Space and Spacetime

M. Lachièze-Rey
Service d'Astrophysique, C.E. Saclay, F-91191 Gif sur Yvette Cedex, France

This lecture comprises two parts. The first one presents a short analysis of the evolution of ideas about space and spacetime. In particular, it suggests an analogy between the introduction of *homogeneous space* by Newton, and the introduction of *spacetime* in special relativity.

The second part provides, in the frame of general relativity, a prescription to define *space*, at a given moment, for an arbitrary observer in an arbitrary (sufficiently regular) curved spacetime. Based on synchronicity arguments, this prescription defines a foliation of spacetime, which provides a natural global reference frame (with space and time coordinates) for the observer, in spacetime, which remains Minkowskian along his world-line. This definition remains valid in curved spacetime, and/or for non inertial observers.

Application to Minkowski spacetime illustrates clearly the fact that different observers see different spaces. It allows, for instance, to define space everywhere without ambiguity, for the Langevin observer (involved in the Langevin pseudoparadox of twins). Applied to the Rindler observer (with uniform acceleration) it leads to the Rindler coordinates, whose choice is so justified with a physical basis. This allows to interpret the Unruh effect as due to the observer dependence of the space-time splitting. Finally an application if given for a rotating observer in circular motion.

We also apply this prescription in cosmology, to inertial observers in the Friedmann-Lemaître models: space constructed in this manner differs from the hypersurfaces of homogeneity, which do not obey the simultaneity requirement. I work out two examples: the Einstein – de Sitter model, in which space, for an inertial observer, is not flat nor homogeneous, and the de Sitter case.

I. FIRST PART: FROM SPACE TO SPACETIME

This part is concerned by the evolution of ideas about Space. To begin, the Cosmology of Aristotle is without a concept of *space*: he notion is replaced by that of a *place* (locus). The World is certainly not homogeneous : there is a center (coinciding with that of the Earth), there is a strong distinction between the *sublunar* part (below the Moon) and the *supralunar* one (the skies beyond the Moon); there is a hierarchy of concentric spheres, up to the last one, which is the *border* of the World.

Also, the geometry of the World is strongly anisotropic: the vertical dimension points towards the center of the World, and is fundamentally distinct from the horizontal ones. Terrestrial bodies fall along the vertical, when they tend to recover their "natural place", which coincides with the center of the Earth (because they contain the element "earth"). This privilegiates the vertical (light bodies also follow the vertical, in the upward direction). On the other hand, the celestial motions, and the surfaces of the celestial spheres may be seen as horizontal: the sky also is strongly anisotropic.

In the language of present physics, this geometry incorporates some aspects of the the (terrestrial) gravitation: rather than an identified physical effect, the latter appears as part of the geometry of the World. But this concerns only the *terrestrial* gravity since its universality will be discovered only two millenaries later.

A. The birth of modern physics

After Antiquity and Middle Age, the physics and astronomy of Aristotle are more and more criticized. After the works of Nicolas Copernic, Giordano Bruno, Tycho Brahe, Johannes Kepler, René Descartes,..., a new vision of the World is adopted. Its birth may be taken at the publication of the *Principia* by Isaac Newton. The apparitions of the new physics, of the new astronomy (now astrophysics) and of the new cosmology are strongly related to the notions of space, time, and Universe. The new born *Physical space* is assimilated to the mathematical *Euclidean* space, $\mathbb{R}^3$ for mathematicians. At the same moment, time is parametrized and geometrized as the real line $\mathbb{R}$.

Newton introduces the absolute space, with no physical property, other than being the passive frame for physical phenomena. The universal gravitation, his main discovery, is treated as a physical interaction - a *force* - not included in the geometry. Since gravitation becomes external to the geometry, this allows the *isotropisation* of space: all dimensions become equivalent.

The apparent specificity of the vertical dimension appears now as a *local* effect only, due to the peculiar direction of the gravitational acceleration here (see Vilain, 2003). It is non geometric in nature. The universality of the gravitation law, of the dynamics, and of all physical interactions leads to the concept of the *Universe*, at the basis of the modern physics. There is no more distinction between terrestrial and celestial physics.

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Space and time

The relations between spatial and temporal dimensions are expressed by *kinematics*. In Newtonian physics, they remain independent: there is no *spacetime*, but separate space and time. The Newtonian kinematics is based on the Principle of inertia and on the law of composition of velocities by addition.

In fact, the Principle of inertia is strongly linked to the isotropy of space: in the absence of any privileged direction in space, a body cannot initiate a motion. This would imply the choice of such a direction, contrary to spatial isotropy. So, motion remains identical. Thus, the nature of the Principle of inertia, and of the resulting kinematics, appear geometrical.

Light and matter

The Newtonian time is universal, and geometrized, although without explicit introduction of the arrow of time. The Newtonian kinematics is based on the *Galilean* relativity of motion. But, during the XIXth century, it is realized that the addition law for velocities does not apply to light. In particular, the light velocity c remains constant in all conditions.

Are they two different kinematics, one for matter and an other one for light ? This would imply two different conceptions of space, of time, and of their links. This seems not acceptable. The crisis will be solved by special relativity.

B. Relativity

The theory of special relativity may be expressed by the replacement of *space and time* by [the Minkowski] *spacetime*.

The temporal and spatial dimensions acquire a similar geometrical status, in a true 4 dimensional geometry, which is a *Lorentzian manifold*. In particular, there is a physical possibility to exchange spatial and temporal dimensions, by [hyperbolic] rotations called *boosts*. Spatial rotations and boosts form the *Lorentz group*, at the basis of the theory.

There is no absolute time, but every observer has a different *time line* (also called world line). There is no universal time line, common to every body, like in Newtonian physics. The composition of velocities is no more given by an addition, but it is the result of an hyperbolic rotation, called a *Lorentz transformation*. This is expressed by a new, *relativistic* kinematics. As it is well known, it leads to peculiar effects like time dilatation and length contraction. These relativistic effects (like the pseudoparadox of the Langevin twins) are of pure geometrical nature in spacetime.

Spacetime

The new kinematics is nothing else than the spacetime geometry. The Principle of Inertia acquires a new and simpler formulation: *Inertial motion is described by a straight line in spacetime.* This includes the prescription of constant velocity, that there is no need to add explicitly. The velocity of light c becomes an absolute quantity, now considered as a part of the structure of spacetime (this will remain true in general relativity). This new geometry corresponds to a gain of geometrical symmetry, as explained in the next section.

Like in Newtonian physics, gravitation is not incorporated in the geometry but remains excluded. This will be the task of general relativity to incorporate it again.

C. Symmetries of space and spacetime

The main point of this talk is to emphasize the analogy:

- The introduction of homogeneous and isotropic *space* by Newton may be seen as the replacement:
  2 horizontal dimensions + 1 vertical dimension
  $\rightarrow$ 3 equivalent dimensions of the homogeneous (3-d) space.
- The special relativity may be seen as the replacement:
  1 time dimension + 3 space dimensions
  $\rightarrow$ 4 (almost) equivalent dimensions
  of the homogeneous (4-d) Minkowski spacetime.

In the frame of Newtonian physics, no effect distinguishes a vertical or horizontal dimension, far from the Earth. A space traveller may define his "vertical direction" as that going from his feet to his head. But it will not coincide with the vertical of another space traveller.

In special relativity, no effect allows to distinguish a time direction among the directions of spacetime (although some of these directions are excluded as being purely spatial). A space traveller defines his "proper time", as the peculiar direction of its world line, in spacetime. But this direction will not coincide with the time direction of another space traveller, with a relative velocity.

In both cases, there is an increase of geometrical symmetry, expressed by the group theory:
Newton:
SO(2) [plane rotations] $\rightarrow$ SO(3) [space rotations]
Special relativity:
SO(3) [space rotations] $\rightarrow$ SO(1,3) = Lorentz group [spacetime rotations]

General relativity

Einstein incorporates the gravitation in the geometry of spacetime. The Minkowski spacetime is replaced by a more general spacetime, a Lorentzian (pseudo Riemannian) 4-dimensional manifold. The gravitation becomes part of its geometrical structure, expressed by the Riemann *curvature* [tensor]. In some sense, the latter plays the role of the elusive *gravitational ether*. Note that the pseudo Riemannian structure, usually expressed by the metric, may also be described by equivalent tools, like the [Levi-Civita] *connection*, the *tetrad coefficients*, which may appear more convenient.

This is not a return to the Aristotle's conception: the *universality* of gravitation is now taken into account. The gravitation is not necessarily that of the terrestrial field. It is not, in general, vertical and, on average (beside *local* irregularities), space remains isotropic. Locally (or rather infinitesimally, i.e., in the tangent space), the symmetries of Minkowski spacetime remain preserved.

II. SECOND PART: FROM SPACETIME TO SPACE

A. Do we need Space ?

The evolution of Physics has led us from space to spacetime. Special relativity learns us that we cannot define uniquely *space* in spacetime (see below), and independently of the choice of an observer. This remains the same in general relativity, with additional difficulties. This is the subject of this work.

On the other hand, we are used to perform and interpret most of our physical experiments and observations (in particular for quantum physics) in terms of space and time, rather than spacetime. Special relativity provides the possibility to define space an time without ambiguity *for an inertial observer*, in Minkowski spacetime. Although this is not exemplified, this possibility is related to synchronization procedures originally proposed by Einstein. This work considers the possibility to extend this procedure to non inertial observer, in Minkowski, or curved spacetimes.

General relativity and relativistic cosmology consider *spacetime* as the arena for physics, and it is an old question to define *space* and *time*. These notions are not covariant and all problems of general relativity and cosmology can be addressed without them, so that they may appear as rather academic. However, on the one hand, the literature refers often to *space*, for instance to affirm that it is flat (or not), or homogeneous (or not) in a given cosmological model. On the other hand, quantum physics, or its interpretation, requires most often a splitting of spacetime into space and time (for instance to define what is a frequency). This points out the necessity of a convenient definition of space in spacetime, or equivalently, the choice of a convenient *global reference frame*. The simple example of two inertial observers in Minkowski spacetime, with different velocities, shows that such a definition must be observer-dependent.

B. Global reference frame

An observer needs a frame to interpret the physics in his environment, in conformity with his physical intuition. In his immediate neighborhood, space is defined without ambiguity at any point of his world line (i.e., at any moment of his story) by orthogonality to his worldline, i.e., to his velocity u. But there are many different ways to extend this definition, i.e., to define space (and time) everywhere, in whole spacetime, or in the largest possible part of it: to define a global frame. Along the world line O, a global frame is constrained to be *Minkowskian* there. This is far from being sufficient to determine the choice. This paper provides a prescription which allows to associates to any observer (defined by his world line) an unique global reference frame (GRF) with special properties. A GRF may be seen as a foliation of spacetime into space plus time. The sheets are timelines (one of them is O). The transverse leaves (constrained to be orthogonal to the time-lines everywhere) are space-like 3-dimensional hypersurfaces, which are identified with the copies of space et the different moments. There is a large number of possibilities to define such foliations. A popular prescription (G) privilegiates the geodesic character of such an hypersurface. This corresponds to the *Fermi coordinates*: the spatial hypersurfaces are generated by the spacelike geodesics orthogonal to O. But, in many situations (even in very simple ones, like for the Langevin observer, see below), these spatial hypersurfaces intersect. This forbids a definition of time valid far from O: different values of time would be associated to the same event. Thus, in general, the validity of this prescription does not extend beyond a very local neighborhood of the observer (which may be sufficient for some applications). Moreover, there is no real motivation to impose a geodesic character, when O itself is not geodesic, i.e., for non inertial motion. Another possibility (H), very popular in cosmology, privilegiates *spatial homogeneity*: the spatial hypersurfaces of homogeneity orthogonal to the world line are selected. But such hypersurfaces do not exist in all spacetimes. Moreover, such a choice has no meaning when the observer himself breaks the spatial symmetries (by his acceleration or rotation for instance). Thus, this proposition has a low range of applications.

The prescription proposed here (S) does not suffer from these drawbacks. It is defined from a "simultaneity criterion", not verified in general by prescriptions G or H (or others), excepted in the immediate neighborhood of O. In the following I will underline space to refer to the result of this prescription: space at instant t is defined as the set of events that the observer O sees as *simultaneous* (see below for a precise definition) at his proper time t. An additional advantage of this prescription results from the fact that the synchronisation procedure depends only on the propagation of light-rays. Thus, only the *conformal* structure of the manifold (not the complete metric, excepted for the proper time of the observer) is used to construct the GRF. This prescription appears valid for a much broader class of spacetimes than those mentioned above (for instance, in the absence of spatial homogeneity). For any observer, inertial or not, in any spacetime (with some restrictive conditions, see below), this procedure allows a canonical global splitting of spacetime into space and time: space is defined uniquely everywhere as a "simultaneity space" Σ_τ, at a value τ of the proper time of the observer, which is so promoted as an universal time function. The Σ_τ never intersect, even in the situations where the Fermi hypersurfaces do, and they are defined even in the absence of spatial homogeneity. This extends the validity of the observer's proper time to the whole spacetime. In the Friedmann-Lemaître models, space does not coincide with some intuitive idea of what space could be (hypersurfaces of homogeneity).

Numerous attempts to define a quantization procedure in curved spacetime, and/or for non inertial observers (see, e.g., Birrel and Davis, 1982), involve, more or less explicitly, a space + time splitting of spacetime. This is especially important for giving a physical interpretation of quantum states in terms of frequencies or particles. For instance, I show below that the present procedure, when applied to the uniformely accelerated observer, leads to the widely used *Rindler coordinates* at the source of the *Unruh effect.* This definition of space provides a good justification otherwise absent, for the use of these coordinates (see also Dolby, 2000).

All the quantities appearing in this work are covariant. This includes all the observer dependent quantities like his velocity, acceleration, world-line and the special reference frame introduced here. In general relativity, an observable quantity (e.g., the energy) is a combination of a covariant quantity associated to the observer (e.g., its velocity u) with a covariant quantity associated to the observed system (e.g., its momentum-energy tensor). But this combination has a *local* character, i.e., it is simply a tensorial product (contraction). On the other hand, the quantum field theory involves *non local* observables, which may include integrals over [part of] space in Minkowski spacetime. Thus, an extension of quantum field theory to general relativity involves the definition of non local observables in curved spacetime, which requires most often a definition of space and time.

The goal of this work is to construct the reference frame associated to an observer in cosmological situation. Thus, it only concerns that part of spacetime which is causally related to him, in past and in future, what is called the "causal diamond". Throughout this paper, by an abuse of language, I call "spacetime" the causal diamond, i.e., the set $\mathcal{M}_0$ of events inside the particle horizon and the event horizon of the observer, if they exist. In the following, I will assume that the causal structure of spacetime admits only light cones without folding and conjugate points (no gravitational lensing; no multi-connected spacetime). These restrictions are appropriate for cosmology, and characterize a background spacetime convenient for quantization. They also appear with the other prescriptions, which all appear more restrictive that the one here. Moreover, the validity of the latter can be extended to many situations including conjugate points.

In Section 3, I implement the definition of space, and the related notions. I show how they allow to define a GRF convenient to the observer, and a congruence of canonically associated observers. Section 4 applies these results to observers in Minkowski spacetime. Section 5 considers inertial observers in the Friedmann-Lemaître cosmological models.

III. A GLOBAL REFERENCE FRAME FOR OBSERVERS

A. The accelerated Observer

In a spacetime $\mathcal{M}$, the most general observer is defined by his timelike world line $O(\tau)$, parametrized by proper time τ. The velocity $u(\tau) \equiv \partial_\tau$, defined everywhere on O, verifies $u \cdot u = 1$. Hereafter, we note the covariant derivative along the curve $\nabla_\tau X = u \cdot \nabla X = \dot{X}$ with a dot. Writing the acceleration

$$a(\tau) \equiv \dot{u} = A\ h_1, \text{ with } h_1 \cdot h_1 = -1,\ A \in \mathbb{R}^+, \tag{1}$$

we may complete a moving tetrad along O with $h_0 \equiv u$, h_2 and h_3 defined by

$$\dot{a} \equiv \dot{A}\ h_1 + A\ \dot{h}_1,\ \dot{h}_1 \equiv A\ u + R\ h_2, \tag{2}$$

$$(h_3)^\mu = \epsilon^{\mu\nu\rho\sigma}\ (h_0)_\nu\ (h_1)_\rho\ (h_2)_\sigma.$$

We have $h_\mu \cdot h_\nu = \eta_{\mu\nu}$.The unit vector h_2 characterizes the spatial rotation of the observer. For a non rotating (NR) observer (defined as having $R = 0$), h_2 may be chosen as an arbitrary vector orthogonal to h_0 and h_1.After calculations, the ON frame h naturally associated to the arbitrary observer, obeys the Frenet-Serret equations Synge (1967)Pauri & Vallisneri 2000

$$\dot{h_0} = A\ h_1 \tag{3}$$
$$\dot{h_1} = A\ h_0 + R\ h_2 \tag{4}$$
$$\dot{h_2} = -R\ h_1 + C\ h_3 \tag{5}$$
$$\dot{h_3} = -C\ h_2, \tag{6}$$

where A, R, C vary along O, in general.Transport along the world line corresponds to a Lorentz rotation, which may be seen as the combination of a boost, in the plane (u, a) and, for the rotating observer, a spatial rotation in a space like plane orthogonal to u and to the space-like vector $\omega \equiv C\ h_1 + R\ h_3$. The frame h is rotating with the observer, when the later does. We will associate to an observer an other non-rotating frame f.

1. *The Fermi-derivative*

For an arbitrary vector V field defined along O, we define the *Fermi-derivative* (along O) as

$$d_F V \equiv \dot{V} - [(u \cdot V)\ a - (a \cdot V)\ u], \tag{7}$$

$$(d_F V)^\mu = \dot{V}^\mu - a^{[\mu}\ u^{\nu]}\ V_\nu. \tag{8}$$

The vector is said to be *Fermi-transported* when $d_F V = 0$. It is easy to check that

$$d_F u = 0,\ d_F h_1 = R\ h_2,\ d_F h_2 = \dot{h}_2,\ d_F h_3 = \dot{h}_3. \tag{9}$$

Thus, h_1 is Fermi transported only if the observer is NR. For the vectors h_μ, the transport (Lorentz rotation) along O combines a boost in the plane u, a with the spatial rotation $\mathcal{R}$ represented by the vectors u and ω:

$$V \mapsto \mathcal{R}(V) :\ [\mathcal{R}(V)]^\mu = u_\alpha\ \omega_\beta\ V_\gamma\ \epsilon^{\mu\alpha\beta\gamma}. \tag{10}$$

For the vectors above, we have $d_F h_\mu = \mathcal{R}(h_\mu)$, so that the Fermi derivative expresses their spatial rotation. Also, $\mathcal{R}(\omega) = \mathcal{R}(u) = 0$.Although the frame h defined above is rotating with the observer, it is possible to associate to O a non rotating frame $f = (f_\mu)$ (an "ideal gyroscope"), such that each f_μ is Fermi transported, i.e., $d_F f_\mu = 0$. We chose $f_0 = u$, but the spatial vectors do not coincide with the h_i when the observer is rotating. Then, the goal will be to extend the definition of f in the whole spacetime, or, at least, in an extended part of it.

B. Definition of space

A spacetime $\mathcal{M}$ admits many time-like foliations compatible with the world-line of a given observer O. I will show that synchronicity arguments allow to select an unique one, and provide therefore a global definition of space. In the whole paper, I denote $\tilde{v}$ the one-form metric-dual to a vector v, i.e., such that $\tilde{v}(v) = g(v, v) \equiv v \cdot v$.

As it is well known, it is impossible to define *absolute* simultaneity in special or general relativity. However, special relativity allows to define simultaneity in Minkowski spacetime, *from the point of view of an inertial observer.* In general relativity, this corresponds to a prescription of simultaneity or, better, *synchronicity*, which is *local* and *relative to an observer* (see, e.g., Landau and Lifshitz, 1966).This defines a *local* space + time splitting for this observer, with the only constraint that space and time are orthogonal where they meet on O. The construction S presented here extends this prescription beyond a local neighborhood, using perfectly operational arguments of synchronicity.

Given an observer O, I define Σ_τ, the *hypersurface of synchronicity* (HS) of O at proper time τ, as the set of events related by a null geodesics to both $O(\tau+\delta)$ and $O(\tau-\delta)$, where δ is an arbitrary interval of proper time for O:

$$\Sigma_\tau = \cup_\delta \; [I^{future}(\tau-\delta) \cap I^{past}(\tau+\delta)], \tag{11}$$

where $I^{past}(\tau)$ [resp. $I^{future}(\tau)$] denotes the null past [future] light-cone of the observer at proper time τ. The prescription S considers Σ_τ as the space for $O(\tau)$.

Given the restrictions above, the surfaces Σ_τ for different values of τ completely fill $\mathcal{M}_0$. This allows to extend the vector field u to the totality of $\mathcal{M}_0$ by requiring that it is everywhere unit ($u.u = 1$) and orthogonal to Σ_τ (Fig.1). The vector field u constitutes a foliation of $\mathcal{M}_0$, with the Σ_τ as transverse (orthogonal) surfaces. Each integral line of u can be labelled by its intersection $\mathbf{R}$ with Σ_1 (for instance), so that any point of $\mathcal{M}_0$ can be written $(\tau, \mathbf{R})$. The value $R(x) = \delta$ associated to any point x through (11) represents half the interval of observer's proper time between $\tau_1 = \tau + \delta$ and $\tau_2 = \tau - \delta$. These two events correspond to the emission of a flash light (or radar signal) which illuminates a cosmic object at x, and the observation of the resulting image by the observer, after mirror reflection. For any point x, $R(x) = \delta$ defines a natural radial space coordinate, that I call its " proper time interval " (PTI). The integral lines of u define an unique family of observers, that I will call *the canonical observers* associated to O. Care must be taken that they do not necessarily share the properties of O. For instance, they are not geodesic, even when O is, when there is expansion. The word lines generated by u are labelled by $\mathbf{R}$. Along them, the proper time t (with $t = \tau$ along O) corresponds to the *radar time* originally defined by Bondi, and reintroduced by Dolby (2000) and Dolby and Gull (2001). Since the congruence of "associated observers" is completely defined from the world line of the unique observer O, the foliation introduced here defines an unique space + time slicing, from the unique observer only.

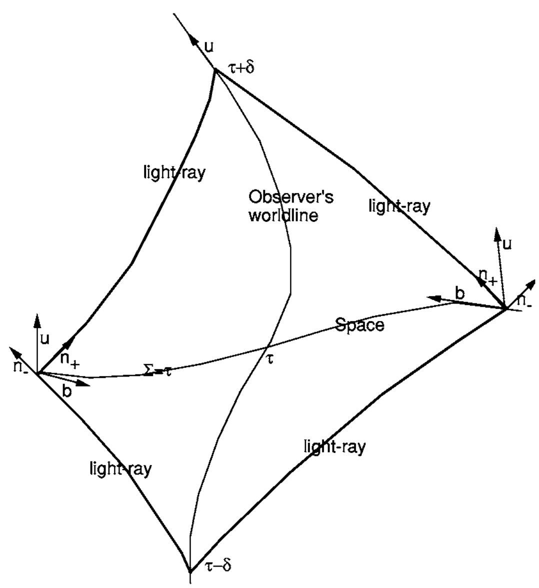

FIG. 1: At each point x, the light-rays from the observer (in the past and in the future) define the vectors n^+ and n^-. The velocity u of the observer is (non parallely) transported to give the vectors u and b at x.

C. Transport along the light-rays

The properties of space are defined from those of the world-line O, transported by the past and future light-rays. To explore them, we define the two null functions $\mathcal{N}_-(x)$ and $\mathcal{N}_+(x)$ such that the value of $\mathcal{N}_-(x)$ [resp. $\mathcal{N}_+(x)$] is the proper time $\tau - \delta$ of the observer O, when he emits a light-ray reaching x [resp. $\tau + \delta$, when he receives a light-ray emitted from x]. In other words the null hypersurface $\mathcal{N}_-(x) = \tau$ [resp. $\mathcal{N}_+(x) = \tau$] is the future [resp. past] light cone of the observer at proper time τ. We define their (null) (past and future) generators as $\tilde{n}_\pm = \nabla\mathcal{N}_\pm = d\mathcal{N}_\pm$. Both are future directed, and normalized so that the frequency emitted or received by the observer is unity (see below).

It is easy to show that Σ_τ is defined by the equation

$$T(x) \equiv [\mathcal{N}_-(x) + \mathcal{N}_+(x)]/2 = \tau. \tag{12}$$

For any point, T constitutes a natural time-coordinate. In addition we define the deformed cylindric hypersurface

$$R(x) \equiv [\mathcal{N}_+(x) - \mathcal{N}_-(x)]/2 = \delta \tag{13}$$

as the set of events at a constant PTI value δ from the observer, when he describes his world-line.

Given the normalization above, we have

$$dT = (\tilde{n}_- + \tilde{n}_+)/2 \text{ and } dR = (\tilde{n}_+ - \tilde{n}_-)/2. \tag{14}$$

It is easy to check that $dT \cdot dR = 0$, and

$$dT \cdot dT = -dR \cdot dR := N^{-2} = n_+ \cdot n_-/2, \tag{15}$$

which defines the *lapse function* N associated to this foliation. Since dT is orthogonal to Σ, we have $\tilde{u} = N\ dT$. From $u^2 = 1$, we have $N\ u \cdot dT = 1$. Since, along $\mathcal{O}$, $\tau = T$, this implies $N = 1$ on $\mathcal{O}$.

Everywhere (except on $\mathcal{O}$), we define

$$\tilde{b} \equiv N\ dR = \tilde{u} - N\ \tilde{n}_- = -\tilde{u} + N\ \tilde{n}_+. \tag{16}$$

We have $b^2 = -1$, $u \cdot b = 0$, $u + b = N\ n_+$ and $u - b = N\ n_-$. Thus, b is a unit space like vector, tangent to Σ_τ and orthogonal to the level surfaces of $R(x)$. In some sense, it points towards the observer O. In general, the vector b is not geodesic but it is *chorodesic*, due to the synchronicity property and the congruence of associated observers is *quasi-rigid* (see Bel, 1998).

1. *Canonical observers*

The vector field u is perfectly defined everywhere and characterizes the family of canonical observers. This family defines a "kinematics " in the sense of Smarr and York (1978). All the relevant formalism of projectors, lapse and shift functions, intrinsic curvature, etc. applies.

The vector fields u and b are not transported parallely along the light rays. In Lachièze-Rey (2001, hereafter MLR), I introduced the two vector fields U^+ and U^- which are, by definition, parallely transported along n^+ and n^- respectively, $n^\varepsilon.\nabla U^\varepsilon = 0$, and which both coincide with u along the world line of O (Fig.2). They allow to define a "future frame", and a "past frame" which, although not obeying synchronicity, appear convenient in some circumstances (Marzlin, 1994). "bisector frame", whose extension may provide a *local* surface of synchronicity for two different observers. This is for instance useful for the study of the quantum evolution of two interacting particles in spacetime (Ali et al., 1990).

2. *Towards a global frame*

This definition of space and time constitutes a first step towards that of a GRF but it is not the whole story. At each point of $\mathcal{M}_0$, we have defined a time like vector u and a 3-dimensional manifold that we consider as space. It remains to precise the spatial part of our frame. This can be done through the integral lines of the vector field b: through each point $x \in \mathcal{M}_0$, there is an unique line of this type. It crosses O at $O(\tau)$, with unit tangent vector $\hat{B}$. It will be possible to define angular coordinates from the 3 scalar products $\hat{B} \cdot f_i$. This will be developed in future work.In many cases of interest, the procedure becomes particularly simple: when spacetime has symmetries, like Minkowski spacetime or cosmological models; and when the observer has simple motion (inertial, non rotating, confined). We treat some specific cases below.

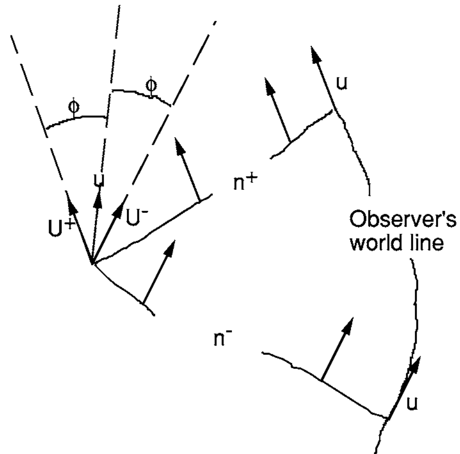

FIG. 2: The field u is not parallely transported. At each point x, U^+ is defined by the parallel transport of u from the future, i.e., by n^+, U^- is defined by the parallel transport of u from the past, i.e., by n^-.

3. Redshifts and Metric

Let us consider a congruence of (non necessarily canonical) objects with velocity $V(x)$ at the point (event) x in spacetime. Each of these objects, at x, is seen with a redshift $z^+ = (n^+(x) \cdot V(x))^{-1}$ by the observer (in the future) and sees the observer (in his past) with a redshift $z^- = n^-(x) \cdot V(x)$. For the congruence of *canonical* observers defined above, $V = u$, and $z^+ = N(x)$ and $z^- = N(x)^{-1}$. These observers are comoving with respect to the coordinate R, i.e., they keep a constant value of R. On the other hand, there is a unique congruence of objects for which $z^+ = 1$ [resp. $z^- = 1$], those with velocity U^+ [resp. U^-] (see MLR). Thus, N appears as the *lapse function* associated to the foliation, or to the congruence of associated observers. The usual ADM formalism allows to define time and space projectors, as well as the fundamental forms (metric and extrinsic curvature) on the surfaces Σ_τ (see, e.g., Smarr and York, 1978).

IV. OBSERVERS IN MINKOWSKI SPACETIME

A. Inertial observers

The inertial observer O (zero acceleration) has a velocity

$$u^0 = c,\ u^1 = s,\ u^2 = u^3 = 0,$$

where $c \equiv \cosh\psi$ and $s \equiv \sinh\psi$, the *rapidity* ψ being a constant. His world line is

$$x^0 = c\,\tau,\ x^1 = s\,\tau,\ x^2 = x^3 = 0. \tag{17}$$

Calculations of the light-ray trajectories (given in MLR) provide the surface Σ_τ as the plane of equation $c\,x^0 - s\,x^1 = \tau$, inclined by ψ with respect to the vertical, and thus orthogonal to O. Thus, space is different for all inertial observers.

B. The Langevin observer

The solution of the celebrated "Langevin's twin paradox" lies in geometry. I define a *Langevin observer* as an observer which is initially inertial, then (at $t = 0$) suffers an instantaneous acceleration, and then is inertial again (Fig.4). Such an observer is able to meet his twin, which remained always inertial, with a different lapse of proper time. Is is often quoted (see, e.g., Misner et al., 1973) that it is impossible to define space globally for such an observer. But the synchronicity prescription applies perfectly in this case, and provides an unambiguous definition of space for this observer. This has been firstly shown by Dolby and Gull (2001).

The trajectory is defined as

$$x^0 = \tau, \quad x^1 = x^2 = x^3 = 0, \qquad \text{for} \quad t < 0, \tag{18}$$

$$x^0 = c\,\tau,\ x^1 = s\,\tau, x^2 = x^3 = 0, \qquad \text{for} \quad t > 0, \tag{19}$$

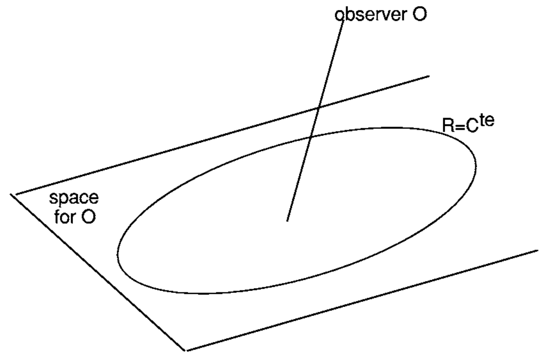

FIG. 3: For the inertial observer in Minkowski, with arbitrary velocity (rapidity ψ), space is the hyperplane with inclination ψ. We have drawn a curve $R = C^{te}$, in this plane.

with $c := \cosh\psi$ and $s := \sinh\psi$.

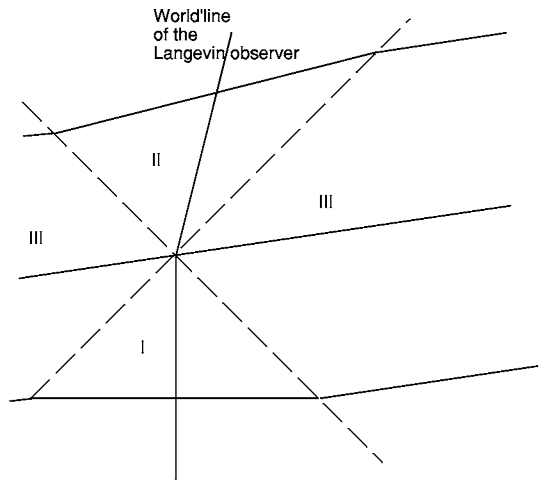

FIG. 4: World's line and (cuts of) space at various moments for the Langevin observer. Its light cone is indicated by the dashed lines.

The light cone of the observer at $t = 0$, $\mathcal{L}^0$, divides the spacetime into three sectors I, II and III (see Fig.4) corresponding to the past, future and spatially related regions to the acceleration point. Study of the light-rays and calculations (see MLR) lead to the values of $\mathcal{N}^\varepsilon(x)$, $T(x)$ and $R(x)$ in the three regions. The surfaces Σ_τ of equation $T(x) = \tau$ defines the spaces for the observer at proper time τ. Fig.4 gives their projections in the (x^0, x^1) plane: In Region I, they are straight horizontal lines, where $R(x) = x^1$. In Region II, they are lines inclined of ψ with respect to the vertical, and thus orthogonal to the world line of the observer in that region. In Region III, they are lines inclined of $\psi/2$ with respect to the vertical, i.e., at equal hyperbolic angle $\psi/2$ of the two previous lines (Fig.4).

For the observer at an arbitrary moment, space is made of a plane disk S^I [or S^{II}] up to the light cone $\mathcal{L}^0$ and is continued by a composite surface S^{III} beyond. Except at the single moment when the observer experiences the instantaneous acceleration, space is not flat, nor homogeneous.

This is the simplest example where our prescription S differs from F and H. As it is well known, it is impossible to extend the Fermi coordinates outside the conical regions, and no homogeneous hypersurfaces would be convenient. Thus, in this simple case, the prescription S is the only one providing a reference frame associated to the observer valid in the whole spacetime, to extend the validity of his proper time, and to consider unambiguous synchronicity procedures (a similar conclusion has been reached by Dolby and Gull, 2001).

C. The Rindler observer

The Rindler observer in Minkowski spacetime has constant acceleration a. His velocity is defined by

$$u^0 = \cosh(a\ \tau),\ u^1 = \sinh(a\ \tau) =,\ u^2 = u^3 = 0, \tag{20}$$

with acceleration $a^0 = a\ \sinh(a\ \tau),\ a^1 = a\ \cosh(a\ \tau),\ a^2 = a^3 = 0$ (with no loss of generality, I have taken the x^2 direction parallel to the acceleration). The world line has the equation $x^0 = a^{-1}\ \sinh(a\tau),\ x^1 = a^{-1}\ \cosh(a\tau),\ x^2 = x^3 = 0$, an hyperbola in spacetime. Since (anticipating) the solution requires $(x^1)^2 - (x^0)^2 > 0$, we can introduce the Rindler coordinates

$$x^0 := a^{-1}\ \exp(a\ \xi)\ \sinh(a\ \eta)$$

$$x^1 := a^{-1}\ \exp(a\ \xi)\ \cosh(a\ \eta). \tag{21}$$

The problem is usually treated in two dimensions, where it appears particularly simple and pedagogic: the hypersurface Σ_τ is the line of equation $\eta = \tau$, which implies $x^0 = \tanh \tau x^1$: a straight line through the origin. The level surfaces of R, $R(x) = \delta$ are the hyperbolae of equation $\xi = \delta$, or $(x^1)^2 - (x^0)^2 = a^{-2}\ \exp(2a\ \delta)$.

In four-dimensions, the hypersurface Σ_τ is the [flat] hyperplane through the origin, of equation $x^0 = \tanh \tau x^1$, which projects to the line seen in the previous section. The (spatial) metric on Σ_τ is given by

$$\begin{aligned} d\sigma^2 &= N^2\ dR^2 + (dx^2)^2 + (dx^3)^2 \\ &= a^{-2}\ [d\exp(a\ \xi)]^2 + (dx^2)^2 + (dx^3)^2 \\ &= (d\frac{x^1}{\cosh\tau})^2 + (dx^2)^2 + (dx^3)^2, \end{aligned} \tag{22}$$

the latter form showing that its hypersurface ($\tau = Ct$) is flat and homogeneous.

The surfaces of constant PTI δ are given by $\xi' = \delta$, or

$$2\ a^{-1}\ \cosh(a\ \delta)\ \sqrt{(x^1)^2 - (x^0)^2} - a^{-2}\ = (x^1)^2 - (x^0)^2 + (x^2)^2 + (x^3)^2. \tag{23}$$

This calculation shows that the widely used Rindler coordinates correspond in fact to the definition of space and time introduced here for the accelerated observer. This justifies their use, and sheds some light on the Unruh effect, which appears as a consequence of the different space-time splittings for the two observers (inertial and Rindler): they associate different frequencies to the same state, namely, the Minkowski inertial vacuum. This has led Pauri and Vallisneri (1999) classical (not quantum) origin for this effect, not discussed here.

D. Rotation in Minkowski spacetime

The previous observers are non rotating. Here we consider an observer which describes a circle in space (the radius of the circle is taken as an unit for all space and time coordinates). In spacetime, he describes the helix with coordinates

$$x^0 \equiv t = \gamma\ \tau,\ x^1 = \cos\Omega\tau,\ x^2 = \sin\Omega\tau,\ x^3 = 0,\ \gamma \equiv \sqrt{1+\Omega^2}.$$

Its velocity is thus

$$u = (\gamma,\ \ -\Omega\ \sin\Omega\tau,\ \Omega\ \cos\Omega\tau,\ 0).$$

Derivation leads to

$$h_1 = (0, -\cos\Omega\tau, -\sin\Omega\tau, 0),\ A = \Omega^2,$$

$$R = \gamma\ \Omega,\ h_2 = (-\ \Omega, \gamma\ \sin\Omega\tau, -\gamma\ \cos\Omega\tau, 0),$$

$h_3 = (0,0,0,1)$, $C = 0$, and the relations above are verified. It is easy to verify that u coincides, on O, with the Killing vector

$$\xi = w\ \partial_t + \Omega\ (x\ \partial_y - y\ \partial_x) = w\ \partial_t + \Omega\ \partial_\phi,$$

if we use the polar coordinates ρ, ϕ with $\tan\phi \equiv y/x$. Also, the acceleration a coincides (along O) with the Killing vector $\xi' = -\rho\ \Omega^2 \partial_\rho$. These Killing vectors Lie-transport each other, $\mathcal{L}_\xi \xi' = 0$.

1. *The Fermi transported frame*

Since the observer is NR, the frame h defined above is not Fermi-transported. To define aFermi-transported frame f, we first put $f_0 = u$ and $f_3 = h_3$. We then define the 2 following as arbitrary combinations of h_1 and h_2, with orthonormality conditions and the requirement of Fermi-transport. This leads to

$$f_1 = \cos(R\ \tau + \psi)\ h_1 - \sin(R\ \tau + \psi)\ h_2,\ \ f_2 = \sin(R\ \tau + \psi)\ h_1 + \cos(R\ \tau + \psi)\ h_2, \tag{24}$$

where ψ is an arbitrary phase. Note the apparition of the new frequency $R = \Omega\ \gamma$ linked with the Thomas precession (see for instance Misner et al., 1973, p. 175). Note also that $R\ \tau = \Omega\ x^0 = \Omega\ t$.

2. *The synchronicity surface*

There is no analytical expression of T as a function of the Cartesian coordinates. However, it is possible to coordonize the points of spacetime with τ, δ and two angular coordinates θ, ϕ. A 3-dimensional expression can be found in Pauri & Vallisneri (2000). Here we extend it in 4 dimensions:

$$\begin{aligned}
t &= \sin\Omega\delta\ \sin\theta + \gamma\ \tau, && (25)\\
x &= \cos\Omega\tau\ [b(\delta)\ \cos\theta\ \cos\phi + \ \cos\Omega\delta] - \gamma\ \sin\Omega\tau\ \delta\ \sin\theta, && (26)\\
y &= \sin\Omega\tau\ [b(\delta)\ \cos\theta\ \cos\phi + \ \cos\Omega\delta] + \gamma\ \cos\Omega\tau\ \delta\ \sin\theta, && (27)\\
z &= \sin\phi\ \cos\theta\ b(\delta), && (28)
\end{aligned}$$

with $b(\delta) \equiv \sqrt{\gamma^2\ \delta^2 - \ \sin^2\Omega\delta}$.These equations also give the parametrization of the surface $R(x) = \delta$.Calculations of the transport by the light-rays give the extension of the velocity field

$$u^0 = \gamma\ \delta/b(\delta),\ u^1 = -\sin\Omega\tau\ \sin\Omega\delta/b(\delta),\ u^2 = \cos\Omega\tau\ \sin\Omega\delta/b(\delta),\ u^3 = 0, \tag{29}$$

the b field

$$\begin{aligned}
b^0 &= -\sin\Omega\delta\ \sin\theta/b(\delta) && (30)\\
b^1 &= [-\cos\Omega\tau\ b(\delta)\ \cos\theta\ \cos\phi + \sin\Omega\tau\ \gamma\ \delta\ \sin\theta]/b(\delta) && (31)\\
b^2 &= -[\sin\Omega\tau\ b(\delta)\ \cos\theta\ \cos\phi + \cos\Omega\tau\ \gamma\ \delta\ \sin\theta]/b(\delta) && (32)\\
b^3 &= -\sin\phi\ \cos\theta, && (33)
\end{aligned}$$

and the lapse function

$$N = (\delta\gamma - \Omega\ \sin\Omega\delta\ b\ \cos\theta\ \cos\phi - \Omega\ \ \sin\Omega\delta\ \cos\Omega\delta)/b. \tag{34}$$

V. THE COSMOLOGICAL OBSERVER

Turning to cosmology, I consider the Friedmann-Lemaître models, i.e., spacetimes admitting spatial sections of maximal symmetry. There exists a special system of coordinates in which the metric takes the form

$$ds^2 = A(\eta)^2\ (d\eta^2 + [d\sigma^2 - S(\sigma)^2\ (d\alpha^2 + \sin^2\alpha\ d\beta^2)]), \tag{35}$$

where A is the usual scale factor, the expression between quotes is the metric of a spatial section with maximal symmetry (thus $\mathbb{R}^3$, S^3 or H^3) and η is the *conformal time*. Although different systems of coordinates would be as well convenient, I will perform calculations with the coordinates $(\eta, \sigma, \alpha, \beta)$. I consider here only a cosmological *inertial* observer (CIO) O_I which follows the line $\sigma = 0$, so that spherical symmetry is preserved. The proper time τ of the observer is defined by $d\tau = A\ d\eta$. The functions $\eta(\tau)$ and its inverse f such that $f[\eta(\tau)] := \tau$ will play an important role. Since $\eta > 0$, the CIO has a particle horizon and $\mathcal{M}_0$ is defined inside it, i.e., by $\sigma < \eta$.

For the CIO,

$$\mathcal{N}_\varepsilon(\eta, \sigma) = f[\eta + \varepsilon\ \sigma], \tag{36}$$

$$2T(\eta, \sigma) = f[\eta + \sigma] + f[\eta - \sigma], \tag{37}$$

$$2R(\eta,\sigma) = f[\eta+\sigma] - f[\eta-\sigma]. \tag{38}$$

Differentiation gives

$$n_\varepsilon = A^\varepsilon \ (d\eta + \varepsilon \ d\sigma), \tag{39}$$

where I have defined $A^\varepsilon(\eta,\sigma) := A(\eta + \varepsilon \ \sigma)$. Sum and difference lead to

$$dT = (A^+ + A^-)/2 \ d\eta + (A^+ - A^-)/2 \ d\sigma \tag{40}$$

and

$$dR = (A^+ - A^-)/2 \ d\eta + (A^+ + A^-)/2 \ d\sigma, \tag{41}$$

and thus

$$N^2(\eta,\sigma) = \frac{A(\eta)^2}{A^+ \ A^-}. \tag{42}$$

The parallel transport of the velocity of the CIO along the light rays leads to

$$\tilde{U}^\varepsilon = \frac{1}{A^\varepsilon} \ [((A^\varepsilon)^2 + A^2) \ d\eta + \varepsilon \ ((A^\varepsilon)^2 - A^2) \ d\sigma]. \tag{43}$$

3. Space for the inertial cosmological observer

Space, i.e., the surface Σ_τ, has the equation

$$f[\eta+\sigma] + f[\eta-\sigma] = 2\tau, \text{ with } \sigma < \eta. \tag{44}$$

Excepted in the case without expansion ($\eta = \tau$), this is *not* the surface $\eta = C^{te}$: space is not a spatial section with maximal symmetry, since the cosmic expansion breaks the spatial homogeneity (although not its isotropy when the observer is inertial). The spatial sections $\eta = C^{te}$, sometimes quoted as "space" do not verify the synchronicity condition (they verify a kind of synchronicity condition, but in the conformal time without physical relevance for the observer, rather than in its proper time).

Also, the cosmic expansion imprints a curvature onto space: even when spacetime admits spatial sections of constant curvature (like for instance flat in the Einstein – de Sitter case), this is not the case for the space. This appears clearly in the case of the Einstein – de Sitter model, for which detailed calculations are given in MLR. Space is limited by the horizon $\sigma = \eta$, or $T = R$. On the horizon, $A^- \to 0$, $N \to \infty$, and Space tends to become light-like.

4. Associated observers, time and distances

The *comoving* observers are defined by $\sigma = C^{te}$, and obey the equation $dT = \frac{A^+ + A^-}{A^+ - A^-} \ dR$. They differ from the associated (canonical) observers, which keep a constant PTI and obey the equation

$$dR = \frac{A^+ - A^-}{2} \ d\eta + \frac{A^+ + A^-}{2} \ d \mid \sigma \mid = 0. \tag{45}$$

An associated observer at the horizon is seen by the CIO with a redshift $z^+ \to \infty$.

All measurements made by an observer, local or not, refer to his proper time. Thus, when a CIO considers an event in spacetime, the relevant time to measure durations, or to date the event, is not t or η but T defined above (I recall that T and t coincide *on the world line of the CIO*).

On the other hand, the proper distance is intended to measure the interval between two objects considered *simultaneously*. This means, at a common value of time. But, again, no observer has access to the conformal time η. Thus, simultaneity (relative to the observer) must be defined not by η but by T as we have explained. This leads to use the *proper time distance* (I introduce this specific terminology to avoid confusion) between two objects, calculated by integration of the metric element, not along a spatial section $t = C^{te}$ (or $\eta = C^{te}$), but along Σ_T, i.e.,

$$d_{PT}(g) = \int_{\Sigma_T} ds = \int_{\Sigma_T} N \ dR. \tag{46}$$

The PT-distance is thus really the distance between two objects in space, at a given moment for the CIO. T and R appear as convenient coordinates for the CIO to measure space and time.

5. *Inertial observer in de Sitter spacetime*

The case of the de Sitter spacetime is particularly interesting since, because of its maximal symmetry, it has been widely considered as a frame for quantization. The metric is written as

$$\begin{aligned} ds^2 &= dt^2 - \rho^2\ (\cosh \rho^{-1} t)^2\ [d\sigma^2 + \sin^2 \sigma\ d\Omega^2] \\ &= A^2(\eta)\ [d\eta^2 - d\sigma^2 - \sin^2 \sigma\ d\Omega^2]. \end{aligned} \tag{47}$$

The CIO is defined by $\sigma = 0$. The conformal time is $\eta = 2 \tan^{-1}[e^{t/\rho}]$ and $A(\eta) = (\rho/2)\ [\tan(\eta/2) + 1/\tan(\eta/2)$. The constant ρ characterizes the curvature of spacetime.

The proper time of the CIO is $t = \rho\ \ln[\tan(\eta/2)]$ so that $f(y) \equiv \rho\ \ln[\tan(y/2)]$. Calculation (see MLR) show that $\underline{\text{space}}$, for the CIO at (proper) time τ, is given by

$$\tan \frac{\eta + \sigma}{2}\ \tan \frac{\eta - \sigma}{2} = e^{2\tau/\rho} \tag{48}$$

or

$$\sinh \frac{S - t}{\rho} = e^{2\tau/\rho}\ \sinh \frac{S + t}{\rho}, \tag{49}$$

where we defined $e^{S/\rho} := \tan(\sigma/2)$. Again, this is *not* the surface of constant (positive) curvature $t = C^{te}$. Thus we sugget to perform quantization with $\underline{\text{space}}$ and the orthogonal time (see Dolby and Gull 2001).

VI. DISCUSSION

The prescription S based on synchronicity defines $\underline{\text{space}}$ without ambiguity for any given observer, inertial or not, in arbitrary spacetime (without multi-crossing of null geodesics), including Minkowski and the Friedmann-Lemaître models. $\underline{\text{Space}}$ is relative to the observer, and well defined at each instant of its world line. This provides a foliation of spacetime, valid for this observer, which may be interpreted as a class of *canonically associated* observers, or a "kinematics" of spacetime (Smarr and York 1978). This provides also a natural reference frame, i.e., global space and time coordinates in the whole spacetime, which remains Minkowskian along the world line of the observer (thus, time coincides with its proper time there), thus convenient for physical measurements. In many cases (in particular for Rindler observers; see all references concerning the Rindler effect, and Sriramkumar and Padmanabhan, 1999), the coordinate system introduced here coincides with that used in the literature with no other justification than being "natural", and thus provides an *a posteriori* justification. Also, the prescription presented here applies to a range wider than other reference frames.

Application to Minkowski spacetime confirms that $\underline{\text{space}}$ and time differ for inertial observers with different velocities. It provides an unambiguous and global definition of $\underline{\text{space}}$ and time for the Langevin observer, for which the other prescriptions do not apply. For the Rindler observer (with uniform acceleration), $\underline{\text{space}}$ and time coordinates coincide with the usual Rindler coordinates. This provides a justification of their use. The corresponding interpretation of the Unruh effect involves the observer-dependent character of space and time.

In cosmology, this prescription provides, for the inertial observer in the general Friedmann-Lemaître model, an unambiguous definition of $\underline{\text{space}}$, which *does not coincide with a spatial section of maximal symmetry*. Thus, in the Friedmann-Lemaître models, no inertial observers "sees" a homogeneous space. The lack of homogeneity of $\underline{\text{space}}$ is due to the curvature corresponding to the expansion law. In particular, $\underline{\text{space}}$ is not flat nor homogeneous (although the inertial character of the observer preserves its isotropy) in the Einstein – de Sitter model, sometimes called a "flat universe" ! I have also calculated $\underline{\text{space}}$ for the inertial observer in de Sitter spacetime, which, again, is not a hypersurface of maximal symmetry.

These results do not modify the cosmological formulae when they are expressed in a covariant form and do not involve a definition of space. However they change those interpretations of observational results, which involve a reference to space (like "*space* is homogeneous, flat" etc.). This modifies also the interpretation of the usual *proper distance*: it does not appear as the proper spatial interval between two events occurring at the same time, but rather as a mixed interval between two events which are not synchronous for the observer which performs the measurement (they would be synchronous if the observer's watch were indicating conformal time). The "proper time-distance", introduced here, represents a spatial interval between two events which are synchronous for the observer. It corresponds to the result of a practical measurement that the observer may perform with his watch indicating his proper time. Its value differs, in general, form the usual proper distance.

This prescription for space could have important implications for interpreting quantum effects in curved spacetime, and/or from the point of view of non inertial observers. Its application to the Rindler observer confirms the usual results of the Unruh effect, and provides a clearer comprehension. In other cases, the prescription adopted here differs from most attempts up to now, since the use of spatial sections with maximal symmetry (rather than space) does not obey the synchronicity requirements.

A new prescription for quantum field theory, based on the radar time and concepts very similar to those introduced here can be found in Gull's thesis (2000). This will also be the subject of a forthcoming paper. Also, subsequent work will explore in more detail the extension to arbitrary acceleration and rotation.

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