Space–time intervals as light rectangles

N. David Mermin
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853-2501

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Two inertial observers in relative motion must each see the other's clock running at the same rate. The representation of this symmetry of the Doppler effect in a two-dimensional space–time diagram reveals an important geometrical fact: The squared interval between two events is proportional to the area of the rectangle of photon lines with the events at diagonally opposite vertices.

Recently I described in these pages[1] an approach to the space–time diagrams of Minkowski that leads naturally to a simple geometric interpretation of the invariant interval between two events: *The squared interval between two events, represented as points in a two-dimensional space–time diagram, is proportional to the area of the unique rectangle of four photon lines having those two points at diagonally opposite vertices*. The proportionality constant (of less interest for many purposes) is given in terms of the invariant product of two frame-dependent scale factors,[2] λ and μ, used by various observers to relate certain distances in the diagram to the separations the observers assign to events in real space and time (see Fig. 1).

I describe here a simpler route[4] to these conclusions. For this purpose it is best to define the interval between two time-like separated events (I return below to space-like or light-like separated events) as the time between the events in a frame in which they happen at the same place—i.e., as the amount by which a uniformly moving clock, present at both events, advances between the first and the second. Clearly the squared time between any two events that both happen at the same place in a *single* frame is proportional to the area of *any* geometric figure of fixed shape that scales linearly with the distance in the diagram between the events. The rectangle of photon lines is special because its area remains proportional to the interval between two events, even when the events in different pairs are at the same place in *different* frames. What must therefore be established is that if events P_1 and P_2 happen at the same place in one frame (Alice's), and events Q_1 and Q_2 happen at the same place in another (Bob's), then equality of the time between P_1 and P_2 in Alice's frame and the time between Q_1 and Q_2 in Bob's is the same as equality of the area of the two photon rectangles. This follows directly from a gedanken experiment.

Let Alice be present at events P_1 and P_2, with a clock that reads 0 at the first event and T at the second, and let Bob be present at Q_1 and Q_2, with a clock that reads 0 and T at those two events (Fig. 2). To compare the areas of the photon rectangles determined by the two pairs of events, we may translate Bob and his pair so that the points representing Q_1 and P_1 coincide (Fig. 3). The resulting figure can be interpreted as describing two clocks, in relative motion, which are together when both read 0. I have added to the figure two solid photon lines, which determine the time that Alice and Bob each *sees* the other's clock reading when their own clock reads T. I have also extended those solid photon lines (with dashed photon lines) into two rectangles, one of which has Alice's two events at opposite vertices, and the other of which has Bob's.

Note that Alice and Bob each bears the same relation to the clock of the other: each is watching a clock moving away at the same speed. The principles of relativity and the constancy of the velocity of light require that each must see the moving clock running at a rate that differs from that of their own by the same factor. In particular, Alice and Bob must each see the other's clock reading the same time t at the moment their own clock reads T. It then follows directly from some elementary proportions (explained in the caption of Fig. 3) that the two photon rectangles do indeed have the same area. Therefore the squared interval between any two time-like separated events is indeed proportional to the area of the rectangle of photon lines having the events at diagonally opposite vertices.

To establish from this the invariance of the product of scale factors and its relation to the proportionality constant, note (Fig. 4) that two copies of either of the rectangles of Fig. 2 can be cut up and reassembled into a rhombus that has the two events at adjacent vertices. The area of the rhombus is equal to the product of the length μT of any side with the distance λT between sides. The area of the original rectangle is therefore $\Omega=\frac{1}{2}\lambda\mu T^2$, where λ and μ are the scale factors

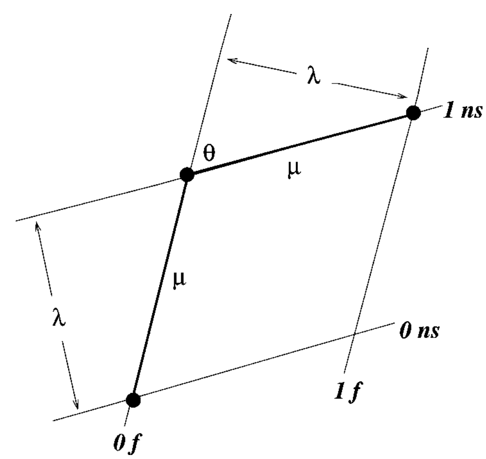

Fig. 1. The scale factors λ and μ for a frame of reference with the indicated lines of constant time and position. Events represented by points on the line labeled 1 ns happen one nanosecond after events represented by points on the line labeled 0 ns; events represented by points on the line labeled 1 f happen one foot away (Ref. 3) from events represented by points on the line labeled 0 f. The scale factors (in length of diagram per nanosecond or foot) are the distances indicated in the diagram. Although λ and μ depend on frame of reference, it turns out that their product $\lambda\mu$, the area of the rhombus bounded by the four lines, does not. Evidently the scale factors for a given frame of reference are related by $\lambda=\mu \sin \theta$, where θ is the angle between lines of constant time and position used in that frame.

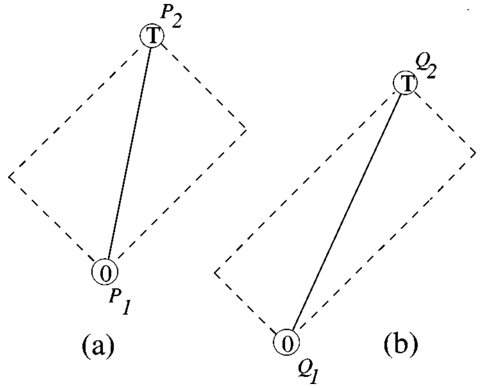

Fig. 2. (a) The world line of a clock present at two events P_1 and P_2 that are at the same place and a time T apart in Alice's frame. (b) The same as (a) for two other events Q_1 and Q_2 that are at the same place and are the same time T apart in Bob's frame. The distance in the diagram $\mu_B T$ between the points representing the events Q_1 and Q_2 differs from the distance in the diagram $\mu_A T$ between the points representing the events P_1 and P_2, because Alice and Bob use different scale factors μ to relate separation in time to distance in the diagram between events that happen at the same place. However the *areas* of the two rectangles formed by photon trajectories emerging from the events are the same. This is established in Fig. 3.

for the frame in which the events happen at the same place. Since a pair of events at the same place and separated by a time T in Alice's frame gives rise to a photon rectangle with the same area Ω as does a pair of events at the same place and separated by the same time T in Bob's, it follows that the product of scale factors $\lambda\mu = 2\Omega/T^2$ must be the same for Alice and Bob: $\lambda_A\mu_A = \lambda_B\mu_B$.

If two events are space-like rather than time-like separated, then the interval can be defined as the distance between them in a frame in which they happen at the same time. Given two space-like separated events, one can reflect them in a photon line to get two other events which are

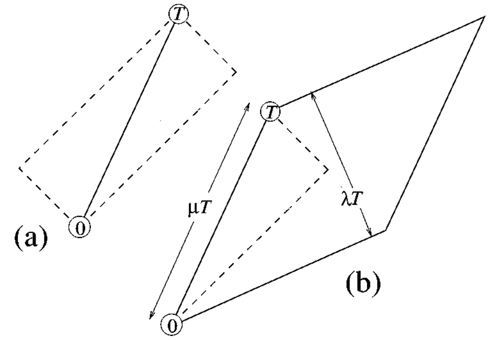

Fig. 4. The area of the photon rectangle in (a) is half the area of the rhombus in (b). But the area of the rhombus is the length μT of a side times the distance λT between sides. So the area of the rectangle is $\frac{1}{2}\lambda\mu T^2$.

time-like separated (Fig. 5). The time (in nanoseconds) between the time-like separated events in the frame in which they happen at the same place is the same as the distance (in feet)[3] between the space-like separated events in the frame in which they happen at the same time. The square of that time, however, is $2/\lambda\mu$ times the area of the photon rectangle determined by the time-like separated events. Since that rectangle goes under reflection into the photon rectangle determined by the two space-like separated events, and since the area of a rectangle is invariant under reflection, the same geometric interpretation of the squared interval holds for space-like separated events: It is proportional (with the same proportionality constant $2/\lambda\mu$) to the area of the rectangle of photon lines having those events at diagonally opposite vertices.

For light-like separated events (events that can be joined by a light trajectory) the photon rectangle degenerates to a single line, as it should since the interval between such events is zero.

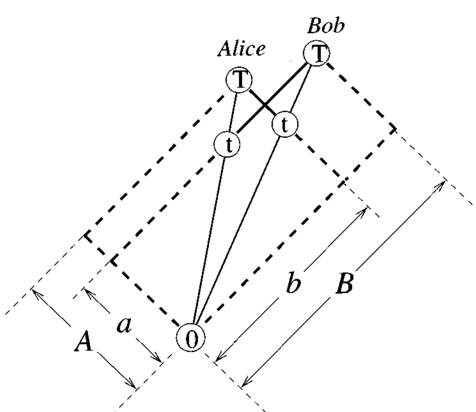

Fig. 3. The two parts of fig. 1 have been translated to describe a situation in which both clocks read 0 at the same place and time. At the moment each clock reads T, Alice and Bob (who are with their clocks) each *looks* at the other's clock and *sees* it reading t. For either clock, the ratio of t and T is just the ratio of the distances in the diagram μt and μT from the events in which the clock had those readings, back to the event at which it read 0. (One uses μ_A for Alice's clock and μ_B for Bob's.) It is evident that this ratio (as read from the trajectory of Alice's clock) is the same as the ratio of a to A or (as read from the trajectory of Bob's clock) the same as the ratio of b to B. But if $a/A = b/B$ then $Ba = bA$—i.e., the rectangle of photon lines with Bob's clock reading 0 and T at opposite vertices has the same area as the corresponding rectangle for Alice's clock.

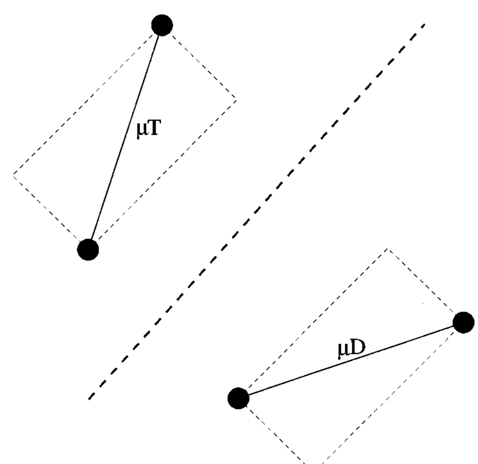

Fig. 5. The black dots above the heavy dashed light line represent a pair of time-like separated events, a time T apart in the frame in which they happen at the same place. The length of the line connecting them is μT and the area of the photon rectangle on whose vertices they lie is $\frac{1}{2}\lambda\mu T^2$ (where λ and μ are the scale factors for that frame). This entire structure appears mirrored below the heavy dashed photon line. Now it represents two space-like separated events, a distance D apart in the frame in which they happen at the same time. Since the lengths μT and μD are the same and the areas of the rectangles are the same, the area of the photon rectangle on whose vertices the space-like separated events lie is $\frac{1}{2}\lambda\mu D^2$.

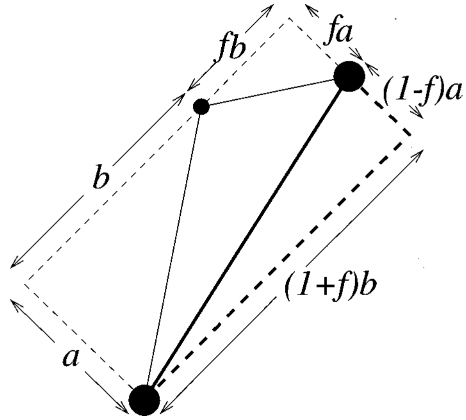

Fig. 6. The two large black dots are two events. The two thin solid lines are lines of constant time and constant position in a frame in which the events happen a time T and a distance D apart. The difference between the areas of the thin-sided right triangles formed by those lines and the thin photon lines is $\frac{1}{2}ba-\frac{1}{2}(fb)(fa)$. This can be re-expressed as $\frac{1}{2}[(1+f)b][(1-f)a]$, which the figure reveals to be the area of the thick-sided right triangle formed by the thick solid line joining the events and the thick photon lines. But the area of the thick-sided triangle is $\frac{1}{4}\mu\lambda I^2$, the area of the large thin-sided triangle is $\frac{1}{4}\mu\lambda T^2$, and the area of the small thin-sided triangle is $\frac{1}{4}\mu\lambda D^2$. Therefore $I^2=T^2-D^2$.

Figure 6 provides the connection with the coordinate-based definition of interval. It shows (using triangular halves of the relevant light rectangles, so as not to clutter the figure) that if two time-like separated events are a time T and a distance D apart in a frame in which they do not happen at the same place, then the value I^2 of the squared interval is indeed equal to $|T^2-D^2|$. (The same figure, reflected in a 45° photon line, demonstrates the same conclusion for two space-like separated events.)

What about the aspect ratio of the photon rectangle determined by a pair of events? Unlike its area, this depends on the flexibility observers have, in setting up a diagram, to choose the angles between lines of constant time and position. The aspect ratio can, however, be simply related to the velocity v, in the frame in which lines of constant time and position are orthogonal, of the frame in which the events happen at the same place (or at the same time). Figure 7 shows that the aspect ratio is $(1-v)/(1+v)$. Finally, Fig. 8 extracts from this the usual expression for the longitudinal Doppler shift.

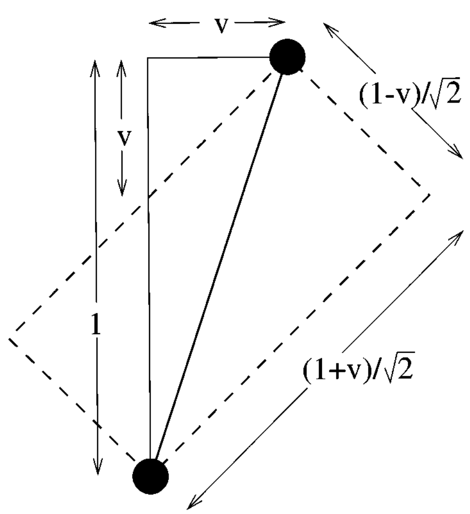

Fig. 7. This demonstrates that if v is the velocity, in the frame using vertical and horizontal lines of constant position and time, of the frame in which two events happen at the same place, then the aspect ratio of the photon rectangle determined by the events is $(1+v)/(1-v)$.

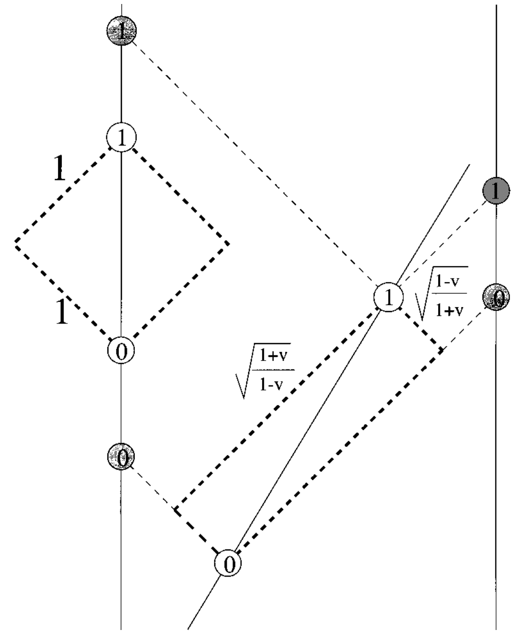

Fig. 8. It follows that the longitudinal Doppler shift is given by $\sqrt{(1-v)/(1+v)}$. Take the unit of area in the diagram to be that of a photon rectangle associated with events separated by an interval of 1 ns. Then a unit square of photon lines determines the 1-ns scale on a vertical line of constant position (shown on the left vertical line). The unit photon rectangle associated with successive nanosecond readings of a clock moving with velocity v in that frame then has sides $\sqrt{(1-v)/(1+v)}$ and $\sqrt{(1+v)/(1-v)}$ (since its area is 1 and its aspect ratio is $(1+v)/(1-v)$. Consequently the clock is *seen* by somebody on a vertical line of constant position as running slowly (left vertical line) or fast (right vertical line) by these factors. (The unit square is drawn with thicker photon lines, as are the segments of photon lines making up the unit rectangle. The pairs of white circles containing 0's and 1's represent events in the history of two actual clocks at which they read 0 and 1 ns. The grey circles represent events in which the clock with the nonvertical world-line is *seen* to be reading 0 or 1 by people whose world-lines are the two vertical lines.)

ACKNOWLEDGMENT

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[1] N. David Mermin, ''An introduction to space–time diagrams,'' Am. J. Phys. **65**, 476–486 (1997). The points made in the present note can be understood without consulting this paper. It is, however, important for instilling the proper *attitude* toward space–time diagrams.

[2] Multiplication by the scale factor λ converts the time between two events to the (perpendicular) distance between the lines of constant time on which lie the points that represent those events. (And it converts the actual spatial distance between two events to the distance in the diagram between the lines of constant position on which the points lie). The scale factor μ for a given frame converts the time between two events that happen at the same place in that frame to the distance in the diagram between the points that represent the events. (And it converts the actual spatial distance between two events that happen at the same time to the distance in the diagram between the points.) The relation between the two scale factors belonging to a given frame is just $\lambda=\mu\sin\theta$, where θ is the angle in the diagram between its lines of constant time and constant position. The units of the scale factors could be, for example, centimeters of diagram per nanosec-

ond of time (or per foot—i.e., light-nanosecond—of distance).

[3]As used here the ''foot'' is the light nanosecond, 29.979 245 8 cm. See, for example, N. David Mermin, ''Light Feet,'' The New Yorker, May 16, 1994, p. 10.

[4]The argument here improves on that in Secs. IV and V of Ref. 1 by focusing not on how fast Alice and Bob each *says* the other's clock runs, but on something that transcends frame-dependent conventions: how fast each *sees* the other's clock run. It is also simpler directly to establish the representation of the squared interval as the area of a rectangle of photon lines, from which the invariance of the product of scale factors follows effortlessly, rather than deriving these results in the opposite order, as in Ref. 1.