(d\vartheta^2 + \sin(\vartheta)^2 d\varphi^2)\right\} \tag{2.13.12}$$

Christoffel symbols:

$$\Gamma^r_{tr} = \frac{\dot{R}}{R}, \qquad \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, \qquad \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R},$$
$$\Gamma^t_{rr} = 16\frac{R\dot{R}}{c^2(4+kr^2)^2}, \qquad \Gamma^r_{rr} = -\frac{2kr}{4+kr^2}, \qquad \Gamma^\vartheta_{r\vartheta} = \frac{4-kr^2}{(4+kr^2)r},$$
$$\Gamma^\varphi_{r\varphi} = \frac{4-kr^2}{(4+kr^2)r}, \qquad \Gamma^t_{\vartheta\vartheta} = 16\frac{Rr^2\dot{R}}{c^2(4+kr^2)^2}, \qquad \Gamma^r_{\vartheta\vartheta} = \frac{r(kr^2-4)}{4+kr^2}, \tag{2.13.13}$$
$$\Gamma^\varphi_{\vartheta\varphi} = \cot(\vartheta), \qquad \Gamma^t_{\varphi\varphi} = 16\frac{Rr^2\sin(\vartheta)^2\dot{R}}{c^2(4+kr^2)^2}, \qquad \Gamma^r_{\varphi\varphi} = \frac{r\sin(\vartheta)^2(kr^2-4)}{4+kr^2},$$
$$\Gamma^\vartheta_{\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta).$$

Riemann-Tensor:

$$R_{trtr} = -16\frac{R\ddot{R}}{(4+kr^2)^2}, \qquad R_{t\vartheta t\vartheta} = -16\frac{Rr^2\ddot{R}}{(4+kr^2)^2}, \tag{2.13.14a}$$

$$R_{t\varphi t\varphi} = -16\frac{Rr^2\sin^2(\vartheta)\ddot{R}}{(4+kr^2)^2}, \qquad R_{r\vartheta r\vartheta} = 256\frac{R^2r^2\left(\dot{R}^2+kc^2\right)}{c^2(4+kr^2)^4}, \tag{2.13.14b}$$

$$R_{r\varphi r\varphi} = 256\frac{R^2r^2\sin^2(\vartheta)\left(\dot{R}^2+kc^2\right)}{c^2(4+kr^2)^4}, \qquad R_{\vartheta\varphi\vartheta\varphi} = 256\frac{R^2r^4\sin(\vartheta)^2\left(\dot{R}^2+kc^2\right)}{c^2(4+kr^2)^4}. \tag{2.13.14c}$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad R_{rr} = 16\frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2(4+kr^2)^2},$$
$$R_{\vartheta\vartheta} = 16r^2\frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2(4+kr^2)^2}, \qquad R_{\varphi\varphi} = 16r^2\sin(\vartheta)^2\frac{R\ddot{R}+2(\dot{R}^2+kc^2)}{c^2(4+kr^2)^2}. \tag{2.13.15}$$

The *Ricci scalar and Kretschman scalar read:*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2+kc^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4+2\dot{R}^2kc^2+k^2c^4}{R^4c^4}. \tag{2.13.16}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(r)} = \frac{1+\frac{k}{4}r^2}{R}\partial_r, \qquad e_\vartheta = \frac{1+\frac{k}{4}r^2}{Rr}\partial_\vartheta, \qquad e_\varphi = \frac{1+k/4r^2}{Rr\sin\vartheta}\partial_\varphi. \tag{2.13.17}$$

Ricci rotation coefficients:

$$\begin{aligned} &\gamma_{(r)(t)(r)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(r)(\vartheta)} = \gamma_{(\varphi)(r)(\varphi)} = -\frac{\frac{k}{4}r^2-1}{Rr}, \\ &\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{(\frac{k}{4}r^2+1)\cot\vartheta}{Rr}. \end{aligned} \tag{2.13.18}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = 2\frac{1-\frac{k}{4}r^2}{Rr}, \qquad \gamma_{(\vartheta)} = \frac{(\frac{k}{4}r^2+1)\cot\vartheta}{Rr}. \tag{2.13.19}$$

Riemann-Tensor with respect to local tetrad:

$$\begin{aligned} &R_{(t)(\eta)(t)(\eta)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2} \\ &R_{(\eta)(\vartheta)(\eta)(\vartheta)} = R_{(\eta)(\varphi)(\eta)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2+kc^2}{R^2c^2}. \end{aligned} \tag{2.13.20}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(r)(r)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2\dot{R}^2+2kc^2}{R^2c^2}. \tag{2.13.21}$$

2.13.3 Form 3

The following forms of the metric are obtained from 2.13.2 by setting $\eta = \sin(\psi), \psi, \sinh(\psi)$ for $k = 1, 0, -1$ respectively.

Positive Curvature

$$ds^2 = -c^2dt^2 + R^2\left\{d\psi^2 + \sin(\psi)^2\left(d\vartheta^2 + \sin(\vartheta)^2 d\varphi^2\right)\right\} \tag{2.13.22}$$

Christoffel symbols:

$$\begin{aligned} &\Gamma^\psi_{t\psi} = \frac{\dot{R}}{R}, && \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, && \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R}, \\ &\Gamma^t_{\psi\psi} = \frac{R\dot{R}}{c^2}, && \Gamma^\vartheta_{\psi\vartheta} = \cot(\psi), && \Gamma^\varphi_{\psi\varphi} = \cot(\psi), \\ &\Gamma^t_{\vartheta\vartheta} = \frac{R\sin(\psi)^2\dot{R}}{c^2}, && \Gamma^\psi_{\vartheta\vartheta} = -\sin(\psi)\cos(\psi), && \Gamma^\varphi_{\vartheta\varphi} = \cot(\vartheta), \\ &\Gamma^t_{\varphi\varphi} = \frac{R\sin(\psi)^2\sin(\vartheta)^2\dot{R}}{c^2}, && \Gamma^\psi_{\varphi\varphi} = -\sin(\psi)\cos(\psi)\sin(\vartheta)^2, && \Gamma^\vartheta_{\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta). \end{aligned} \tag{2.13.23}$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \qquad R_{t\vartheta t\vartheta} = -R\sin(\psi)^2\ddot{R},$$
$$R_{t\varphi t\varphi} = -R\sin(\psi)^2\sin^2(\vartheta)\ddot{R}, \qquad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\sin(\psi)^2\left(\dot{R}^2+c^2\right)}{c^2}, \tag{2.13.24}$$
$$R_{\psi\varphi\psi\varphi} = \frac{R^2\sin(\psi)^2\sin^2(\vartheta)\left(\dot{R}^2+c^2\right)}{c^2}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\sin(\psi)^4\sin^2(\vartheta)\left(\dot{R}^2+c^2\right)}{c^2}.$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad R_{\psi\psi} = \frac{R\ddot{R}+2(\dot{R}^2+c^2)}{c^2},$$
$$R_{\vartheta\vartheta} = \sin(\psi)^2\frac{R\ddot{R}+2(\dot{R}^2+c^2)}{c^2}, \qquad R_{\varphi\varphi} = \sin(\vartheta)^2\sin(\psi)^2\frac{R\ddot{R}+2(\dot{R}^2+c^2)}{c^2}. \tag{2.13.25}$$

The *Ricci scalar and Kretschman read*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2+c^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4+2\dot{R}^2c^2+c^4}{R^4c^4}. \tag{2.13.26}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\psi)} = \frac{1}{R}\partial_\psi, \qquad e_\vartheta = \frac{1}{R\sin(\psi)}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\sin(\psi)\sin\vartheta}\partial_\varphi. \tag{2.13.27}$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{\cot\psi}{R},$$
$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\theta}{R\sin\psi}. \tag{2.13.28}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = 2\frac{\cot\psi}{R}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\sin\psi}. \tag{2.13.29}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2},$$
$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2+c^2}{R^2c^2}. \tag{2.13.30}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2(\dot{R}^2+c^2)}{R^2c^2}. \tag{2.13.31}$$

Vanishing Curvature

$$ds^2 = -c^2dt^2 + R^2\left\{d\psi^2 + \psi^2\left(d\vartheta^2 + \sin(\vartheta)^2d\varphi^2\right)\right\} \tag{2.13.32}$$

The *Christoffel symbols:*

$$\Gamma^\psi_{t\psi} = \frac{\dot{R}}{R}, \qquad \Gamma^\vartheta_{t\vartheta} = \frac{\dot{R}}{R}, \qquad \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R},$$
$$\Gamma^t_{\psi\psi} = \frac{R\dot{R}}{c^2}, \qquad \Gamma^\vartheta_{\psi\vartheta} = \frac{1}{\psi}, \qquad \Gamma^\varphi_{\psi\varphi} = \frac{1}{\psi},$$
$$\Gamma^t_{\vartheta\vartheta} = \frac{R\psi^2\dot{R}}{c^2}, \qquad \Gamma^\psi_{\vartheta\vartheta} = -\psi, \qquad \Gamma^\varphi_{\vartheta\varphi} = \cot(\vartheta), \tag{2.13.33}$$
$$\Gamma^t_{\varphi\varphi} = \frac{R\psi^2\sin(\vartheta)^2\dot{R}}{c^2}, \qquad \Gamma^\psi_{\varphi\varphi} = -\psi\sin(\vartheta)^2, \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta).$$

Riemann-Tensor:

$$\begin{aligned} R_{t\psi t\psi} &= -R\ddot{R}, & R_{t\vartheta t\vartheta} &= -R\psi^2\ddot{R}, \\ R_{t\varphi t\varphi} &= -R\psi^2\sin^2(\vartheta)\ddot{R}, & R_{\psi\vartheta\psi\vartheta} &= \frac{R^2\psi^2\dot{R}^2}{c^2}, \\ R_{\psi\varphi\psi\varphi} &= \frac{R^2\psi^2\sin^2(\vartheta)\dot{R}^2}{c^2}, & R_{\vartheta\varphi\vartheta\varphi} &= \frac{R^2\psi^4\sin^2(\vartheta)\dot{R}^2}{c^2}. \end{aligned} \tag{2.13.34}$$

Ricci-Tensor:

$$\begin{aligned} R_{tt} &= -3\frac{\ddot{R}}{R}, & R_{\psi\psi} &= \frac{R\ddot{R}+2\dot{R}^2}{c^2}, \\ R_{\vartheta\vartheta} &= \psi^2\frac{R\ddot{R}+2\dot{R}^2}{c^2}, & R_{\varphi\varphi} &= \sin(\vartheta)^2\psi^2\frac{R\ddot{R}+2\dot{R}^2}{c^2}. \end{aligned} \tag{2.13.35}$$

The *Ricci scalar and Kretschman read*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4}{R^4c^4}. \tag{2.13.36}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\psi)} = \frac{1}{R}\partial_\psi, \qquad e_\vartheta = \frac{1}{R\psi}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\psi\sin\vartheta}\partial_\varphi. \tag{2.13.37}$$

Ricci rotation coefficients:

$$\begin{aligned} &\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{1}{R\psi}, \\ &\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot(\vartheta)}{R\psi}. \end{aligned} \tag{2.13.38}$$

The contractions of the Ricci rotation coefficients read

$$\begin{aligned} &\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = \frac{2}{R\psi}, \\ &\gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\psi}. \end{aligned} \tag{2.13.39}$$

Riemann-Tensor with respect to local tetrad:

$$\begin{aligned} &R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2}, \\ &R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2}{R^2c^2}. \end{aligned} \tag{2.13.40}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2\dot{R}^2}{R^2c^2}. \tag{2.13.41}$$

Negative Curvature

$$ds^2 = -c^2dt^2 + R^2\left\{d\psi^2 + \sinh(\psi)^2\left(d\vartheta^2 + \sin(\vartheta)^2d\varphi^2\right)\right\} \tag{2.13.42}$$

Christoffel symbols:

$$\Gamma^{\psi}_{t\psi} = \frac{\dot{R}}{R}, \qquad \Gamma^{\vartheta}_{t\vartheta} = \frac{\dot{R}}{R}, \qquad \Gamma^{\varphi}_{t\varphi} = \frac{\dot{R}}{R},$$
$$\Gamma^{t}_{\psi\psi} = \frac{R\dot{R}}{c^2}, \qquad \Gamma^{\vartheta}_{\psi\vartheta} = \coth(\psi), \qquad \Gamma^{\varphi}_{\psi\varphi} = \coth(\psi),$$
$$\Gamma^{t}_{\vartheta\vartheta} = \frac{R\sinh(\psi)^2\dot{R}}{c^2}, \qquad \Gamma^{\psi}_{\vartheta\vartheta} = -\sinh(\psi)\cosh(\psi), \qquad \Gamma^{\varphi}_{\vartheta\varphi} = \cot(\vartheta),$$
$$\Gamma^{t}_{\varphi\varphi} = \frac{R\sinh(\psi)^2\sin(\vartheta)^2\dot{R}}{c^2}, \qquad \Gamma^{\psi}_{\varphi\varphi} = -\sinh(\psi)\cosh(\psi)\sin(\vartheta)^2, \qquad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin(\vartheta)\cos(\vartheta). \tag{2.13.43}$$

Riemann-Tensor:

$$R_{t\psi t\psi} = -R\ddot{R}, \qquad R_{t\vartheta t\vartheta} = -R\sinh(\psi)^2\ddot{R},$$
$$R_{t\varphi t\varphi} = -R\sinh(\psi)^2\sin^2(\vartheta)\ddot{R}, \qquad R_{\psi\vartheta\psi\vartheta} = \frac{R^2\sinh(\psi)^2\left(\dot{R}^2-c^2\right)}{c^2}, \tag{2.13.44}$$
$$R_{\psi\varphi\psi\varphi} = \frac{R^2\sinh(\psi)^2\sin^2(\vartheta)\left(\dot{R}^2-c^2\right)}{c^2}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{R^2\sinh(\psi)^4\sin^2(\vartheta)\left(\dot{R}^2-c^2\right)}{c^2}.$$

Ricci-Tensor:

$$R_{tt} = -3\frac{\ddot{R}}{R}, \qquad R_{\psi\psi} = \frac{R\ddot{R}+2(\dot{R}^2-c^2)}{c^2},$$
$$R_{\vartheta\vartheta} = \sinh(\psi)^2\frac{R\ddot{R}+2(\dot{R}^2-c^2)}{c^2}, \qquad R_{\varphi\varphi} = \sin(\vartheta)^2\sin(\psi)^2\frac{R\ddot{R}+2(\dot{R}^2-c^2)}{c^2}. \tag{2.13.45}$$

The *Ricci scalar and Kretschman read*

$$\mathscr{R} = 6\frac{R\ddot{R}+\dot{R}^2-c^2}{R^2c^2}, \qquad \mathscr{K} = 12\frac{\ddot{R}^2R^2+\dot{R}^4-2\dot{R}^2c^2+c^4}{R^4c^4}. \tag{2.13.46}$$

Local tetrad:

$$e_{(t)} = \frac{1}{c}\partial_t, \qquad e_{(\psi)} = \frac{1}{R}\partial_\psi, \qquad e_\vartheta = \frac{1}{R\sinh(\psi)}\partial_\vartheta, \qquad e_\varphi = \frac{1}{R\sinh(\psi)\sin\vartheta}\partial_\varphi. \tag{2.13.47}$$

Ricci rotation coefficients:

$$\gamma_{(\psi)(t)(\psi)} = \gamma_{(\vartheta)(t)(\vartheta)} = \gamma_{(\varphi)(t)(\varphi)} = \frac{\dot{R}}{Rc} \qquad \gamma_{(\vartheta)(\psi)(\vartheta)} = \gamma_{(\varphi)(\psi)(\varphi)} = \frac{\coth\psi}{R},$$
$$\gamma_{(\varphi)(\vartheta)(\varphi)} = \frac{\cot\theta}{R\sinh\psi}. \tag{2.13.48}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = \frac{3\dot{R}}{Rc}, \qquad \gamma_{(r)} = 2\frac{\coth\psi}{R}, \qquad \gamma_{(\vartheta)} = \frac{\cot\vartheta}{R\sinh\psi}. \tag{2.13.49}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(\psi)(t)(\psi)} = R_{(t)(\vartheta)(t)(\vartheta)} = R_{(t)(\varphi)(t)(\varphi)} = -\frac{\ddot{R}}{Rc^2},$$
$$R_{(\psi)(\vartheta)(\psi)(\vartheta)} = R_{(\psi)(\varphi)(\psi)(\varphi)} = R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = \frac{\dot{R}^2-c^2}{R^2c^2}. \tag{2.13.50}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -\frac{3\ddot{R}}{Rc^2}, \qquad R_{(\psi)(\psi)} = R_{(\vartheta)(\vartheta)} = R_{(\varphi)(\varphi)} = \frac{R\ddot{R}+2(\dot{R}^2-c^2)}{R^2c^2}. \tag{2.13.51}$$

Further reading:
Rindler[Rin01]

2.14 Straight spinning string

The metric of a straight spinning string in cylindrical coordinates (t, ρ, φ, z) reads[Per04]

$$ds^2 = -(c\,dt - a\,d\varphi)^2 + d\rho^2 + k^2\rho^2 d\varphi^2 + dz^2, \tag{2.14.1}$$

where a and k are two parameters.
Christoffel symbols:

$$\Gamma^t_{\rho\varphi} = \frac{a}{c\rho}, \qquad \Gamma^\varphi_{\rho\varphi} = \frac{1}{\rho}, \qquad \Gamma^\rho_{\varphi\varphi} = -k^2\rho. \tag{2.14.2}$$

The Riemann-, Ricci-, and Weyl-tensors as well as the Ricci- and Kretschman-scalar vanish identically.
Static Local tetrad:

$$\mathbf{e}_{(0)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(1)} = \partial_\rho, \qquad \mathbf{e}_{(2)} = \frac{1}{k\rho}\left(\frac{a}{c}\partial_t + \partial_\varphi\right), \qquad \mathbf{e}_{(3)} = \partial_z. \tag{2.14.3}$$

Dual tetrad:

$$\theta^{(0)} = c\,dt - a\,d\varphi, \qquad \theta^{(1)} = d\rho, \qquad \theta^{(2)} = k\rho\,d\varphi, \qquad \theta^{(3)} = dz. \tag{2.14.4}$$

Ricci rotation coefficients and their contractions read

$$\gamma_{(2)(1)(2)} = \frac{1}{\rho}, \qquad \gamma_{(0)} = \gamma_{(2)} = \gamma_{(3)} = 0, \qquad \gamma_{(1)} = \frac{1}{\rho}. \tag{2.14.5}$$

Comoving local tetrad:

$$\mathbf{e}_{(0)} = \frac{\sqrt{k^2\rho^2 - a^2}}{k\rho}\left(\frac{1}{c}\partial_t - \frac{a}{k^2\rho^2 - a^2}\partial_\varphi\right), \qquad \mathbf{e}_{(1)} = \partial_\rho, \tag{2.14.6a}$$

$$\mathbf{e}_{(2)} = \frac{1}{\sqrt{k^2\rho^2 - a^2}}\partial_\varphi, \qquad \mathbf{e}_{(3)} = \partial_z. \tag{2.14.6b}$$

Dual tetrad:

$$\theta^{(0)} = \frac{k\rho}{\sqrt{k^2\rho^2 - a^2}}c\,dt, \qquad \theta^{(1)} = d\rho, \qquad \theta^{(2)} = \frac{ac\,dt}{\sqrt{k^2\rho^2 - a^2}} + \sqrt{k^2\rho^2 - a^2}d\varphi, \qquad \theta^{(3)} = dz. \tag{2.14.7}$$

Ricci rotation coefficients and their contractions read

$$\gamma_{(0)(1)(0)} = \frac{a^2}{\rho\,(k^2\rho^2 - a^2)}, \quad \gamma_{(2)(1)(0)} = \gamma_{(0)(2)(1)} = \gamma_{(0)(1)(2)} = \frac{ak}{k^2\rho^2 - a^2}, \tag{2.14.8a}$$

$$\gamma_{(2)(1)(2)} = \frac{k^2\rho}{k^2\rho^2 - a^2}, \tag{2.14.8b}$$

$$\gamma_{(1)} = \frac{1}{\rho}. \tag{2.14.8c}$$

2.15 Kasner

The Kasner spacetime in cartesian coordinates (t,x,y,z) is represented by the line element[MTW73, Kas21] $(c=1)$

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}tz^2, \tag{2.15.1}$$

where p_1, p_2, p_3 have to fulfill the two conditions

$$p_1 + p_2 + p_3 = 1 \qquad \text{and} \qquad p_1^2 + p_2^2 + p_3^2 = 1. \tag{2.15.2}$$

These two conditions can also be represented by the Khalatnikov-Lifshitz parameter u with

$$p_1 = -\frac{u}{1+u+u^2}, \qquad p_2 = \frac{1+u}{1+u+u^2}, \qquad p_3 = \frac{u(1+u)}{1+u+u^2}. \tag{2.15.3}$$

Christoffel symbols:

$$\Gamma^x_{tx} = \frac{p_1}{t}, \qquad \Gamma^y_{ty} = \frac{p_2}{t}, \qquad \Gamma^z_{tz} = \frac{p_3}{t}, \tag{2.15.4a}$$

$$\Gamma^t_{xx} = \frac{p_1 t^{2p_1}}{t}, \qquad \Gamma^t_{yy} = \frac{p_2 t^{2p_2}}{t}, \qquad \Gamma^t_{zz} = \frac{p_3 t^{2p_3}}{t}. \tag{2.15.4b}$$

Riemann-Tensor:

$$R_{txtx} = \frac{p_1(1-p_1)t^{2p_1}}{t^2}, \qquad R_{tyty} = \frac{p_2(1-p_2)t^{2p_2}}{t^2}, \qquad R_{tztz} = \frac{p_3(1-p_3)t^{2p_3}}{t^2}, \tag{2.15.5a}$$

$$R_{xyxy} = \frac{p_1 p_2 t^{2p_1} t^{2p_2}}{t^2}, \qquad R_{xzxz} = \frac{p_1 p_3 t^{2p_1} t^{2p_3}}{t^2}, \qquad .R_{yzyz} = \frac{p_2 p_3 t^{2p_2} t^{2p_3}}{t^2}. \tag{2.15.5b}$$

The Ricci tensor as well as the Ricci scalar vanish identically. The Kretschman scalar reads

$$\mathscr{K} = \frac{4}{t^4}\left(p_1^2 - 2p_1^3 + p_1^4 + p_2^2 - 2p_2^3 + p_2^4 + p_1^2 p_3^2 + p_3^2 - 2p_3^3 + p_3^4 + p_1^2 p_2^2 + p_2^2 p_3^2\right) \tag{2.15.6a}$$

$$= \frac{16u^2(1+u)^2}{t^4(1+u+u^2)^3}. \tag{2.15.6b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \partial_t, \qquad \mathbf{e}_{(x)} = t^{-p_1}\partial_x, \qquad \mathbf{e}_{(y)} = t^{-p_2}\partial_y, \qquad \mathbf{e}_{(z)} = t^{-p_3}\partial_z. \tag{2.15.7}$$

Dual tetrad:

$$\theta^{(t)} = dt, \qquad \theta^{(x)} = t^{p_1}dx, \qquad \theta^{(y)} = t^{p_2}dy, \qquad \theta^{(z)} = t^{p_3}dz. \tag{2.15.8}$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(r)} = \frac{p_1}{t}, \qquad \gamma_{(t)(\vartheta)(\vartheta)} = \frac{p_2}{t}, \qquad \gamma_{(t)(\varphi)(\varphi)} = \frac{p_3}{t}. \tag{2.15.9}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = -\frac{1}{t}. \tag{2.15.10}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(x)(y)(x)} = \frac{p_1(1-p_1)}{t^2}, \qquad R_{(t)(y)(t)(y)} = \frac{p_2(1-p_2)}{t^2}, \qquad R_{(t)(z)(t)(z)} = \frac{p_3(1-p_3)}{t^2}, \tag{2.15.11a}$$

$$R_{(x)(y)(x)(y)} = \frac{p_1 p_2}{t^2}, \qquad R_{(x)(z)(x)(z)} = \frac{p_1 p_3}{t^2}, \qquad R_{(y)(z)(y)(z)} = \frac{p_2 p_3}{t^2}. \tag{2.15.11b}$$

2.16 Bertotti-Kasner

The Bertotti-Kasner spacetime in spherical coordinates $(t, r, \vartheta, \varphi)$ is represented by the line element[Rin98]

$$ds^2 = -c^2 dt^2 + e^{2\sqrt{\Lambda}ct} dr^2 + \frac{1}{\Lambda}\left(d\vartheta^2 + \sin^2\vartheta d\varphi^2\right). \tag{2.16.1}$$

Christoffel symbols:

$$\Gamma^r_{tr} = c\sqrt{\Lambda}, \qquad \Gamma^t_{rr} = \frac{\sqrt{\Lambda}}{c} e^{2\sqrt{\Lambda}ct}, \qquad \Gamma^{\varphi}_{\vartheta\varphi} = \cot\vartheta, \qquad \Gamma^{\vartheta}_{\varphi\varphi} = -\sin\vartheta\cos\vartheta. \tag{2.16.2}$$

Riemann-Tensor:

$$R_{trtr} = -\Lambda c^2 e^{2\sqrt{\Lambda}ct}, \qquad R_{\vartheta\varphi\vartheta\varphi} = \frac{\sin^2\vartheta}{\Lambda}. \tag{2.16.3}$$

Ricci-Tensor:

$$R_{tt} = -\Lambda c^2, \qquad R_{rr} = \Lambda e^{2\sqrt{\Lambda}ct}, \qquad R_{\vartheta\vartheta} = 1, \qquad R_{\varphi\varphi} = \sin^2\vartheta. \tag{2.16.4}$$

The Ricci and Kretschman scalars read

$$\mathscr{R} = 4\Lambda, \qquad \mathscr{K} = 8\Lambda^2. \tag{2.16.5}$$

Weyl-Tensor:

$$C_{trtr} = -\frac{2}{3}\Lambda c^2 e^{2\sqrt{\Lambda}ct}, \qquad C_{t\vartheta t\vartheta} = \frac{c^2}{3}, \qquad C_{t\varphi t\varphi} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct}, \tag{2.16.6a}$$

$$C_{r\vartheta r\vartheta} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct}, \qquad C_{r\varphi r\varphi} = -\frac{1}{3}e^{2\sqrt{\Lambda}ct}\sin^2\vartheta, \qquad C_{\vartheta\varphi\vartheta\varphi} = \frac{2}{3}\frac{\sin^2\vartheta}{\Lambda}. \tag{2.16.6b}$$

Local tetrad:

$$\mathbf{e}_{(t)} = \frac{1}{c}\partial_t, \qquad \mathbf{e}_{(r)} = e^{-\sqrt{\Lambda}ct}\partial_r, \qquad \mathbf{e}_{(\vartheta)} = \sqrt{\Lambda}\partial_\vartheta, \qquad \mathbf{e}_{(\varphi)} = \frac{\sqrt{\Lambda}}{\sin\vartheta}\partial_\varphi. \tag{2.16.7}$$

Dual tetrad:

$$\theta^{(t)} = c\,dt, \qquad \theta^{(r)} = e^{\sqrt{\Lambda}ct}dr, \qquad \theta^{(\vartheta)} = \frac{1}{\sqrt{\Lambda}}d\vartheta, \qquad \theta^{(\varphi)} = \frac{\sin\vartheta}{\sqrt{\Lambda}}d\varphi. \tag{2.16.8}$$

Ricci rotation coefficients:

$$\gamma_{(t)(r)(r)} = \sqrt{\Lambda}, \qquad \gamma_{(\vartheta)(\varphi)(\varphi)} = -\sqrt{\Lambda}\cot\vartheta. \tag{2.16.9}$$

The contractions of the Ricci rotation coefficients read

$$\gamma_{(t)} = -\sqrt{\Lambda}, \qquad \gamma_{(\vartheta)} = \sqrt{\Lambda}\cot\vartheta. \tag{2.16.10}$$

Riemann-Tensor with respect to local tetrad:

$$R_{(t)(r)(t)(r)} = -R_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\Lambda. \tag{2.16.11}$$

Ricci-Tensor with respect to local tetrad:

$$R_{(t)(t)} = -R_{(r)(r)} = -R_{(\vartheta)(\vartheta)} = -R_{(\varphi)(\varphi)} = -\Lambda. \tag{2.16.12}$$

Weyl-Tensor with respect to local tetrad:

$$C_{(t)(r)(t)(r)} = -C_{(\vartheta)(\varphi)(\vartheta)(\varphi)} = -\frac{2\Lambda}{3}, \tag{2.16.13a}$$

$$C_{(t)(\vartheta)(t)(\vartheta)} = C_{(t)(\varphi)(t)(\varphi)} = -C_{(r)(\vartheta)(r)(\vartheta)} = -C_{(r)(\varphi)(r)(\varphi)} = \frac{\Lambda}{3}. \tag{2.16.13b}$$

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