Geometry of spacetime founded on spacelike metric

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The first part of this paper contains new mathematical techniques for describing a spacetime anisotropy as suggested by the violation of parity conservation. Geometric measures of spacetime involve both the laboratory doing them and the events upon which these measures are done. The time form c and the spacelike length γ are the basic issues of those measures. Both depend on events and also on the timelike direction of the laboratory. Relativity tells that the field $\gamma - c \otimes c$ depends, on the contrary, on events only; in this sense, relativistic spacetime is isotropic. If γ and c do not have that property, the manifold where the observable geometry takes place must be the set of timelike directions. The geometric structure of this manifold given by c and γ is studied in detail. The second part of the paper contains the study of a line of thought opposite to chronogeometry: Building the geometry from lengths instead of times. The datum is γ; through the conditions of stationary spacelike volume and of stationary proper time, a class of time forms and a gauge are obtained under some weak restrictions. Newtonian and relativistic spacelike metrics fulfill these restrictions. Standard connections are induced; they define the absolute derivative of physical fields and the geometric structure of the manifold of timelike directions. The paper ends with some comments about the remaining problem: to suggest and justify field equations.

I. MATHEMATICAL TECHNIQUES FOR SPACETIME PHYSICS

1. INTRODUCTION

We will consider spacetime as an n-dimensional differentiable manifold M, whose underlying set is the set of events. We call timelike the nonvanishing vectors tangent to the possible world line of particles. Let $\tilde{\mathscr{T}}M$ be the set of timelike vectors. Then, if $\bar{x}\in\tilde{\mathscr{T}}M$ and $\alpha > 0$, we have $\alpha\bar{x}\in\tilde{\mathscr{T}}M$. Also we admit that $\tilde{\mathscr{T}}M$ is an open subset of T^1_0M, $\pi^1_0: T^1_0M \to M$ being the tangent bundle over M. In ordinary language, this assumption corresponds to the following experimental fact: Given a particle, it is possible to have particles whose relative movement (with respect to the former) has arbitrary direction. We put $\tilde{\pi}: \tilde{\mathscr{T}}M \to M$, where $\tilde{\pi} = \pi^1_0|\tilde{\mathscr{T}}M$, and suppose $\tilde{\mathscr{T}}_m = \tilde{\pi}^{-1}(m)$ to be nonempty for each $m\in M$.

We emphasize that $\tilde{\mathscr{T}}M$ is not related here to a Lorentz metric, because we are looking for a wider mathematical ground than the relativistic one.

Any physical quantity must be measured from some laboratory, and every physical experiment must be devised referring it to several instruments. These instruments constitute the laboratory, and they are built by particles following their respective world lines. Let U be the spacetime neighborhood where the experiment takes place. Then, we can provide a rough description of the laboratory as a cross section $\bar{r}$ of $\tilde{\pi}$ on U, where $\bar{r}_m$ stands for the tangent to the world line of the particle (belonging to the involved instruments) at $m\in U$. Thus, one could expect the result to be a function of $\bar{r}$ and other parameters. Obviously, this happens in practice: for example, the Doppler effect of a signal received in earth from a satellite.

However, this experiment and others like it are too far from our geometric goal. So, we shall fix our attention upon the measurement of geometric features of spacetime: (a) time elapsed between two events, as measured by clocks (laboratories) following different world lines connecting both events; (b) spacelike distance between two events as measured by different meter sticks (laboratories), such that both events occur on each meter stick. In both cases, the resulting quantity depends on the laboratory, i.e., on the local cross section of $\tilde{\pi}$ attached to each clock or meter stick.

The wondrous thing would be that one could find, from that type of measures, a magnitude depending on events of spacetime only, and not also or $\bar{r}$. If this did occur, we could say that spacetime geometry was isotropic, since it did not depend on the timelike directions of the laboratories measuring it. Einstein's standard relativity is, of course, the best example.

But spacetime is not isotropic in its mass or charge distribution, at least on local scale. Moreover, the violation of parity conservation suggests an anisotropic spacetime at the microscopic level, as it has been explained by Horváth.[1] So, *one could regard general relativity as a first approximation that neglects anisotropy, and consider the manifold of timelike directions as the proper ground for the measurable spacetime geometry.* We say directions instead of vectors because $\bar{r}$ and $a\bar{r}$ do represent the same laboratory if $a: M \to \mathbb{R}$ is a positive function. Thus, the true manifold must be $\mathscr{T}M$, the quotient of $\tilde{\mathscr{T}}M$ under the equivalence relation given by homotheties.

Now, what could one expect to find out as measurable quantities? Of course, the same we are obtaining until now, that is, ordinary numbers, vectors, or tensors. Thus, our physical fields will be maps from $\mathscr{T}M$ to $\mathbb{R}$ (scalar fields), or to T^r_sM (tensor fields).

The goal of Part I is to develop a suitable mathematical formalism for the treatment of these "mixed" fields also depending on directions. It provides a common geometric framework for the study and comparison of different spacetime theories (Newtonian and relativistic for instance). As far as I am aware, it constitutes a new mathematical technique; however, for the sake of brevity, we shall restrict our-

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selves to the concepts we will directly use in spacetime theory; the risk of such restriction is to conceal somewhat the mathematical reasons for giving certain definitions. Anyway, our paper (Montesinos[2]) could serve as an introduction to those techniques.

Through Part II, my own physical theory is developed under the formalism of Part I. The fundamental field will be the spacelike metric. From it, we build simultaneity and, partially, time length; in this sense, my theory is somewhat new, since it no longer takes time or light signals as fundamental. It could be looked at as the opposite veiwpoint of chronogeometry.

Besides this Introduction, Part I has nine sections. In Sec. 2. we briefly describe the notation and some mathematical notations which we will use.

As for Sec. 3, let $\mathscr{F}M$ be the quotient manifold of $\tilde{\mathscr{F}}M$ under the equivalence relation given by positive homotheties. If $\pi:\mathscr{F}M\to M$ is the induced projection, then the manifold of timelike directions, $\mathscr{F}M$, becomes an open submanifold of the sphere bundle over M. Physical fields are maps as $h:\mathscr{F}M\to T^r_sM$, satisfying $\pi^r_s\circ h=\pi$, where $\pi^r_s:T^r_sM\to M$ is the tangent tensor bundle of type (r,s). This condition tells us that a physical field assigns to each timelike direction r_m a tensor lying in the tensor space tangent to M at m, the event where that timelike direction lies. We can consider physical fields as included in the algebra of Finsler tensor fields over $\tilde{\mathscr{F}}M$ because there is a one-to-one correspondence with homogeneous degree zero Finsler fields.

This material serves for describing the basic geometric features of spacetime, namely the time function $\tilde{f}$, the time form c, and the spacelike metric γ (Sec. 4). We discuss the physical meaning of these fields and give two examples, Newtonian and relativistic spacetimes, clearing up the wide range of spacetime models where this scheme applies.

The mixed nature of physical fields makes a direct treatment difficult. So, *we shall submit it to the techniques for usual fields over $\mathscr{F}M$. Besides the physical motivations for my viewpoints, that is the main objective of this part.* Thus, in Sec. 5 we define horizontal and vertical homomorphisms from the module of physical vector fields to that of ordinary vector fields over $\mathscr{F}M$.

In Sec. 6, these homomorphisms are extended to be graded tensor algebra homomorphisms (lifts) from ΠM, the algebra of physical fields, to $V\mathscr{F}M$, the algebra of ordinary tensor fields over $\mathscr{F}M$. Each lift has a unique lowering that is its transpose map. We define crossed pairs of lift lowerings. They induce the horizontal and vertical projectors. The main result of this section tells that a pair of horizontal and vertical homomorphisms, in the sense of Sec. 5, do define a unique pair of crossed lifts.

In Sec. 7 we define and interpret several types of connections we will use later, namely horizontal and vertical connections on ΠM, physical connections, and the j-connection D, an important mathematical tool. We interpret $\eta=Dc$ as the rate of time retardation when the relative speed increases. This field plays an important role in the existence problem for connections.

Section 8 is devoted to the definition, explanation, existence, and uniqueness of horizontal and vertical torsionless metric connections. They have a suggestive meaning: The vertical connection measures the absolute directional dependence of physical fields; the horizontal one, the absolute along spacetime dependence.

In Sec. 9, we lift the pair of these horizontal and vertical connections for having a unique physical connection. It defines the absolute dependence of physical fields along the time manifold. This connection is also lifted for having the linear connection **D**, that yields the final geometric structure of the time manifold itself. These results are briefly resumed in the conclusion (Sec. 10).

2. NOTATION

M, n-dimensional Hausdorff second countable real C^∞ manifold, briefly manifold. It stands for spacetime.

$\pi^r_s:T^r_sM\to M$, tangent tensor bundle over M of type (r,s); M_m, tangent space at $m\in M$.

V^0_0M, the ring of C^∞ real functions on M; V^r_sM, the V^0_0M-module of C^∞ cross sections of π^r_s; $VM=\oplus V^r_sM$, tensor $\mathbb{R}$ algebra, graded by the indexes (r,s).

$\tilde{\mathscr{F}}M$, the set of timelike vectors, is an open submanifold of T^1_0M; $\tilde{\pi}:\tilde{\mathscr{F}}M\to M$ is defined by $\tilde{\pi}=\pi^1_0|\tilde{\mathscr{F}}M$. We suppose that $0\notin\tilde{\mathscr{F}}_m=\tilde{\pi}^{-1}(m)\neq\emptyset$ for every $m\in M$. In addition, we require that if $\bar{x}\in\tilde{\mathscr{F}}M$, then $\alpha\bar{x}\in\tilde{\mathscr{F}}M$ for every $0<\alpha\in\mathbb{R}$.

Since $\tilde{\mathscr{F}}M$ is itself a manifold, we use $V^r_s\tilde{\mathscr{F}}M$ and $V\tilde{\mathscr{F}}M$ to denote the module of ordinary tensor fields of type (r,s) over $\tilde{\mathscr{F}}M$, and the respective graded tensor $\mathbb{R}$ algebra.

$\tilde{\Pi}^0_0M=V^0_0\tilde{\mathscr{F}}M$, the ring of real C^∞ functions on $\tilde{\mathscr{F}}M$; $\tilde{\Pi}^r_sM$ is the $\tilde{\Pi}^0_0M$ module of Finsler tensor fields of type (r,s), that is C^∞ maps $\tilde{h}:\tilde{\mathscr{F}}M\to T^r_sM$ satisfying $\pi^r_s\circ\tilde{h}=\tilde{\pi}$; $\tilde{\Pi}M=\oplus\tilde{\Pi}^r_sM$, graded tensor $\mathbb{R}$ algebra of Finsler fields. We say that a Finsler tensor field $\tilde{h}$ is homogeneous of degree $\alpha\in\mathbb{R}$ if $\tilde{h}_{q\bar{x}}=q^\alpha\tilde{h}_{\bar{x}}$ for every $0<q\in\mathbb{R}$ and $\bar{x}\in\tilde{\mathscr{F}}M$. That property will be denoted $h(\alpha)$.

$\tilde{u}:\tilde{\mathscr{F}}M\to T^1_0M$, the canonic Finsler vector field, is defined as the inclusion. Hence, $\tilde{u}$ is $h(1)$.

$i:\tilde{\Pi}^1_0M\to V^1_0\tilde{\mathscr{F}}M$, the vertical injection. That is, if $\bar{v}\in M_m$ and $\bar{x}\in\tilde{\mathscr{F}}_m$, then $i_{\bar{x}}(\bar{v})$ is the tangent at $t=0$ to the curve $\sigma:t\to\bar{x}+\bar{v}t$. Since $\tilde{\mathscr{F}}M$ is open in T^1_0M, then $\tilde{\mathscr{F}}_m$ is open in M_m; therefore, for some $\epsilon>0$ that curve lies in $\tilde{\mathscr{F}}_m\subset\tilde{\mathscr{F}}M$ if $-\epsilon<t<\epsilon$. Thus, $\sigma(t)$ is a curve on $\tilde{\mathscr{F}}M$, whence its tangent $i_{\bar{x}}(\bar{v})$ at $t=0$ is a vector belonging to $(\tilde{\mathscr{F}}M)_{\bar{x}}$. Hence, if $\tilde{v}:\tilde{\mathscr{F}}M\to T^1_0M$ is a Finsler vector field, we define $i\tilde{v}\in V^1_0\tilde{\mathscr{F}}M$ by means of $(i\tilde{v})_{\bar{x}}=i_{\bar{x}}(\tilde{v}_{\bar{x}})$.

$\langle s,v\rangle$, the contraction of the 1-form s (belonging to V^0_1M, $V^0_1\tilde{\mathscr{F}}M$, $\tilde{\Pi}^0_1M$, etc.) with the vector field v (belonging to V^1_0M, $V^1_0\tilde{\mathscr{F}}M$, $\tilde{\Pi}^1_0M$, etc., respectively).

3. THE TIME MANIFOLD. PHYSICAL FIELDS

On $\tilde{\mathscr{F}}M$ we define an equivalence relation $\sim$ by means of $\bar{x}\sim\bar{y}$ if $\tilde{\pi}(\bar{x})=\tilde{\pi}(\bar{y})$ and $\bar{x}=\alpha\bar{y}$ for some $\alpha>0$. Let $\mathscr{F}M$ be the set of equivalence classes, and $p:\tilde{\mathscr{T}}M\to\mathscr{T}M$ the natural projection, which applies an element $\tilde{x}\in\tilde{\mathscr{T}}M$ into its class $p\tilde{x}$. Then, $\mathscr{T}M$ can be given a unique differentiable manifold structure making p a submersion. We will call $\mathscr{T}M$, with that structure, the *time manifold*. It represents the manifold of timelike directions. The map $\pi:\mathscr{T}M\to M$, where $\pi\circ p=\tilde{\pi}$, defines the *time bundle*. Note that a cross section r of π on $U\subset M$ can be looked at as a laboratory whose instruments have at $m\in U$ a particle with speed r_m.

Let Π^0_0M be the ring of C^∞ real functions on $\mathscr{T}M$. We use Π^r_sM to denote the set of *physical fields* of type (r,s), i.e., C^∞ maps $h:\mathscr{T}M\to T^r_sM$ satisfying $\pi^r_s\circ h=\pi$. Then, if for example $v\in\Pi^1_0M$, its value v_{r_m} at $r_m\in\mathscr{T}_m=\pi^{-1}(m)$ is a vector of M_m, the tangent space to M at m. Thus, Π^r_sM becomes a Π^0_0M-module, and we can build the graded tensor algebra ΠM of physical fields.

If $h\in\Pi^r_sM$, we put $e_ah=h\circ p:\tilde{\mathscr{T}}M\to T^r_sM$. Then $\pi^r_s\circ e_ah=\pi^r_s\circ h\circ p=\pi\circ p=\tilde{\pi}$; therefore, e_ah is a Finsler tensor field of type (r,s). Since $p(\alpha\tilde{x})=p\tilde{x}$ for every $\alpha>0$ and $\tilde{x}\in\tilde{\mathscr{T}}M$, we conclude that e_ah is h (0). Hence, $e_a:\Pi M\to\tilde{\Pi}M$ is a graded $\mathbb{R}$ algebra homomorphism mapping ΠM onto the graded subalgebra of h (0) Finsler tensor fields. Conversely, if $\tilde{h}\in\tilde{\Pi}M$ is h (0), it defines $e_b\tilde{h}\in\Pi M$ by means of $e_b\tilde{h}\circ p=\tilde{h}$. Thus, $e_b\circ e_a=\text{id}$ on ΠM, and $e_a\circ e_b=\text{id}$ on the subalgebra of h (0) Finsler tensor fields. So we have bridge between Finsler techniques and those we present here.

4. SPACE AND TIME FORMS

Spacetime geometry involves two main concepts, spacelike and timelike length, and a link between them: synchronization. This last is the troubling point because since Einstein's relativity, light signals came in. The trouble is: timelike length defines by itself a synchronization, as we shall see at once; spacelike length also does that (see Part II). So, what do light signals do in all this matter? This question is purposely bold, but I think it is not merely rhetorical. It aims to raise doubts about the role light signals must play on spacetime geometry, and to make more plausible the viewpoint of this paper. In fact, my methodological way is the following: to look at space and timelike length as the basic (related between them or not) geometric data of spacetime, and to consider gravitational or electromagnetic phenomena (light signals among them) as desirable *dynamical* issues from the *static* (geometric) description. So, *in this paper light signals do not play any direct role among the basic geometric features of spacetime.* Of course, electromagnetic signals are the best practical tool for the study of spacetime in several areas. I simply say they are unnecessary for our theoretical purposes.

Let us consider time length first. As it has been pointed out by chronogeometry, time length must be defined by a h (1) function $\tilde{f}\in\tilde{\Pi}^0_0M$, such that if $\sigma:[a,b]\to M$ is the world line of an atomic clock, then $\int_a^b\tilde{f}_{\dot{\sigma}}dt$ is the time measured by that clock between $\sigma(a)$ and $\sigma(b)$. The function $\tilde{f}$ must be h (1) for time elapsed could be invariant under parametrization changes of σ. We will call $\tilde{f}$ the *time function*.

A synchronization is given by a *time form*, that is a field $c\in\Pi^0_1M$ such that $\langle\tilde{c},\tilde{u}\rangle$ is everywhere nonzero (we always will put $\tilde{c}=e_ac$). Briefly, if $\tilde{x}\in\tilde{\mathscr{T}}M$, then the hyperplane of M_m spanned by the vectors $\bar{v}\in M_m$ satisfying $\langle\tilde{c}_{\tilde{x}},\bar{v}\rangle=0$ defines the simultaneity relative to the timelike direction $p\tilde{x}$. Note that $\langle\tilde{c},\tilde{u}\rangle_{\tilde{x}}=\langle c_{p\tilde{x}},\tilde{x}\rangle\neq0$ because $\tilde{x}$ stands for a tangent to the world line of the particle defining the simultaneity $c_{p\tilde{x}}$; since that line is timelike, different events on it cannot be simultaneous. Note also that if c is multiplied by any nonvanishing function $q\in\Pi^0_0M$, then qc defines the same simultaneity than c.

Let us relate $\tilde{f}$ and c. If $\tilde{f}$ and c are given, then c can be multiplied by some function $q\in\Pi^0_0M$ such that $\langle e_a(qc),\tilde{u}\rangle=\tilde{f}$, since it is enough to take $q=e_b(\tilde{f}/\langle\tilde{c},\tilde{u}\rangle)$. Thus, *an arbitrary given time function can be defined on this way from an arbitrary simultaneity.* The choice of a "length" for a simultaneity c (the multiplication by q) fixes a time scale on each synchronized laboratory. That is, if r_m is a timelike direction at m, then c_{r_m} stratifies on equitime hyperplanes the affine tangent space M_m. Thus, if $\bar{v}\in M_m$, then $\langle c_{r_m},\bar{v}\rangle$ stands for the time shift between the tail and the head events determining $\bar{v}$. This time shift depends on the inclination (synchronization) of c_{r_m}, and also depends on the separation of equitime hyperplanes (the lenght of c_{r_m}). Now, if $\sigma:[a,b]\to M$ is a world line and $r:M\to\mathscr{T}M$ is a laboratory, then $\int_a^b\langle c_{r\circ\sigma},\dot{\sigma}\rangle dt$ is the time inverted by the particle σ from $\sigma(a)$ to $\sigma(b)$, as measured by the laboratory r. If r is the particle itself, that is $r\circ\sigma=p\dot{\sigma}$, and $\langle\tilde{c},\tilde{u}\rangle=\tilde{f}$, then $\int_a^b\langle c_{r\circ\sigma},\dot{\sigma}\rangle dt=\int_a^b\langle\tilde{c},\tilde{u}\rangle_{\dot{\sigma}}dt=\int_a^b\tilde{f}_{\dot{\sigma}}dt$. In other words, the condition $\langle\tilde{c},\tilde{u}\rangle=\tilde{f}$ means that *we have picked for the synchronized laboratories the same time scale which measures proper times by means of $\tilde{f}$.*

Let us consider the inverse problem: Given $\tilde{f}$, find out a time form c such that $\langle\tilde{c},\tilde{u}\rangle=\tilde{f}$. A solution is the element of Π^0_1M defined through $\langle\tilde{c},\tilde{v}\rangle=i\tilde{v}(\tilde{f})$ for every $\tilde{v}\in\tilde{\Pi}^1_0M$. In fact we have $\langle\tilde{c},\tilde{u}\rangle=i\tilde{u}(\tilde{f})=\tilde{f}$ because $\tilde{f}$ is h (1); also $\tilde{c}$ is h (0) because $\tilde{u}$ is h (1). Therefore, $e_b\tilde{c}=c$ is a solution. Now, if $b\in\Pi^0_1M$ satisfies $\langle e_ab,\tilde{u}\rangle=0$, then $c+b$ is another solution. But only the first one has a decisive property: *The simultaneity if furnishes corresponds to that of infinitely slow clock transport.* In fact, we will see in Part II Sec. 4 that this correspondence is characterized by the property $i\tilde{v}(\langle\tilde{c},\tilde{u}\rangle)=\langle\tilde{c},\tilde{v}\rangle$ for every $\tilde{v}\in\tilde{\Pi}^1_0M$. So, we can say that *a time function $\tilde{f}$ gives raise to a unique compatible time form c*, the one satisfying $\langle\tilde{c},\tilde{u}\rangle=\tilde{f}$, $i\tilde{v}(\langle\tilde{c},\tilde{u}\rangle)=\langle\tilde{c},\tilde{v}\rangle$. Due to this, in the following we will use time forms instead of time functions.

As for spacelike length, it is given by a field $\gamma\in\Pi^0_2M$, symmetric, of signature $(0,+,\ldots,+)$, and such that $\tilde{\gamma}(\tilde{u},\ )=0$, where $\tilde{\gamma}=e_a\gamma$ Along Part II we will justify this assertion and see in what manner γ defines a time form. So, we shall then reach another puzzling point: the compatibility of the time forms obtained from time functions or from spacelike metrics (II.1). Until then, we will leave this question and go on to describe two typical examples under this formalism.

Let $\bar{g}\in V^0_2M$ be a Lorentz metric. Then it defines the time form $\tilde{c}=-\tilde{g}(\tilde{u},\ )/(-\tilde{g}(\tilde{u},\tilde{u}))^{1/2}$ where $\tilde{g}=\bar{g}\circ\tilde{\pi}$, and the spacelike metric $\gamma=\bar{g}\circ\pi+c\otimes c$. This is the *relativistic model.*

As for a *generalized Newtonian spacetime* (locally absolute time and length), let M admit a symmetric field $\bar{q}\in V^0_2M$ of signature (0, +, ..., +), and a field $\bar{b}\in V^0_1M$, everywhere nonzero, such that if $0\neq\bar{v}\in M_m$ and $\langle\bar{b}_m,\bar{v}\rangle = 0$, then $\bar{q}_m(\bar{v},\bar{v})>0$. Thus, $\bar{b}$ defines the local absolute time and $\bar{q}$ the local absolute length. We put $\mathscr{F}M = \{\bar{x}_m\in T^1_0M:\langle\bar{b}_m,\bar{x}_m\rangle\neq 0\}$. Then, the time form is given by $c = \bar{b}\circ\pi$ and the spacelike metric by

$$e_a\gamma = \tilde{q} + \frac{\tilde{q}(\tilde{u},\tilde{u})}{\langle\tilde{c},\tilde{u}\rangle^2}\tilde{c}\otimes\tilde{c} - \frac{\tilde{q}(\tilde{u},\ )\otimes\tilde{c}}{\langle\tilde{c},\tilde{u}\rangle} - \frac{\tilde{c}\otimes\tilde{q}(\tilde{u},\ )}{\langle\tilde{c},\tilde{u}\rangle},$$

where $\tilde{q} = \bar{q}\circ\bar{\pi}$.

See also Ref. 3.

5. VERTICAL AND HORIZONTAL HOMOMORPHISMS

A vector field $\mathbf{v}\in V^1_0\mathscr{F}M$ is said to be vertical if $\mathbf{v}(\bar{a}\circ\pi) = 0$ for every $\bar{a}\in V^0_0M$. That is, vertical vector fields are tangent to the fibres $\pi^{-1}(m)$. The set of vertical vector fields is a $V^0_0\mathscr{F}M$-module, locally $(n-1)$-dimensional, for it is the annihilator of the $V^0_0\mathscr{F}M$-module spanned by the elements $d(\bar{a}\circ\pi)\in V^0_1\mathscr{F}M$, and this last module is clearly n-dimensional [take for example $\bar{a} = \bar{x}^i$, where $\{\bar{x}^i\}$ is a coordinate system on $U\subset M$, and note that $\mathscr{F}M$ is $(2n-1)$-dimensional].

Suppose that a time form c is given. Then, it defines in a natural way a homomorphism $j:\Pi^1_0M\to V^1_0\mathscr{F}M$ such that its image, $j(\Pi^1_0M)$, equals the module of vertical vector fields (in this sense we say that j is a *vertical homomorphism*). In face, let $v\in\Pi^1_0M$; then $\tilde{v} = e_a v\in\tilde{\Pi}^1_0M$, and $\langle\tilde{c},\tilde{u}\rangle i\tilde{v}$ is a vertical field of $V^1_0\tilde{\mathscr{F}}M$. By its own definition, $(\langle\tilde{c},\tilde{u}\rangle i\tilde{v})_{\alpha\bar{x}}$ is the tangent, at $t = 0$, to the curve $t\to\alpha\bar{x} + \langle\tilde{c}_{\alpha\bar{x}},\alpha\bar{x}\rangle t\tilde{v}_{\alpha\bar{x}}$. Now, because the factor $\langle\tilde{c}_{\alpha\bar{x}},\alpha\bar{x}\rangle = \alpha\langle\tilde{c}_{\bar{x}},\bar{x}\rangle,p$ projects all these curves (varying the number α) upon the same curve $p(\bar{x} + \langle c_{p\bar{x}},\bar{x}\rangle tv_{p\bar{x}})$, whose tangent at $t = 0$ defines $j_{p\bar{x}}(v_{p\bar{x}})$. Thus, we put $(jv)_{p\bar{x}} = j_{p\bar{x}}(v_{p\bar{x}})$. Hence we have $(jv)\circ p = p_*\circ(\langle\tilde{c},\tilde{u}\rangle ie_a v)$, where p_* stands for the derived map of p. If $a\in\Pi^0_0M = V^0_0\mathscr{F}M$, then $jv(a)$ defines a derivation along the fibres; that is, *jv measures the dependence of functions on directions, not on events of spacetime.* We have that $\ker j$ is spanned by $k = e_b(\tilde{u}/\langle\tilde{c},\tilde{u}\rangle)$, because $p_*\circ i\tilde{u} = 0$.

Now, let $\Lambda:\Pi^1_0M\to V^1_0\mathscr{F}M$ be a homomorphism. Then we say it is *horizontal* if $(\Lambda v)_r(\bar{a}\circ\pi) = v_r(\bar{a})$ for every $r\in\mathscr{F}M$ and $\bar{a}\in V^0_0M$. The definition tells that Λ is injective. Note that our condition is equivalent to $\langle d(\bar{a}\circ\pi),\Lambda v\rangle = \langle(d\bar{a})\circ\pi,v\rangle$. Let us give an interpretation of Λ. We have that the elements $v\in\Pi^1_0M$ can be locally written as $v^i(\partial/\partial\bar{x}^i\circ\pi)$, where $v^i\in\Pi^0_0M$ and $\{\bar{x}^i\}$ is a coordinate system on $U\subset M$. Since Λ is Π^0_0M-linear, we shall only give the interpretation of Λ upon associated fields, that is such as $\bar{v}\circ\pi$, with $\bar{v}\in V^1_0M$.. A horizontal homomorphism Λ is an assignment of a field $\Lambda v\in V^1_0\mathscr{F}M$ to the field $v = \bar{v}\circ\pi$ such that Λv projects upon v under the map π_*. In other words, integral curves of Λv are projectecd by π on integral curves of v. Or roughly speaking, *a horizontal homomorphism is an interpretation of derivatives along M as derivatives along $\mathscr{F}M$.*

6. LIFTS AND LOWERINGS

Our purpose is now to extend a pair of vertical and horizontal homomorphisms to vertical and horizontal lifts for arbitrary fields of ΠM.

The map $A:\Pi M\to V\mathscr{F}M$ is called a *horizontal* (*vertical*) *lift* if: (a) $A\,|\Pi^1_0M$ is horizontal (vertical) homomorphism; (b) A is a type preserving graded $\mathbb{R}$ algebra homomorphism; (c) if v is in annihilator of $\ker A\,|\Pi^0_1M$ and $s\in\Pi^0_1M$, then $\langle As,Av\rangle = \langle s,v\rangle$; and if s is in annihilator of $\ker A\,|\Pi^1_0M$ and $v\in\Pi^1_0M$, then $\langle As,Av\rangle = \langle s,v\rangle$.

Note that if A is horizontal, then $A\,|\Pi^1_0M$ is injective; thus every $s\in\Pi^0_1M$ belongs to the annihilator of $\ker A\,|\Pi^1_0M$; hence, if A is horizontal, condition (c) tells us that $\langle As,Av\rangle = \langle s,v\rangle$ for every s,v. Note also that for every lift we have that $Aa = a$ if $a\in\Pi^0_0M$.

The *lowering* B of a lift A is its transpose map $B:V\mathscr{F}M\to\Pi M$. In other words, B is the graded $\mathbb{R}$ algebra homomorphism such that $Ba = a$, $\langle B\mathbf{s},v\rangle = \langle\mathbf{s}, A\mathbf{v}\rangle$, $\langle s,B\mathbf{v}\rangle = \langle As,\mathbf{v}\rangle$.

If A is horizontal, then $\langle s,BAv\rangle = \langle As,Av\rangle = \langle s,v\rangle = \langle BAs,v\rangle$, whence $BA = \text{id}$. If A is vertical we have $BAB = B$, $ABA = A$. In fact, if $v\in\Pi^1_0M$ and $\mathbf{s}\in V^0_1\mathscr{F}M$, then $\langle\mathbf{s},ABAv\rangle = \langle B\mathbf{s},BAv\rangle$. But if $z\in\ker A\,|\Pi^1_0M$, then $\langle B\mathbf{s},z\rangle = \langle\mathbf{s},Az\rangle = 0$. Hence $B\mathbf{s}$ belongs to the annihilator of $\ker A\,|\Pi^1_0M$. Therefore, $\langle B\mathbf{s},BAv\rangle = \langle AB\mathbf{s},Av\rangle = \langle B\mathbf{s},v\rangle = \langle\mathbf{s},Av\rangle$. Since $\mathbf{s}$ is arbitrary we have $ABA = A$ on Π^1_0M; in the same way $ABA = A$ on Π^0_1M; therefore, this relation holds on the whole ΠM. The proof for $BAB = B$ is similar.

The maps A,B have a local character, as is easily proved as customary. This means that if $v_r = w_r$, then $(Av)_r = (Aw)_r$ and so on.

The following definition will be useful for our purposes. Let A_1 be horizontal, A_2 vertical, and B_1, B_2 their respective lowerings. Then we say they form a *crossed lift pair* if $B_1A_2 = B_2A_1 = 0$ on Π^r_sM for $(r,s)\neq(0,0)$ (on Π^0_0M, these homomorphisms are always the identity), and $A_1B_1 + A_2B_2 = \text{id}$ on $V^1_0\mathscr{F}M$ and on $V^0_1\mathscr{F}M$.

Then we shall put $\mathbf{H} = A_1B_1$, $\mathbf{V} = A_2B_2$. Thus we have $\mathbf{H}^2 = \mathbf{H}$, $\mathbf{V}^2 = \mathbf{V}$, $\mathbf{HV} = \mathbf{VH} = 0$ on $V^r_s\mathscr{F}M$ with $(r,s)\neq(0,0)$. Thus, $\mathbf{H}$ and $\mathbf{V}$ project fields of $V\mathscr{F}M$ into their horizontal and vertical components. These components sum the given field if it is a vector field or a 1-form because then $\mathbf{H}+\mathbf{V} = \text{id}$.

Now we reach the fundamental result of this section:

Theorem: Given the vertical and horizontal homomorphisms j and Λ, they define a unique crossed lift pair A_1, B_1, A_2, B_2 satisfying $A_1|\Pi^1_0M = \Lambda$, $A_2|\Pi^1_0M = j$, $\langle c,B_2\mathbf{v}\rangle = 0$ for every $\mathbf{v}\in V^1_0\mathscr{F}M$. Moreover, then $B_1|V^1_0\mathscr{F}M = \pi_*$.

Proof: First we prove $j(\Pi^1_0 M) \oplus \Lambda(\Pi^1_0 M) = V^1_0 \mathscr{F}M$. In fact, if $\mathbf{v} = jv = \Lambda w$, then for every $\bar{a} \in V^0_0 M$ we have $\mathbf{v}_r(\bar{a}\circ\pi) = (jv)_r(\bar{a}\circ\pi) = 0 = (\Lambda w)_r(\bar{a}\circ\pi) = w_r(\bar{a})$. Hence $w_r = 0$ and $\mathbf{v}_r = 0$. Thus, the intersection of those submodules is zero. Now, Λ is injective and the image of j equals the submodule of vertical vector fields. Therefore, the maps Λ_r, j_r defined at each $r \in \mathscr{F}M$ by $\Lambda_r v_r = (\Lambda v)_r$, $j_r v_r = (jv)_r$ have rank n and $n-1$, respectively. Hence $j_r(M_{\pi r}) \oplus \Lambda_r(M_{\pi r}) = (\mathscr{F}M)_r$ because $(\mathscr{F}M)_r$ is $(2n-1)$-dimensional. Now, it is a simple matter to extend this direct sum globally for having our first claim. As a consequence, if $\mathbf{v} \in V^1_0 \mathscr{F}M$, it can be written in a unique manner as $\mathbf{v} = \Lambda v_1 + jv_2$, where $v_1, v_2 \in \Pi^1_0 M$ and $\langle c, v_2\rangle = 0$ (note that $\ker j$ is spanned by k, and $\langle c,k\rangle = 1$). We put $\langle A_1 s, \mathbf{v}\rangle = \langle s, v_1\rangle$, $\langle A_2 s, \mathbf{v}\rangle = \langle s, v_2\rangle$. These maps, together with Λ and j, in fact define the whole lifts A_1, A_2 satisfying our requirements. The proof is rather mechanical and is left to the reader. As for the assertion $B_1 | V^1_0 \mathscr{F}M = \pi_*$, we have $\langle s, B_1 \mathbf{v}\rangle = \langle A_1 s, \mathbf{v}\rangle = \langle s, v_1\rangle$ if $\mathbf{v} = \Lambda v_1 + jv_2$. Then $\pi_* \mathbf{v} = \pi_*\circ\Lambda v_1 + \pi_*\circ j v_2 = \pi_*\circ\Lambda v_1 = v_1$ as we have seen in our interpretation of horizontal homomorphisms. Therefore, $B_1 \mathbf{v} = \pi_*\circ\mathbf{v}$.

Note that if $v \in \Pi^1_0 M$, it can be written as $v = (v - \langle c,v\rangle k) + \langle c,v\rangle k$. Thus, $A_2 v = j(v - \langle c,v\rangle k)$ and $\langle s, B_2 A_2 v\rangle = \langle A_2 s, A_2 v\rangle = \langle A_2 s, j(v - \langle c,v\rangle k)\rangle = \langle s, v - \langle c,v\rangle k\rangle$ because $\langle c, v - \langle c,v\rangle k\rangle = 0$. Hence, $B_2 A_2$ is the identity on annihilator of c. On a similar way, $B_2 A_2$ is the identity on the annihilator of k.

7. CONNECTIONS

The map $\nabla : (v,h) \in \Pi^1_0 M \times \Pi M \to \nabla_v h \in \Pi M$ is called a *horizontal* (*vertical*) *connection on* ΠM if: (a) $\nabla_v : \Pi M \to \Pi M$ is a derivation of degree zero on the graded $\mathbb{R}$ algebra ΠM; (b) $\nabla_v a = Av(a)$, $A : \Pi^1_0 M \to V^1_0 \mathscr{F}M$ being a horizontal (vertical) homomorphism and $a \in \Pi^0_0 M$; (c) it is $\Pi^0_0 M$-linear in v, that is $\nabla_{av+bw} = a\nabla_v + b\nabla_w$; (d) if $s \in \Pi^0_1 M$ and $w \in \Pi^1_0 M$, then $\nabla_v\langle s,w\rangle = \langle \nabla_v s, w\rangle + \langle s, \nabla_v w\rangle$.

The map $\Delta : (\mathbf{w}, h) \in V^1_0 \mathscr{F}M \times \Pi M \to \Delta_{\mathbf{w}} h \in \Pi M$ is called a *physical connection* if: (a) $\Delta_{\mathbf{w}}$ is a derivation of degree zero on ΠM; (b) $\Delta_{\mathbf{w}} a = \mathbf{w}(a)$ for $a \in \Pi^0_0 M$; (c) it is $V^0_0 \mathscr{F}M$-linear in $\mathbf{w}$; (d) $\Delta_{\mathbf{w}}\langle s,v\rangle = \langle \Delta_{\mathbf{w}} s, v\rangle + \langle s, \Delta_{\mathbf{w}} v\rangle$.

From a geometric and physical viewpoint, physical connections are more natural than connections on ΠM, but these are easier to handle. We will use them as a tool for finding physical metric connections. However, both types have a physical significance. The meaning of physical connections is that they give the covariant derivatives of physical fields along the directions $\mathbf{w}$, that is, along curves on $\mathscr{F}M$; in other words, when we move from a point m at which the laboratory has direction r_m, to a point m' where the laboratory has direction $r_{m'}$, in such a manner than the points r_m and $r_{m'}$ of $\mathscr{F}M$ are detached between them by the vector $\mathbf{w}$ (roughly speaking).

Now, as another useful tool, we build the j connection D, which is a vertical connection on ΠM. It is defined by $D_v a = jv(a)$ if $a \in \Pi^0_0 M$, and $D_v(\bar{h}\circ\pi) = 0$ if $\bar{h} \in VM$. It is not difficult to prove the consistency of this definition. We have:

Theorem: If c is a time form such that $i\tilde{v}(\langle\tilde{c},\tilde{v}\rangle) = \langle\tilde{c},\tilde{v}\rangle$ (see Sec. 4), then $\langle D_k c, w\rangle = \langle D_w c, k\rangle = 0$, and $\langle D_v c, w\rangle = \langle D_w c, v\rangle$ for $v, w \in \Pi^1_0 M$.

Proof: Since these expressions are $\Pi^0_0 M$-linear in v,w, we can suppose that they are associated fields, that is $Dv = Dw = 0$. Thus, $\langle D_k c, w\rangle = D_k\langle c,w\rangle - \langle c, D_k w\rangle = D_k\langle c,w\rangle = jk(\langle c,w\rangle) = 0$ because $jk = 0$. Now $(D_w\langle c,v\rangle)\circ p = \langle\tilde{c},\tilde{u}\rangle i\tilde{w}(i\tilde{v}(\langle\tilde{c},\tilde{u}\rangle)) = \langle\tilde{c},\tilde{u}\rangle i\tilde{v}(i\tilde{w}(\langle\tilde{c},\tilde{u}\rangle))$ because v, w are associated fields and iv, iw are ordinary derivatives (in the same sense used in $\mathbb{R}^n$) on the fibres of $\tilde{\pi}$. Hence $0 = D_w\langle c,v\rangle - D_v\langle c,w\rangle = \langle D_w c, v\rangle - \langle D_v c, w\rangle$. Therefore, $\langle D_w c, k\rangle = \langle D_k c, w\rangle = 0$.

This theorem tells us that $\eta = Dc$ defines a symmetric element of $\Pi^0_2 M$ such that $\eta(k, \ ) = 0$. This field gives the rate of time retardation when the relative speed increases. In fact, let $\sigma(t)$ be the world line of a particle, and r a cross section of π, that is a laboratory. If $\dot\sigma(t)$ is the tangent to σ, we can roughly think of $\dot\sigma(t)$ as a vector joining two events in the world line, namely $\sigma(t)$ and $\sigma(t) + \dot\sigma(t)$. Then $\tau_r = \langle c_{r\circ\sigma(t)}, \dot\sigma(t)\rangle$ is the time interval, measured by the synchronized laboratory r, for the track of that particle between $\sigma(t)$ and $\sigma(t) + \dot\sigma(t)$. Thus, if σ remains fixed, this time interval depends on r only. Thus, $D_v\tau_r$ is the rate of variation of τ_r with respect to s, at $s = 0$, when we take laboratories $p(\bar{r} + \langle c_{r\circ\sigma(t)}, \bar{r}\rangle s\bar{v})$ measuring it (see Sec. 5), where we suppose $p\bar{r} = r\circ\sigma(t)$ and $\bar{v} = v_{r\circ\sigma(t)}$. That is, $D_v\tau_r$ is the rate of variation of τ_r when the speed of the laboratory changes towards the $\bar{v}$ direction. But $D_v\tau_r = \langle (D_v c)_{r\circ\sigma(t)}, \dot\sigma(t)\rangle = \eta_{r\circ\sigma(t)}(\bar{v}, \dot\sigma(t))$. If $\bar{v}$ is a positive multiple of $\dot\sigma(t)$, this means we are approaching the laboratory speed to that of the particle because $\langle c_{r\circ\sigma(t)}, \bar{r}\rangle$ (proper time) is supposed to be positive. Then, if $\eta_{r\circ\sigma(t)}(\dot\sigma(t), \dot\sigma(t)) < 0$, we have that clocks relatively retard with respect to each other when their relative speed increases (as a thinking guide, bear in mind special relativity).

In relativity we have $\gamma + \eta = 0$. In Newtonian spacetime $\eta = 0$.

8. METRIC CONNECTIONS

If ${}^1\nabla, {}^2\nabla$ are horizontal and vertical connections on ΠM, respectively, then we have that ${}^1T(v,w) = {}^1\nabla_v w - {}^1\nabla_w v - B_1[A_1 v, A_1 w]$ and ${}^2T(v,w) = B_2 A_2({}^2\nabla_v w - {}^2\nabla_w v - B_2[A_2 v, A_2 w])$ are $\Pi^0_0 M$-bilinear operators, where A_1, B_1, A_2, B_2 is the crossed lift pair defined through the Theorem in Sec. 6 from the homomorphisms associated to these connections. These operators define elements ${}^1T, {}^2T \in \Pi^1_2 M$, called the *horizontal and vertical torsion*, respectively.

Thus, we say that ${}^1\nabla$ (${}^2\nabla$) is a *horizontal* (*vertical*) *metric connection* if ${}^1\nabla c = 0$, ${}^1\nabla\gamma = 0$, ${}^1T = 0$ (${}^2\nabla_c = 0$, ${}^2\nabla\gamma = 0$, ${}^2T = 0$).

Then, since $\gamma(k, \ ) = 0$, we have that ${}^1\nabla_v\gamma(k \ ,) = ({}^1\nabla_v\gamma)(k, \ ) + \gamma({}^1\nabla_v k, \ ) = \gamma({}^1\nabla_v k, \ ) = 0$. Hence, ${}^1\nabla_v k$ must be a multiple of k; but $\langle c,k\rangle = 1$ and ${}^1\nabla c = 0$. Therefore, ${}^1\nabla k = 0$, and in the same way we can prove ${}^2\nabla k = 0$.

The problem of existence for horizontal metric connections is rather difficult; in Appendix A is proved that *if the signature of* γ *is* $(0, +, ..., +)$, *the signature of* η *is* $(0, -, ..., -)$ *and c satisfies* $i\bar{v}(\langle\tilde{c},\tilde{u}\rangle) = \langle\tilde{c},\tilde{v}\rangle$, *then there is a unique horizontal metric connection on* IIM. The root of the difficulty is that we do not know *a priori* the horizontal homomorphism associated with that connection. It must be determined from our requirements together with the action of ${}^1\nabla$ upon vector fields.

Now, the increasing half-lifes of particles has been verified for many speeds and directions. Thus, we have an experimental reason for taking $(0, -, ..., -)$ as the signature of η. Assuming this for granted, there is a unique horizontal metric connection. In Newtonian spacetime, $\eta = 0$ and that connection, if it exists, is not unique; the existence condition is that $\bar{b}$ be an exact 1-form. This means a universal absolute time. The proof of that assertion is too long for bringing it here.

The physical meaning of ${}^1\nabla$ is the following: *It defines the absolute derivative of physical fields along spacetime* (cf. the interpretation of homomorphisms given in Sec. 5) *from a laboratory whose particles are each other at relative rest* (at the limit when these particles are close to the event where the derivative is taken). The reason for this last remark is that ${}^1\nabla k = 0$, and k could be looked at, in some respects, as the laboratory field. An account for this interpretation is given in Ref. 4.

As for the vertical metric connection, it defines the absolute derivative of fields along the fibres of π, having γ and c as an absolute measure for the directional dependence of fields. That vertical metric connection also is uniquely determined, and given by

$$
\begin{aligned}
{}^2\nabla_v w = D_v w + \tfrac{1}{2}g^{-1}((D_v g)(w, \,) + (D_u g)(v, \,) - (Dg)(v,w), \,) \\
- g(v,w)k - \langle c,w\rangle v,
\end{aligned}
$$

where we have put $g = \gamma - c \otimes c$ (see Appendix B).

9. LIFTING CONNECTIONS

If Δ is a physical connection and A,B is a lift lowering, then $\nabla_v w = \Delta_{Av} w$ defines a connection on IIM. The following assertion justifies our use of connections on IIM:

Given the horizontal and vertical metric connections ${}^1\nabla$, ${}^2\nabla$ *there is a unique physical connection* Δ *giving* ${}^1\nabla$ *and* ${}^2\nabla$ *through the above process. It is metric in the sense that* $\Delta c = 0$ *and* $\Delta\gamma = 0$.

For if Δ satisfies that condition, then $\Delta_{\mathbf{w}} h = \Delta_{\mathbf{Hw}} h + \Delta_{\mathbf{Vw}} h = {}^1\nabla_{B_1\mathbf{w}} h + {}^2\nabla_{B_2\mathbf{w}} h$ for every $h \in IIM$, $\mathbf{w} \in V^1_0\mathscr{F}M$. Now it is a trivial matter to prove this formula effectively gets a physical connection. Moreover $\Delta\gamma = 0$ and $\Delta c = 0$ because ${}^2\nabla_\gamma = {}^1\nabla_\gamma = 0$ and ${}^1\nabla c = {}^2\nabla c = 0$. Also we have $\Delta k = 0$.

The formula giving Δ is rather striking: It manifests itself our way to get it. It splits in two terms, corresponding to the horizontal and vertical components of $\mathbf{w}$, that is, of the tangent to the curve on $\mathscr{F}M$ along which we compute the derivative. Thus, it does not require a more detailed explanation.

Unfortunately, physical connections are awkward to handle because of their mixed nature. Due to this, we shall lift Δ for having an ordinary linear connection on $V\mathscr{F}M$. The process is the following.

Let A_1, B_1 be the horizontal lift lowering given by the metric connections ${}^1\nabla$, ${}^2\nabla$. Then, if $\mathbf{D}$ is a linear connection on $V\mathscr{F}M$, we have that $\Delta_{\mathbf{w}} h = B_1\mathbf{D}_{\mathbf{w}} A_1 h$ defines a physical connection. We shall demand that Δ should be *the* physical metric connection we have just defined.

As for A_2, B_2, the formula $B_2\mathbf{D}_{\mathbf{w}} A_2 h$ does not define a physical connection because B_2A_2 is not the identity. But $B_2A_2B_2 = B_2$, whence the preceding formula defines a physical connection on $B_2(V\mathscr{F}M)$ that is a subalgebra of IIM. But our physical metric connection also is a connection on this subalgebra, because $B_2(V^1_0\mathscr{F}M)$ is the annihilator of c, and $B_2(V^0_1\mathscr{F}M)$ is the annihilator of k. For if $\langle s,k\rangle = 0$, then $\langle\Delta_{\mathbf{w}} s,k\rangle = -\langle s,\Delta_{\mathbf{w}} k\rangle = 0$; also, if $\langle c,v\rangle = 0$, then $\langle c,\Delta_{\mathbf{w}} v\rangle = 0$. So we shall demand that $\Delta_{\mathbf{w}} B_2\mathbf{h} = B_2\mathbf{D}_{\mathbf{w}} A_2B_2\mathbf{h}$ for every $\mathbf{h} \in V\mathscr{F}M$. In addition, we demand that the parallel displacement given by $\mathbf{D}$ should apply horizontal vectors into horizontal vectors; in other words, that $\mathbf{DH} = 0$.

Theorem: There is a unique linear connection $\mathbf{D}$ on $V\mathscr{F}M$ such that $\Delta_{\mathbf{w}} h = B_1\mathbf{D}_{\mathbf{w}} A_1 h$, $\Delta_{\mathbf{w}} B_2\mathbf{h} = B_2\mathbf{D}_{\mathbf{w}} A_2B_2\mathbf{h}, \mathbf{DH} = 0$.

Proof: Note first that $\mathbf{H}$ linearly applies $V^r_s\mathscr{F}M$ into $V^r_s\mathscr{F}M$; hence, each restriction $\mathbf{H}|V^r_s\mathscr{F}M$ can be looked at as a tensor field of type $(r + s, r + s)$; in this sense, $\mathbf{DH}$ has a definite meaning. If $\mathbf{D}$ is the required connection, then $\mathbf{D}_{\mathbf{w}}\mathbf{v} = +\mathbf{D}_{\mathbf{w}}\mathbf{Hv} + \mathbf{D}_{\mathbf{w}}\mathbf{Vv} = \mathbf{D}_{\mathbf{w}}\mathbf{H}^2\mathbf{v} + \mathbf{D}_{\mathbf{w}}\mathbf{V}^2\mathbf{v}$ $= \mathbf{HD}_{\mathbf{w}}\mathbf{Hv} + \mathbf{VD}_{\mathbf{w}}\mathbf{Vv}$ because $\mathbf{H} + \mathbf{V} = \mathrm{id}$ on $V^1_0\mathscr{F}M$ and as a consequence $\mathbf{DV} = 0$. Thus $\mathbf{D}_{\mathbf{w}}\mathbf{v}$ $= A_1B_1\mathbf{D}_{\mathbf{w}}A_1B_1\mathbf{v} + A_2B_2\mathbf{D}_{\mathbf{w}}A_2B_2\mathbf{v} = A_1\Delta_{\mathbf{w}}B_1\mathbf{v} + A_2\Delta_{\mathbf{w}}B_2\mathbf{v}$. Hence, if such a linear connection exists, it is unique and given by the above formula (valid for elements of $V^1_0\mathscr{F}M$ and $V^0_1\mathscr{F}M$; for other tensor types, the expression is more complicated). Now it is a trivial exercise to prove that formula fulfill our demands.

10. GEOMETRY ON THE TIME BUNDLE

We look at γ and c as the primordial geometric features of spacetime. From them, we build unique vertical and horizontal metric connections, and they define the physical metric connection, which describes the absolute derivative of physical fields along the time manifold. Also we have the linear connection $\mathbf{D}$ that could be regarded as getting *the geometry of the time bundle itself*; in fact, the torsion of $\mathbf{D}$, its curvature and Ricci fields, Bianchi identities, etc., can now be computed as customary. Thus, our goal has been reached: we have translated the problem of spacetime geometry to the geometry of the time manifold, the manifold where the observable physics takes place. This lifting process has the advantage of recovering the usual techniques of differential geometry.

However, to tell the truth, I have some doubts about this process, in the following sense. One could also say that η

measures the increasing relative energy when the relative speed increases, because time retardation and relative energy are directly related in relativity and quantum mechanics. Therefore, η would stand for the *vertical potential* in the manner as in relativity $\bar{g}$ stands for the gravitational (horizontal) potential. Then, this symmetry lends some strength to the definition of vertical metric connections through ${}^2\nabla c = 0$, ${}^2\nabla \eta = 0$, ${}^2T = 0$. On this assumption the lifting process for connections becomes the same because we also have that $\eta(k, \ ) = 0$; but then $\Delta\gamma$ and $\Delta\eta$ are in general different from zero. Thus, what is the appropriate field, γ or η, to be used for defining a metric on the fibres of π? Relativity is not an aid because then $\gamma + \eta = 0$, whence the choice does not matter. But in Newtonian spacetime, $\eta = 0$; thus, no vertical distance among velocities?, no relative energy?, no inertia? These strange outcomes and the nonmetric character of Δ compel me to prefer γ instead η.

Disregarding these doubts, I believe this process is not merely a desperate issue from an unnecessarily puzzled starpoint; on the contrary, it seems to me more natural than the relativistic one, because it allows a step by step construction of different models of spacetime, clearing up the different options one must take for having different theories.

II. SPACELIKE LENGTH AND SPACETIME
1. INTRODUCTION

Until now, we have considered as independent data the time form and the spacelike metric. But are they independent magnitudes? In relativity the answer is no, because then $\gamma + Dc = \gamma + \eta = 0$, and there is experimental evidence favoring some link between γ and η—the Michelson–Morley experiment for instance.

Let us accept that link, but suppose that $\tilde{f}$ is a general time function, perhaps not a relativistic one. From $\tilde{f}$ we build c and $Dc = \eta$. Suppose the signature of η is everywhere $(0, -, ..., -)$. Increasing half-lifes is the experimental support for this assumption. Then, it seems a suggestive attitude to postulate that the relation between γ and η is the same as the relativistic one, i.e., $\gamma + \eta = 0$. In other words, we are *defining* the spacelike metric as $\gamma = -Dc$. From this point, we could apply the techniques of Part I for reaching a geometry of the time manifold. That would be the track of a *pure chronogeometry: to reject meters, adopt clocks and build lengths from times.* Classical chronogeometry in additon postulates that $\eta + c \otimes c = \bar{g} \circ \pi$, with $\bar{g} \in V^0_2 M$, that is we can mix these magnitudes for having a Lorentz metric.

So far I do not know examples of the opposite viewpoint: *to reject clocks, adopt meters, and build times from lengths.* My own position is the construction of a very general spacetime geometry from the datum of a spacelike metric. At least I judge this task convenient, as complementary with respect to chronogeometry. Moreover, I find some physical arguments favoring my position. First, η and γ have very different physical meanings: η stands for the rate of time retardation, and γ for spacelike length as measured by metersticks; thus, the relation $\gamma + \eta = 0$ seems rather accidental. Second, I think of time as a more dynamic feature than spacelike length, whence also a more secondary datum from our methodological viewpoint (see Part I, Sec. 4); I believe that is in the same line of thought as the way in which superspace theories are going; that is, spacelike length carries information about time, but can we say that time carries information about space? Third, Pythagoras' theorem, on which our theory leans, has always been verified at the macroscopic level, and always supposed at the microscopic one.

Thus, our departure point is a spacelike metric, that is a field $\gamma \in \Pi^0_2 M$, symmetric, of signature $(0, +, ..., +)$, and such that $\tilde{\gamma}(\tilde{u}, \ ) = 0$. This field describes Pythagoras' theorem at each laboratory (Sec. 2).

The key point of the paper is Sec. 3. On it, we define a simultaneity from γ through the criterion of stationary spacelike volume. It is a generalization of the oldest definition of simultaneity, that given by a person saying: "I cannot be in two places at the same time!" He signifies that he cannot reduce the distance (relative to him) between two events happening at different places if they are simultaneous. We will take volume instead of distance, but the basic point is the same: to take spacelike measures instead of interchanging signals for defining the simultaneity. The criterion of spacelike volume gives a time form c under a multiplicative function.

Each choice of that function defines a time function; we demand that time function to be consistent with the time form by means of infinitely slow clock transport (Sec. 4). However, this requirement does not entirely determine the time form; the equivalence among these consistent time forms gives raise to a gauge (Sec. 5).

In Sec. 6 we characterize our geometric model of space time in terms of a nonsingular symmetric field $g \in \Pi^0_2 M$. Some examples are shown.

Gauge invariance makes the definitions of metric connections on ΠM more difficult. Along the study of this problem, a field $\phi \in \Pi^0_1 M$ arises (Sec. 7). It determines the horizontal metric connection, and perhaps could be interpreted as the electromagnetic potential.

In Sec. 8 we apply the techniques of Part I for lifting connections, and so reach a physical metric connection which is gauge invariant, and a linear connection on $V\mathcal{F}M$ giving the geometry of the time bundle.

Section 9 contains some comments about our results.

2. PYTHAGOREAN SPACELIKE METRIC

For a better understanding, we will translate back and forth our constructive process from the *special* to the *general* case, in a similar manner to that of special and general relativity.

In the special case, spacetime is considered as the four-dimensional affine space. Geometric features of spacetime, that is γ or c, are supposed to be independent of events; they could perhaps depend on laboratory directions. As in special relativity or classical mechanics, if no forces act upon a particle, its world line is straight. An inertial laboratory is now a set of solidary particles, i.e., whose world lines are parallel straight lines. Any nonzero vector tangent to them must be timelike by definition. So, each laboratory is characterized by a timelike vector $\bar{x}$ or by any nonvanishing multiple of it. Let us consider ourselves traveling with the laboratory $\bar{x}$. Given an arbitrary event, it appears *located* at a well defined point of our laboratory. In spacetime language, location is the world line of the particle of our laboratory whose history contains the given event. Events which happened at the same point of our laboratory must have the same location, no matter the time elapsed among them. If two events are given, we can measure the distance between their locations by means of a meter stick *at rest in our laboratory*. Obviously, this is the ordinary method of spacelike length measurements among events: The bottle carrying the help message was found *four thousand miles* away from the wreckage⋯.

The resulting quantity depends on the vector $\bar{y}$ joining both events (4-vector of spacetime). But it is clear that it also depends on the selected laboratory, that is on $\bar{x}$. Now, we assume that Pythagoras' theorem holds at each laboratory. In other words, the spacelike length of $\bar{y}$ at $\bar{x}$ is given by $\tilde{\gamma}_{\bar{x}}(\bar{y},\bar{y})$, where $\tilde{\gamma}_{\bar{x}}$ is a quadratic form that depends on $\bar{x}$, but not (in the special case) on the events of spacetime. Obvious properties of this field $\tilde{\gamma}:\bar{x}\rightarrow\tilde{\gamma}_{\bar{x}}$ are: (a) $\tilde{\gamma}_{\bar{x}}(\alpha\bar{x},\ ) = 0$ for every $\alpha\in\mathbb{R}$, because $\alpha\bar{x}$ stands for the vector joining two events having the same location at the laboratory $\bar{x}$; (b) $\tilde{\gamma}_{\bar{x}} = \tilde{\gamma}_{\alpha\bar{x}}$ for $\alpha > 0$ since $\bar{x}$ and $\alpha\bar{x}$ stand for the same laboratory (hence we say that $\tilde{\gamma}$ is homogeneous of degree zero); (c) $\tilde{\gamma}_{\bar{x}}(\bar{y},\bar{y}) > 0$ if $\bar{y}$ is not a multiple of $\bar{x}$.

By a standard generalization, in the general case a laboratory will be a local cross section of $\pi:\mathscr{T}M\rightarrow M$, the time bundle, and γ will become a symmetric element of $\Pi^0_2 M$, of signature $(0, +, ..., +)$, and satisfying $\tilde{\gamma}(\tilde{u},\ ) = 0$, that is $\tilde{\gamma}(\tilde{u},\ )_{\bar{x}} = \tilde{\gamma}_{\bar{x}}(\bar{x},\ ) = 0$ (see Part I, Sec. 4).

3. SIMULTANEITY FROM SPACELIKE METRIC

Our problem is now the discovery of a simultaneity linked to the spacelike metric γ. The process is performed in two steps: imposing both the condition of stationary spacelike volume and that of infinitely slow clock transport synchronization. In terms of Part I, we look for time forms privileged with respect to γ; let us discuss what kind of privilege it is.

At this point it is interesting to remark that the preceding description of γ is by no means restricted to a particular class; thus, since we are looking for a generalization, it would be desirable that our definition of privileged time forms could be consistently applicable to Newtonian or relativistic spacelike metrics, considered as simple and extreme examples.

In classical Newtonian spacetime or in special relativity we can verify without difficulty the following argument (we are in the special case), whose rigorous proof is the theorem in Sec. 6. Let A, B, C, D be four events determining a hyperplane. If $\bar{x}$ is a laboratory, we can measure by means of $\tilde{\gamma}_{\bar{x}}$ the volume of the tetrahedron determined by the locations of these events in the laboratory $\bar{x}$ (or $\bar{x}$ locations). Let $V(\bar{x})$ be that volume. We call $V(\bar{x})$ the spacelike volume of the (A,B,C,D)-$\bar{x}$ locations. If the field $\bar{x}\rightarrow\tilde{\gamma}_{\bar{x}}$ is smooth and A,B,C,D remain fixed, then $V:\bar{x}\rightarrow V(\bar{x})$ is a differentiable function; let dV be its differential. Suppose that dV vanishes at some laboratory $\bar{x}_0$, i.e., $(dV)_{\bar{x}_0} = 0$. This means that the spacelike volume of the (A,B,C,D)-$\bar{x}$ locations is stationary at $\bar{x}_0$. If this occurs, then in Newtonian mechanics the four events are *pairwise absolutely simultaneous*. In special relativity, we conclude that *the hyperplane A,B,C,D is spacelike and that $\bar{x}_0$ is orthogonal to that hyperplane*; *or*, equivalently, *the four events are pairwise simultaneous as viewed from the laboratory* $\bar{x}_0$. Moreover, in both spacetime theories, given the timelike vector $\bar{x}_0$, *there is a unique hyperplane whose spacelike volume is stationary at* $\bar{x}_0$, in the above sense (strictly speaking, a distribution of parallel hyperplanes). Thus we can say that such a hyperplane is privileged at $\bar{x}_0$ with respect to $\tilde{\gamma}$, since the laboratory and its corresponding stationary spacelike volume hyperplane are related by simultaneity.

The same idea serves us for defining priviledged time forms from the spacekike metric, though it should not be Newtonian or relativistic. Suppose that c is a 1-form determining a distribution of parallel hyperplanes (we keep in the special case). Choose one of them, say, H. As before, let A,B,C,D be four fixed events determining H. Let $V(\bar{x})$ be the spacelike volume of the (A,B,C,D)-$\bar{x}$ locations, as measured by means of $\tilde{\gamma}_{\bar{x}}$. Suppose $(dV)_{\bar{x}_0} = 0$; obviously this condition does not depend on the chosen four events belonging to the fixed H. Thus, we say the spacelike volume of H is stationary at $\bar{x}_0$, and that *the events belonging to H are by definition pairwise simultaneous with respect to the laboratory* $\bar{x}_0$.

Our basic requirement upon $\tilde{\gamma}$ is: Consider the subset of timelike vectors for each of them, $\bar{x}$, there is one unique (up to a multiplicative nonzero constant) 1-form $\tilde{b}_{\bar{x}}$ whose associated hyperplanes are of stationary spacelike volume at $\bar{x}$, and such that $\langle\tilde{b}_{\bar{x}},\bar{x}\rangle\neq 0$. Then, *this subset is supposed to be nonempty and open, and it constitutes our final set of timelike vectors*. Our additional demand is: there is a representant $\tilde{c}_{\bar{x}}$ of each $\{\alpha\tilde{b}_{\bar{x}}\}_{\alpha\neq 0}$ such that the field $\tilde{c}:\bar{x}\rightarrow\tilde{c}_{\bar{x}}$ is smooth and homogeneous of degree zero. This last field is called *privileged time form*, and it defines the synchronization associated to $\tilde{\gamma}$.

In the general case this question becomes rather technical; a detailed account is given in Ref. 4. A brief sketch is the following. Let $\Sigma\subset M$ be a hypersurface of M, and B a compact regular domain of Σ, contained in the domain of some chart of Σ. Let $\{\bar{f}_\alpha\}$ be the coordinate vector fields of this chart, and $\{\bar{s}^\alpha\}$ the dual base. Let $\bar{r}$ be a cross section of $\tilde{\pi}$ such that $\bar{r}_m$ is not tangent to Σ for $m\in\Sigma$. Then, $\bar{\gamma} = \gamma\circ p\circ\bar{r}$ is a positive definite quadratic form when it acts upon $T^1_0\Sigma$. Thus, $\bar{\gamma}$ defines a volume form on Σ. Hence, the volume of B given by that volume form can be interpreted as the spacelike volume of B, as measured from the laboratory $\bar{r}$. It is given by $V(\bar{r}) = \int_B |\bar{\gamma}(\bar{f}_\alpha\bar{f}_\beta)|^{1/2}\bar{s}^1\wedge\cdots\wedge\bar{s}^{n-1}$, where $|\ |$ stands for determinant. If B remains fixed, this integral defines a functional on the field $\bar{r}$. Let us put $\tilde{\gamma} = \gamma\circ p$ and $\tilde{f}_\alpha = \bar{f}_\alpha\circ\tilde{\pi}$. By applying usual variational techniques, we find that $V(\bar{r})$ is stationary at $\bar{r}$ if we have $(i\tilde{v})_{\bar{r}_m}(|\tilde{\gamma}(\tilde{f}_\alpha\tilde{f}_\beta)|) = 0$ for every $m\in B$ and $\tilde{v}\in\tilde{\Pi}^1_0 M$.

This is a point-by-point condition, and it does not depend on the choice of $\{\bar{f}_\alpha\}$. In other terms, it only depends on the inclination of Σ at each point. Thus, we can build, at each $m\in M$ the set $T_m \subset \tilde{\mathscr{F}}_m$ of vectors as $\bar{x}_m$, for each of them there is unique n-1-dimensional subspace $\{\bar{f}_\alpha\}$ of M_m, not containing $\bar{x}_m$, such that $(i\bar{v})_{\bar{x}_m}(|\tilde{\gamma}(\tilde{f}_\alpha,\tilde{f}_\beta)|) = 0$ for every $\bar{v}\in\tilde{\Pi}{}^1_0M$ (note that $i\bar{v}$ is a derivation along the fibres of $\tilde{\pi}$). Our abstract model of spacetime, the *time-elements space* (TES), consists of a manifold M, a time bundle $\pi:\mathscr{F}M\to M$, a spacelike metric field γ such that $p(T_m) = \pi^{-1}(m)$ for every $m\in M$, and an element $c\in\Pi{}^0_1M$ such that c_{r_m} determines the unique subspace of M_m of stationary spacelike volume at $r_m\in\pi^{-1}(m)$. We call c a privileged time form, and c_{r_m} the simultaneity associated to r_m.

4. CLOCK TRANSPORT SYNCHRONIZATION

If c is a privileged time form, then qc also is a privileged time form, whenever $q\in\Pi{}^0_0M$ is everywhere nonvanishing. Thus, the choice of q defines the time function $\langle e_a(qc),\bar{u}\rangle$, that is the time length scale at each laboratory. Now, can this function q be arbitrarily picked without contradiction?

Let us return to the special case. If $\bar{x}$ is a laboratory, we will call the $\bar{x}$ clock an apparatus, at rest in $\bar{x}$, which computes time intervals among events of its history by means of $\tilde{c}_{\bar{x}}$. Equivalently, $\langle\tilde{c}_{\bar{x}},\bar{x}\rangle$ is the time interval measured by the $\bar{x}$ clock between two events of its history, detached each other by the vector $\bar{x}$.

Consider two laboratories, $\bar{x}$ and $\bar{x}'$. Suppose the $\bar{x}'$ clock lying at the spacelike origin of $\bar{x}'$, passes, at some event, next to the $\bar{x}$-clock of the origin of $\bar{x}$. At that event, both clocks are set to zero. Suppose that all $\bar{x}$ clocks are synchronized among them by the condition of stationary spacelike volume, that is, through $\tilde{c}_{\bar{x}}$. Now, does *that* $\bar{x}'$ clock point to the same hour as the $\bar{x}$ clocks it is passing by? There are few chances for getting this agreement by a suitable choice of q. With a Newtonian spacelike metric, the agreement is possible; thus, *absolute universal time is, from our viewpoint, a consequence of Newtonian spacelike metric*! In special relativity, the answer is no.

However, in relativity an intermediate thing can be achieved, the agreement when the clock transport is "infinitely slow," the limit case when relative speed approaches zero (I believe this is Eddington's idea). So, could we require this weaker agreement with all generality? The answer is affirmative. In fact, suppose that the $\bar{x}'$ clock starts from the event A. Both this clock and the $\bar{x}$ clock at A, point to zero at A. After a while, the $\bar{x}'$ clock reaches another $\bar{x}$ clock at the event B. Let $\bar{z}$ be the vector joining A with B. Then $\bar{z}$ can be decomposed as $\bar{z} = \bar{y} + \alpha\bar{x}$ where $\langle\tilde{c}_{\bar{x}},\bar{y}\rangle = 0$ and $\alpha\in\mathbb{R}$. If all $\bar{x}$ clocks are synchronized, at B the $\bar{x}$ clock points to $\langle\tilde{c}_{\bar{x}},\alpha\bar{x}\rangle$, and the $\bar{x}'$ clock, to $\langle\tilde{c}_{\bar{y}+\alpha\bar{x}},\bar{y}+\alpha\bar{x}\rangle$. Then, we demand that $\lim_{\alpha\to\infty}(\langle\tilde{c}_{\bar{y}+\alpha\bar{x}},\bar{y}+\alpha\bar{x}\rangle - \langle\tilde{c}_{\bar{x}},\alpha\bar{x}\rangle) = 0$. Now, $\tilde{c}$ is $h(0)$; thus that expression becomes
$\lim_{\alpha\to\infty}\langle\tilde{c}_{\bar{x}+\beta\bar{y}} - \tilde{c}_{\bar{x}},\bar{y}+\alpha\bar{x}\rangle = \lim_{\beta\to 0}\langle(\tilde{c}_{\bar{x}+\beta\bar{y}} - \tilde{c}_{\bar{x}})/\beta,\bar{x}\rangle$, where we have put $\beta = 1/\alpha$. Then, since $\lim_{\beta\to 0}(\tilde{c}_{\bar{x}+\beta\bar{y}} - \tilde{c}_{\bar{x}})/\beta = \langle\tilde{c}_{\bar{x}},\bar{x}\rangle^{-1}(D_y c)_{p\bar{x}}$, we conclude that infinitely slow clock transport agrees with stationary volume synchronization iff $\langle D_y c,k\rangle = 0$. Now, it is not difficult to see that this condition is equivalent to that of $i\bar{y}(\langle\tilde{c},\bar{u}\rangle) = \langle\tilde{c},\bar{y}\rangle$.

In the general case, infinitely slow clock transport is nonsense. The best we could do is the following (this new process is equivalent to infinitely slow clock transport in the special case). Let $\sigma:[a,b]\to M$ be a world line, i.e., $\dot{\sigma}(t)$ belongs to $\tilde{\mathscr{F}}M$. Let $\bar{r}$ be a cross section of $\tilde{\pi}$, and c a privileged time form. Then, the time elapsed from $\sigma(a)$ to $\sigma(b)$, as measured by the synchronized laboratory $\bar{r}$ is $\int_a^b\langle\tilde{c}_{\bar{r}\circ\sigma},\dot{\sigma}\rangle dt = \tau(\bar{r})$. If σ remains fixed, this integral is a functional on $\bar{r}$. Infinitely slow clock transport here means that $\bar{r}$ approaches $\dot{\sigma}$ on σ; the forementioned agreement translates into the condition that τ would be stationary *when* $\bar{r}\circ\sigma = \dot{\sigma}$. By requiring this for every world lines we easily find that $\langle D_v c,k\rangle = 0$ is the necessary and sufficient condition. If it is fulfilled, we can say that the "proper time" is an extremum for every world lines (in comparison with the time lapse measures performed from other laboratories); or, in the special case, that c gives the same synchronization as infinitely slow clock transport. In both cases, we say that c is a *fundamental time form*.

5. THE GAUGE

As for existence of fundamental time forms, see Sec. 6.

Suppose that c is fundamental. Then, if $\bar{a}\in V^0_0M$ is everywhere nonzero and we put $a = \bar{a}\circ\pi$, then we have $\langle D_v(ac),k\rangle = 0$ because $D(\bar{a}\circ\pi) = 0$. Therefore, ac also is fundamental. All these fundamental time forms will be regarded as equaly valid for describing geometric features of spacetime. Then, the gauge for deciding if a geometric object is physically consistent must be its invariance under the transformation $c\to ac$, where $a = \bar{a}\circ\pi$ is everywhere nonvanishing.

Since $\bar{a}$ does not depend on directions, that transformation simply means certain change of time unities on each event. But $\bar{a}$ could depend on events of spacetime. Therefore, *we cannot get an absolute comparison among time scales at different events of spacetime*; *however, at the same event, time scales for clocks with different speed can be absolutely compared with respect to each other*. Now assume that two *real clocks* depart from an event A and travel along different paths, so that they meet at B. Someone might ask if the relative tick rythm of both clocks in B is different from that on A. Whereas this question makes sense in itself, it is not relevant here, because my time is a spacewise time, and I do not know whether the time of the *real* clocks agree with it. As it has been suggested to me, perhaps this means that this theory embodies in some nonquantic manner the following quantic assertion: The uncertainty principle prevents one from knowing both the metric of a spacelike slice and its respective extrinsic curvature.

6. THE TIME ELEMENTS SPACE

Now, suppose the γ defines a TES and c is a fundamental time form. Then, $g = \gamma - c\otimes c$ defines a symmetric element of Π^0_2M of signature $(-,+,...,+)$. After rather long computations[4] we can characterize TES's through the following:

Theorem: Consider a given time manifold $\mathscr{F}M$, and let

$g \in \Pi^0_2 M$ be symmetric, of signature $(-,+,\ldots,+)$, and such that $\tilde{g}(\tilde{u},\tilde{u}) < 0$. Define $k = e_b(\tilde{u}/[-\tilde{g}(\tilde{u},\tilde{u})]^{1/2})$, $c = -g(k,\ )$, $\gamma = g + c \otimes c$. Thus, if $\langle D_v c, k \rangle = 0$ and $D_v |g(f_i,f_j)| = 0$ for every $v \in \Pi^1_0 M$ and $f_i = \bar{f}_i \circ \pi$ with $\bar{f}_i \in V^1_0 M$ $(i{:}0,1,\ldots,n-1)$, then γ defines a TES on $\mathscr{F}M$, and c is a fundamental time form. Conversely, if γ defines a TES on $\mathscr{F}M$ and c is a fundamental time form, the field $g = \gamma - c \otimes c$ satisfies the above requirements.

The theorem enables us to build fundamental time forms. If c is a privileged time form, it is enough to multiply c by a function $q \in \Pi^0_0 M$ in order to make $|g|$ constant along each fibre. Also, it facilitates the construction of TES models. Besides the relativistic one, which trivially satisfies the theorem, the generalized Newtonian spacetime (see Part I, Sec. 4) defines another TES.

We also can alloy relativistic and Newtonian theories in the following way. Suppose M admits a Lorentz metric $\bar{g} \in V^0_2 M$, whence also a timelike one-dimensional distribution. We put $\bar{b} = \bar{g}(\bar{x},\ )/(-\bar{g}(\bar{x},\bar{x}))^{1/2}$ and $\bar{q} = \bar{g} + \bar{b} \otimes \bar{b}$, where $\bar{x}$ lies in that distribution. Let $\bar{N},\bar{R} \in V^0_0 M$ be scalar fields such that $\bar{N} + \bar{R} = 1$ (the alloy ratios). By means of $\bar{g}$ we build the relativistic spacelike metric γ_R, and by means of $\bar{q}$ and $\bar{b}$ the Newtonian one, γ_N (see Part I, Sec. 4). Define the time bundle by $p(\bar{y}) \in \mathscr{F}_m$ if $\bar{y} \in M_m$, $\langle \bar{b}_m, \bar{y} \rangle \neq 0$, $\bar{g}_m(\bar{y},\bar{y})(\bar{N}_m \bar{g}_m(\bar{y},\bar{y}) - \bar{R}_m \langle \bar{b}_m,\bar{y} \rangle^2) > 0$. Then $\gamma = N\gamma_N + R\gamma_R$, where $N = \bar{N} \circ \pi$ and $R = \bar{R} \circ \pi$, defines the mixed Newtonian relativistic TES on $\mathscr{F}M$. The proof of this assertion is rather long, and for the sake of brevity I prefer to not write it down. This TES has some bizarre properties: For example, its time bundle admits speeds greater than light.

The preceding theorem excludes from our scheme the old theories with an interval given by $ds = (-g_{ij} dx^i dx^j)^{1/2} + (e/m) A_i dx^i$, and a metric field defined through the Cartan technique. In fact, that metric field would have its determinant constant along each fibre iff $A_i = 0$.

7. GAUGE INVARIANCE AND CONNECTIONS

We are interested on connections that should be compatible with the geometric structure given by γ and fundamental time forms. The gauge invariant properties of this structure are: spacelike metric, simultaneity that is the condition $\langle c,v \rangle = 0$, and fibre constancy of $|\gamma - c \otimes c|$.

Suppose that j is the vertical homomorphism associated to c and that ${}^2\nabla$ is a vertical connection such that ${}^2\nabla\gamma = 0$, ${}^2\nabla c = 0$, ${}^2T = 0$, ${}^2\nabla_v b = jv(b)$ if $b \in \Pi^0_0 M$. If $c \to c' = ac$ is a gauge transformation, we have ${}^2\nabla c' = {}^2\nabla ac = a\, {}^2\nabla c = 0$ because $a = \bar{a} \circ \pi$. Thus, our definition of vertical metric connections goes without changes. That is, to each fundamental time form c we attach a vertical metric connection ${}^2\nabla$ on ΠM, the one satisfying ${}^2\nabla c = 0$, ${}^2\nabla\gamma = 0$, ${}^2T = 0$, ${}^2\nabla_v b = jv(b)$. This connection is unique and defined by the formula which appears in Part I, Sec. 8; but it is not gauge invariant because the connection attached to c' is ${}^2\nabla' = a\, {}^2\nabla$. This is not a bad feature, as we shall see in the following section.

The definition of horizontal metric connections requires more care. We will say that ${}^1\nabla$ is a horizontal metric connection if it is torsionless and:

(a) ${}^1\nabla\gamma = 0$, that is, ${}^1\nabla$ preserves spacelike length;

(b) $\langle {}^1\nabla_v c, w \rangle = 0$ if $\langle c,w \rangle = 0$; hence, ${}^1\nabla$ preserves simultaneity;

(c) $D_v(g^{-1} \lrcorner\, {}^1\nabla_z g) = 0$ whenever $v \in \Pi^1_0 M$ and $z = \bar{z} \circ \pi$ ($\lrcorner$ stands for double contraction). Equivalently, ${}^1\nabla$ preserves the fibre constancy of $|g|$.

Condition (a) tells that ${}^1\nabla_v k$ must be a multiple of k because $\gamma(k,\ ) = 0$ and ${}^1\nabla\gamma = 0$. Thus, there must be some $\phi \in \Pi^0_1 M$ such that ${}^1\nabla_v k = -\langle \phi,v \rangle k$. Condition (b) implies that ${}^1\nabla_v c$ must be a multiple of c; but $\langle c,k \rangle = 1$. Therefore, ${}^1\nabla_v c = \langle \phi,v \rangle c$. Then ${}^1\nabla_z g = {}^1\nabla_z(\gamma - c \otimes c) = -2\langle \phi,z \rangle c \otimes c$. Since $g^{-1}(c,\ )' = -k$, then $g^{-1} \lrcorner\, {}^1\nabla_z g = 2\langle \phi,z \rangle$ and $D_v(g^{-1} \lrcorner\, {}^1\nabla_z g) = 2\langle D_v \phi, z \rangle = 0$, because $z = \bar{z} \circ \pi$. Therefore, ϕ must be associated to some element $\bar{\phi} \in V^0_1 M$, that is, $\phi = \bar{\phi} \circ \pi$.

In Appendix A we prove that such a connection exists and is unique.

Now, let $c \to c' = ac$ be a gauge transformation. Then, if ${}^1\nabla_v c = \langle \phi,v \rangle c$, we have ${}^1\nabla_v c' = \langle (\bar{\phi} + d \ln \bar{a}) \circ \pi, v \rangle c'$ because ${}^1\nabla$ is horizontal. Thus, if we require that ${}^1\nabla$ be gauge invariant, then ϕ must change into $\phi + (d \ln \bar{a}) \circ \pi$ under a gauge transformation. Therefore, *if $\bar{\phi}$ is associated to c in such a manner that $\bar{\phi} + d \ln \bar{a}$ is associated to $(\bar{a} \circ \pi)c$, then there is a unique gauge invariant horizontal metric connection.*

We will think of ϕ, thogther with γ, as the fundamental data of spacetime geometry. We tentatively call ϕ the electromagnetic potential, though its true meaning must be disclosed only after disclosing field equations. Its operational definition is the following. Let $m \in M$ be fixed. Take $r_m \in \mathscr{F}_m$, and extend r_m to a cross section r of π in a neighborhood U of m in such a manner that r be experimentally stationary at m; in other words, *we suppose there is an operational definition for the relative rest of close particles with respect to a given one*. Now, restrict γ and ac to r, that is, take the values of these fields at the laboratory r for having the ordinary fields $\gamma \circ r$, $(ac) \circ r$; build the Lorentz metric $\bar{g} = (\gamma - a^2 c \otimes c) \circ r$, where $a = a \circ \pi$ is to be determined. Compute the Riemann standard connection of $\bar{g}$. Check if the normalized laboratory field $k \circ r$ is stationary at m, i.e., if the covariant derivative of $k \circ r$ is zero at m. If this is not so, pick $\bar{a}$ in a suitable manner in order to have an affirmative answer. Then $\bar{\phi}_m = -(d \ln \bar{a})_m$ is the value at m of the electromagnetic potential associated to c. Therefore, the electromagnetic potential associated to ac is zero. In other words, *the value at m of the electromagnetic potential associated with c is minus the differential at m of the deviation of c from the time form, which correctly gives the observable stationary (at m) character of a laboratory with normalized speed.*

If $\bar{\phi}$ is the differential of a function, it can be globally removed by a suitable election of $\bar{a}$. That is, in such a case we would have an absolute comparison among time unities at different places of spacetime. If $\bar{\phi}$ is not so, that comparison does not globally exist; we only can compare clocks at the limit when they join together. The proof of this interpretation requires additional techniques; it can be found in Ref. 4.

8. THE PHYSICAL METRIC CONNECTION AND THE CONNECTION ON $V\mathscr{F}M$

As in Part I, we now lift the pair ${}^1\nabla, {}^2\nabla$. So, we obtain a gauge invariant physical metric connection $\Delta_{\mathbf{w}}h = {}^1\nabla_{B_1\mathbf{w}}h + {}^2\nabla_{B_2\mathbf{w}}h$. Since ${}^1\nabla$ is gauge invariant and $B_1\mathbf{w} = \pi_* \circ \mathbf{w}$, then the first term is gauge invariant. As for the last term, we have seen that in a gauge transformation, the relation ${}^2\nabla' = a\,{}^2\nabla$ holds. But j changes into $j' = aj$. Hence $B_2|V^1_0\mathscr{F}M$ changes into B_2/a. Therefore, the last term and, as a consequence, Δ are gauge invariant.

Now, we can lift Δ as in Part I, Sec. 9, getting the linear connection $\mathbf{D}$ on $V\mathscr{F}M$, which is defined by $\mathbf{D}_{\mathbf{w}}\mathbf{v} = A_1\Delta_{\mathbf{w}}B_1\mathbf{v} + A_2\Delta_{\mathbf{w}}B_2\mathbf{v}$. Note that $\mathbf{D}$ is not gauge invariant. In fact we have

$$\mathbf{D}'_{\mathbf{w}}\mathbf{v} = A_1\Delta_{\mathbf{w}}B_1\mathbf{v} + A'_2\Delta_{\mathbf{w}}B'_2\mathbf{v} = \mathbf{D}_{\mathbf{w}}\mathbf{v} - \mathbf{w}(\ln a)\mathbf{V}\mathbf{v}.$$

Nevertheless, *the curvature field of* $\mathbf{D}$, *and therefore its Ricci field, are gauge invariant*, as it is easily proved.

9. CONCLUSION

The departure point of this paper is the Pythagorean spacelike metric, a principle which permeates every significant theory, experiment, and technology. The electromagnetic potential appears later, in the study of connections. I believe this point is very coherent in a tentative unified theory. In others, the electromagnetic field appears in the construction of the static geometric description—the metric—under the form of light signals; but it also appears, as a geometric object, in the dynamic description—connections or field equations. Thus, field equations must imply that the electromagnetic field propagates along null directions; otherwise, the theory would be meaningless. In our theory, this objection does not go.

We have reached a number of geometric objects enabling one to study the time manifold geometry. The main remaining problem is to suggest and justify field equations. In my opinion, it is a very difficult one:

(a) because of the horrific computations, even in simple models that perhaps could serve as a guideline for generalization;

(b) because the energy–momentum field depends on the geometry; thus, it must be reinterpreted under our basic assumptions;

(c) our manifold is now $\mathscr{F}M$; hence, usual patterns of field equation techniques cannot directly be translated here.

A naive field equation would be $\delta\int_\Omega |\mathbf{K}|^{1/2}d\tau = 0$, where Ω is a domain of $\mathscr{F}M$, $d\tau$ is the coordinate standard volume form on $\mathscr{F}M$, and $|\mathbf{K}|$ is the Ricci field determinant of $\mathbf{D}$. I have computed this integral for the relativistic model and my results are:

(a) if $\phi = 0$, then $|\mathbf{K}| = 0$, and this field equation is meaningless;

(b) if $\phi = 0$ but the Lorentz metric is constant (special relativity), then $|\mathbf{K}|$ also vanishes;

(c) I have studied a static spherical model (one-charged body problem), and there is no solution for the field equation. I believe this result is general for a relativistic model, but I do not have a proof.

I find three answers to these troubles. First, we must add to the integrand a term (or factor), depending on directions, standing for a mass-energy density. Second, the dependence of fields on directions is an essential property of spacetime, whence it precludes the assumption of a Lorentz metric; or, equivalently, that relativity is not compatible with local mass or charge anisotropy. Third, that the field equation is not appropriate.

I feel this last is the correct answer. So, it seems that this way will be around for a while.

APPENDIX A: EXISTENCE AND UNIQUENESS OF HORIZONTAL METRIC CONNECTIONS

We look for horizontal metric connections, in the sense that ∇ is horizontal, $\nabla c = \phi \otimes c$, $\nabla k = -\phi \otimes k$, $\nabla\gamma = 0$, $T = 0$, where $\phi = \bar{\phi}\circ\pi$. In Part I, ϕ is supposed to be zero.

It seems to me that this problem must be treated through local expressions, at least in a first attempt. But this way gives raise to another difficulty: The charts of the manifold $\mathscr{F}M$ are awkward to handle. So, we shall develop a technique enabling us to translate the problem to ordinary Finsler fields and Laugwitz connection (cf. Ref. 2), whose local expressions are simpler. Analogous techniques can be applied in other computations, for example the curvature or Ricci fields of $\mathbf{D}$.

If ∇ is a solution, and $\tilde{v}$, $\tilde{w}$ are h (0), we can put $\tilde{\nabla}_{\tilde{w}}\tilde{v} = e_a\nabla_{e_b\tilde{w}}e_b\tilde{v}$ for defining a Laugwitz connection; we also need to know the action of $\tilde{\nabla}$ upon fields of $\tilde{\Pi}^0_0M$, that is, the associated homomorphism $\tilde{A}:\tilde{\Pi}^1_0M \rightarrow V^1_0M$. If $\tilde{w}$ is h (0) and $\tilde{\mathbf{s}} \in V^0_1\mathscr{F}M$, we define $\tilde{A}\tilde{w}$ through $\langle\tilde{\mathbf{s}},\tilde{A}\tilde{w}\rangle = \langle\tilde{\mathbf{s}} - \langle\tilde{\mathbf{s}},i\tilde{u}\rangle d\ln\langle\tilde{c},\tilde{u}\rangle, p_*^{-1}Ae_b\tilde{w}\rangle$, where A is the homomorphism associated with ∇ and $p_*^{-1}Ae_b\tilde{w}$ is any element of $V^1_0\mathscr{F}M$ such that $p_*\circ(p_*^{-1}Ae_b\tilde{w}) = (Ae_b\tilde{w})\circ p$. This definition is consistent because $\tilde{\mathbf{s}} - \langle\tilde{\mathbf{s}},i\tilde{u}\rangle d\ln\langle\tilde{c},\tilde{u}\rangle, i\tilde{u}\rangle = 0$ and $i\tilde{u}$ spans $\ker p_*$.

Proposition: With the above notation, $\tilde{\nabla}$ is a torsionless horizontal Langwitz connection such that $\tilde{\nabla}\tilde{\gamma} = 0$, $\tilde{\nabla}\tilde{u} = -\tilde{\phi}\otimes\tilde{u}$, $\tilde{\nabla}\tilde{c} = \tilde{\phi}\otimes\tilde{c}$, where we have put $\tilde{\gamma} = e_a\gamma$, $\tilde{c} = e_a c$, $\tilde{\phi} = e_a\phi$.

Proof: that $\tilde{\nabla}$ is a Laugwitz connection is a trivial matter. It is horizontal because if $\tilde{v}$ is h (0), then $(\tilde{A}\tilde{v})_{\tilde{x}}(\bar{a}\circ\tilde{\pi}) = \langle d{:}(\bar{a}\circ\tilde{\pi}), p_*^{-1}Ae_b\tilde{v}\rangle_{\tilde{x}} = \langle p^*d(\bar{a}\circ\pi), p_*^{-1}Ae_b\tilde{v}\rangle_{\tilde{x}} = \langle d(\bar{a}\circ\pi), Ae_b\tilde{v}\rangle_{p\tilde{x}} = (e_b\tilde{v})_{p\tilde{x}}(\bar{a}) = \tilde{v}_{\tilde{x}}(\bar{a})$. Now, the torsion of $\tilde{\nabla}$ is given by $\tilde{T}(\tilde{v},\tilde{w}) = \tilde{\nabla}_{\tilde{v}}\tilde{w} - \tilde{\nabla}_{\tilde{w}}\tilde{v} - \tilde{\pi}_*\circ[\tilde{A}\tilde{v},\tilde{A}\tilde{w}]$. Then, if $\tilde{v}\tilde{w}$ are h (0), we have $\tilde{T}(\tilde{v},\tilde{w}) = \pi_*\circ[Ae_b\tilde{v},Ae_b\tilde{w}]\circ p - \tilde{\pi}_*\circ\{\tilde{A}\tilde{v},\tilde{A}\tilde{w}\}$, because $T = 0$. Taking account of the definition of $\tilde{A}$, it is not difficult to prove that $p_*\circ\tilde{A}\tilde{w} = (Ae_b\tilde{w})\circ p$. Thus $p_*\circ[\tilde{A}\tilde{v},\tilde{A}\tilde{w}] = [Ae_b\tilde{v},Ae_b\tilde{w}]\circ p$. Hence $\tilde{T} = 0$ because $\tilde{\pi} = \pi\circ p$. Now, $\nabla_{\tilde{w}}\tilde{u} = \nabla_{\tilde{w}}\langle\tilde{c},\tilde{u}\rangle e_a k = \tilde{A}\tilde{w}(\langle\tilde{c},\tilde{u}\rangle)e_a k + \langle\tilde{c},\tilde{u}\rangle e_a\nabla_{e_b\tilde{w}}k = -\langle\tilde{\phi},\tilde{w}\rangle\langle\tilde{c},\tilde{u}\rangle e_a k = -\langle\tilde{\phi},\tilde{w}\rangle\tilde{u}$. The proof of $\tilde{\nabla}\tilde{\gamma} = 0$ and $\tilde{\nabla}\tilde{c} = \tilde{\phi}\otimes\tilde{c}$ is trivial.

Proposition: Let $\widetilde{\nabla}$ be a torsionless horizontal Laugwitz connection such that $\widetilde{\nabla}\tilde{c} = \tilde{\phi} \otimes \tilde{c}$, $\widetilde{\nabla}\tilde{u} = -\tilde{\phi} \otimes \tilde{u}$, $\widetilde{\nabla}\tilde{\gamma} = 0$. Then $\Delta_v w = e_b \widetilde{\nabla}_{c_* v} e_a w$ defines a horizontal metric connection on IIM.

Proof: The formula defining ∇ makes sense if $\widetilde{\nabla}_{\tilde{v}} \tilde{w}$ is $h(0)$ whenever $\tilde{v},\tilde{w}$ are, and this is supposed to be true in our case, as we can verify by means of the formula (1) below. The proof that $\nabla k = -\phi \otimes k$, etc., is straightforward. Now, $\widetilde{\nabla}_{\tilde{v}} \tilde{w} = e_a e_b \widetilde{\nabla}_{c_* c_* \tilde{v}} e_a e_b \tilde{w} = e_a \nabla_{c_* \tilde{v}} e_b \tilde{w}$; therefore, ∇ and $\widetilde{\nabla}$ do induce each other in the sense of our previous proposition if A and $\widetilde{A}$ are related as before. We have $\widetilde{\nabla}\langle \tilde{c},\tilde{u}\rangle = 0$, hence $\langle \tilde{\mathbf{s}},\widetilde{A}\tilde{v}\rangle = \langle \tilde{\mathbf{s}} - \langle \tilde{\mathbf{s}}, i\tilde{u}\rangle d\ln\langle \tilde{c},\tilde{u}\rangle,\widetilde{A}\tilde{v}\rangle$. But $\langle \tilde{\mathbf{s}} - \tilde{\mathbf{s}}, i\tilde{u}\rangle d\ln\langle \tilde{c},\tilde{u}\rangle$ belongs to the $V^0_0\widetilde{\mathscr{F}}M$-module spanned by $p^*(V^0_1\mathscr{F}M)$; thus, if $a \in II^0_0 M$, then $\langle d(a\circ p),\widetilde{A}\tilde{v}\rangle = \langle p^* da,\widetilde{A}\tilde{v}\rangle = \langle (da)\circ p, p_* \circ \widetilde{A}\tilde{v}\rangle$; but if $\tilde{v}$ is $h(0)$, then $\langle d(a\circ p),\widetilde{A}\tilde{v}\rangle = \widetilde{\nabla}_{\tilde{v}}(a\circ p) = e_a \nabla_{c_* \tilde{v}} a = \langle (da)\circ p,(Ae_b \tilde{v})\circ p\rangle$, whence $p_* \circ \widetilde{A}\tilde{v} = (A\, e_b \tilde{v})\circ p$. Then, as before, we can prove the torsion of ∇ is zero.

Thus, our problem of existence and uniqueness can be equivalently stated on $\widetilde{II}M$. Let $\{\bar{x}^i\}$ be a coordinate system on $U \subset M$. We take for $\widetilde{\mathscr{F}}U$ the coordinate functions $\{q^i,p^j\}$ defined by $q^i(\bar{x}_m) = \bar{x}^i(m), p^i(\bar{x}_m) = \langle (d\bar{x}^i)_m,\bar{x}_m\rangle$. Since $\widetilde{A}$ is horizontal, we can write $\widetilde{A}\tilde{e}_i = \partial/\partial q^i + A^j_i(\partial/\partial p^j)$, where $\tilde{e}_i = (\partial/\partial \bar{x}^i)\circ \tilde{\pi}$. We put $\widetilde{\nabla}_{\tilde{e}_i}\tilde{e}_k = \Gamma^j_{ik}\tilde{e}_j$. Then $\widetilde{\nabla}_{\tilde{e}_i}\tilde{u} = \widetilde{\nabla}_{\tilde{e}_i}(u^k\tilde{e}_k) = \Gamma^j_{ik}u^k\tilde{e}_j + \widetilde{A}\tilde{e}_i(u^j)\tilde{e}_j = -\phi_i u^j \tilde{e}_j$. Therefore, $A^j_i = -\Gamma^j_{ik}u^k - \phi_i u^j$. If we put $\tilde{g} = \tilde{\gamma} - \tilde{c}\otimes\tilde{c} = g_{ij}(d\bar{x}^i\circ\tilde{\pi}) \otimes (d\bar{x}^j\circ\tilde{\pi})$, then the matrix (g_{ij}) is everywhere regular. Thus, after some standard computation, we find there is a solution on $\widetilde{\mathscr{F}}U$ if the following linear system has it:

$$2g_{ir}\Gamma^r_{jk} + \frac{\partial g_{ik}}{\partial p^r}\Gamma^r_{jm}u^m + \frac{\partial g_{ij}}{\partial p^r}\Gamma^r_{km}u^m - \frac{\partial g_{jk}}{\partial p^r}\Gamma^r_{im}u^m$$
$$= \frac{\partial g_{ij}}{\partial q^k} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^i} - 2\phi_i c_j c_k + 2\phi_j c_i c_k + 2\phi_k c_i c_j, \qquad (1)$$

where $\tilde{c} = c_i(d\bar{x}^i\circ\tilde{\pi})$. If we contract (1) with u^k, it becomes

$$2g_{ir}B^r_j + c_{ir}B^r_j + \frac{\partial g_{ij}}{\partial p^r}B^r_k u^k - c_{jr}B^r_i = M_{jik}u^k, \qquad (2)$$

where we have put $B^r_j = \Gamma^r_{jk}u^k$, $c_{ir} = (\partial g_{ik}/\partial p^r)u^k$, and M_{jik} is the right-hand side of (1). Since (g_{ij}) is regular, there is a solution for (1) if it occurs for (2). We can verify without difficulty that $\gamma_{ir} + \eta_{ir} = -c_{ir}$. Hence $c_{ir} = c_{ri}$ and $c_{ir}u^r = 0$. Suppose we write (2) taking at $\bar{x}_m \in \widetilde{\mathscr{F}}U$ the values of the different quantities. Then, since $\tilde{\gamma}$ has signature $(0, +, \ldots, +)$, we can choose the coordinates in such a maner that, at $\bar{x}_m$, we would have $\gamma_{\alpha\beta} = \delta_{\alpha\beta}, c_{\alpha\beta} = s_\alpha\delta_{\alpha\beta}, \gamma_{k0} = c_{k0} = 0$ (Greek indexes from 1 to $n-1$). Now (2) has a unique solution at $\bar{x}_m$ if $2 + s_\alpha + s_\beta \neq 0$ for every $\alpha,\beta \in \{1,\ldots,n-1\}$. Thus taking into account that $\gamma_{ir} + \eta_{ir} = -c_{ir}$, we have after some obvious steps:

Theorem: Let $r \in \mathscr{F}M$, and suppose that $(0,1,\ldots,1)$ and $(0,\eta_1,\ldots,\eta_{n-1})$ are the diagonal elements of γ_r and η_r when they are simultaneously diagonalized. Then, if $\eta_\alpha + \eta_\beta \neq 0$ for every $\alpha,\beta \in \{1,\ldots,n-1\}$, $r \in \mathscr{F}M$, there is one unique horizontal metric connection on IIM.

Corollary: If η is supposed to have signature $(0, -, \ldots, -)$ everywhere, there is one unique horizontal metric connection on IIM.

APPENDIX B: *Existence and uniqueness of vertical metric connections*

Our conditions are: $\nabla_v a = jv(a)$, $\nabla c = 0$, $\nabla\gamma = 0$, $T = 0$.

We put $\nabla_w v - D_w v = G(w,v)$; since $\nabla_u a = D_u a = jw(a)$, G is a bilinear operator and it defines an element of $II^1_2 M$. If $\nabla c = 0$, then $0 = \langle \nabla_w c,v\rangle$

$$= \nabla_w\langle c,v\rangle - \langle c,\nabla_w v\rangle$$

$$= D_w\langle c,v\rangle - \langle c,D_w v\rangle - \langle c,G(w,v)\rangle = \eta(w,v) - G(w,v).$$

Thus, $\langle c,G(w,v)\rangle = \eta(w,v)$ for every $v,w \in II^1_0 M$.

Now, if $\langle c,v\rangle = \langle c,w\rangle = 0$, then $B_2[A_2 w,A_2 v] = D_w v - D_v w$, as it is easily proved. Then, we have in general that $B_2[A_2 w,A_2 v] = D_w v - D_v w - \langle c,v\rangle w + \langle c,w\rangle v - \langle c,D_w v\rangle k + \langle c,D_v w\rangle k$.

If $T(w,v) = B_2 A_2(\nabla_w v - \nabla_v w - B_2[A_2 w,A_2 v]) = 0$, then $B_2A_2(G(w,v) - G(v,w) + \langle c,v\rangle w - \langle c,w\rangle v + \langle c,D_w v\rangle k - \langle c,D_v w\rangle k)$

$= B_2A_2(G(w,v) - G(v,w) + \langle c,v\rangle w - \langle c,w\rangle v)$. But if

$\langle c,G(w,v)\rangle = \eta(w,v)$, then

$\langle c,G(w,v) - G(v,w) + \langle c,v\rangle w - \langle c,w\rangle v\rangle = 0$. Thus, our second condition upon G is

$$G(w,v) - G(v,w) = \langle c,w\rangle v - \langle c,v\rangle w.$$

If $\nabla\gamma = 0$, by a standard computation we have:

$$2\gamma(G(v,w),z) = (D_v\gamma)(w,z) + (D_w\gamma)(z,v) - (D_z\gamma)(v,w) + 2\gamma(v,w)\langle c,z\rangle - 2\gamma(v,z)\langle c,w\rangle.$$

But

$$2g(G(v,w),z) = 2\gamma(G(v,w),z) - 2\langle c,G(v,w)\rangle\langle c,z\rangle = 2\gamma(G(v,w),z) - 2\eta(v,w)\langle c,z\rangle.$$

Hence

$$2g(G(v,w),z) = (D_v\gamma)(w,z) + (D_w\gamma)(z,v) - (D_z\gamma)(v,w) + 2(\gamma(v,w) - \eta(v,w))\langle c,z\rangle - 2\gamma(v,z)\langle c,w\rangle.$$

This formula tells us that if the vertical metric connection exists, it is unique. It can be equivalently written

$$2g(G(v,w),z) = (D_v g)(w,z) + (D_w g)(z,v) - (D_z g)(v,w) + 2g(v,w)\langle c,z\rangle - 2g(v,z)\langle c,w\rangle.$$

Hence

$$\nabla_v w = D_v w + \tfrac{1}{2} g^{-1}((D_v g)(w,\ ) + (D_u g)(v,\ ) - (Dg)(v,w),\ ) - g(v,w)k - \langle c,w \rangle v.$$

This formula gives the vertical metric connection, as it is easily proved.

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