Visualizing proper-time in Special Relativity

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We present a new visualization of the proper-time elapsed along an observer's worldline. By supplementing worldlines with light clocks, the measurement of space-time intervals is reduced to the "counting of ticks." The resulting space-time diagrams are pedagogically attractive because they emphasize the relativistic view that "time is what is measured by an observer's clock."

I. INTRODUCTION

Einstein's special relativity[1] forces us to revise our common-sense notions of time. Indeed, clocks in relative motion will generally disagree on the elapsed time interval measured between two meeting events. The discrepancy is practically undetectable for everyday relative speeds, but it is quite significant when the relative speeds are comparable to the speed of light. It is therefore necessary to distinguish these time intervals. Thus, we define for each observer his "proper-time" as the elapsed time interval measured by his clock. The goal of this paper is to find a physically intuitive visualization of this concept of proper-time.

Many textbooks[2,3,4,5,6] introduce proper-time by analyzing the propagation of a light in a light clock,[1,7,8,9,10,11] which consists of a pair of mirrors that face each other and are separated by a proper distance L. One "tick" of this clock is the duration of one round trip of a light ray bouncing back and forth between these mirrors. The analysis is usually done in the context of a simplified Michelson-Morley apparatus[12] whose arms may be regarded as light clocks. Unfortunately, most of these presentations[2,3,5] work in moving frames of reference without making the connection to the space-time formulation, first introduced by Minkowski[13] in 1907 and later extended by Einstein.[14]

Let us recall a quote from J.L. Synge:

> ...We have in the special theory of relativity the Minkowskian geometry of a flat 4-space with indefinite metric... Unfortunately, it has been customary to avoid this geometry, and to reason in terms of moving frames of reference, each with its own Euclidean geometry. As a result, intuition about Minkowskian space-time is weak and sometimes faulty....[15]

Indeed, when studying observers in relative motion, it is advantageous to draw a space-time-diagram of the situation. However, we are immediately faced with an important question: "how does one know where to mark off the ticks of each clock?" More precisely, "given a standard of time marked on an observer's worldline, how does one calibrate the same standard on the other observer's worldline?"

One approach is to use the invariance of the speed of light to algebraically demonstrate the invariance of the
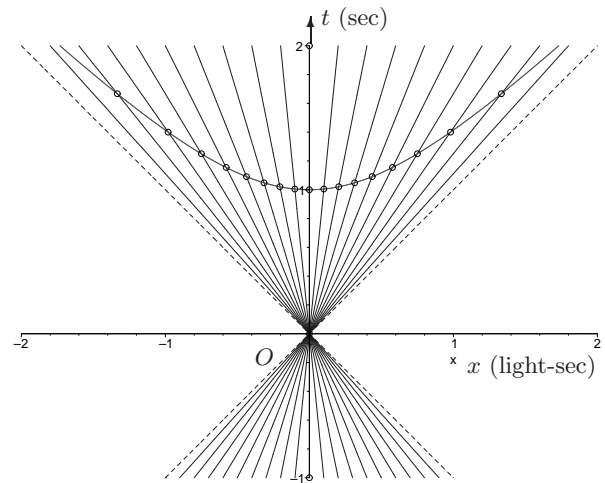

FIG. 1: For inertial observers meeting at event O, the corresponding ticks of their clocks [synchronized at event O] trace out hyperbolas centered about event O.

spacetime-interval, from which the equation of a hyperbola arises.[4,13] Then, for inertial observers that meet at a common event O, it can be shown that the corresponding ticks on their clocks [synchronized at event O] trace out hyperbolas on a spacetime diagram. (See Figure 1.) Once this result is established, many of the results of special relativity follow. This approach, however, is probably too sophisticated for a novice. Its connection with the standard textbook approach, using the more familiar (though provisional) physical concepts of time and space, is not readily apparent.

In this paper, we connect the two approaches by drawing the spacetime-diagram of the Michelson-Morley apparatus. Surprisingly, the only other spacetime-diagram of the apparatus is a rough sketch in Synge's *Relativity: The Special Theory*.[16]

The resulting diagram provides a visualization of proper-time which explicitly incorporates the principle of relativity and the invariance of the speed of light. The standard "effects" of time-dilation, length-contraction, and the relativity of simultaneity are easily inferred from the diagram. In addition, we show that standard calculations[2,3,6,17,18,19] of the Clock Effect and the Doppler Effect can be reduced to the "counting of ticks."

We feel that the resulting diagrams are pedagogically attractive since they emphasize the relativistic view that "time is measured by an observer's clock."

In the last section, we will consider a simplified version of our clock, called the "longitudinal light clock." Although this encodes fewer features than the full light clock, the longitudinal light clock is easy to draw manually.

In this paper, we have provided the detailed calculations used to draw the diagrams. However, we believe that one can first qualitatively construct the diagram for the novice, emphasizing the physical principles first. Then, for those interested, one can continue quantitatively with the analytical construction.

Following the standard conventions for spacetime diagrams, time runs upward on our spacetime diagrams. The scales of the axes are chosen so that light rays are drawn at 45 degrees.

II. A SIMPLIFIED MICHELSON-MORLEY APPARATUS

A. An apparatus at rest

Our simplified Michelson-Morley apparatus has a light source at the origin and two mirrors, each located a distance L along *a set of perpendicular arms.*

First, let us draw the spacetime diagram of the apparatus in its inertial rest-frame, called the "A-frame." The coordinates (x, y, t) will be used to describe the events from this frame. Since relative motion will be taken to be along the x-axis, the mirror along the x-axis will be called the "longitudinal mirror" and the mirror along the y-axis will be called the "transverse mirror."

The worldlines of A's light source and mirrors are described parametrically by

$$
\begin{array}{ll}
\text{A's light source} & \left\{\begin{array}{l} x(t) = 0 \\ y(t) = 0 \end{array}\right. \\
\\
\text{A's transverse mirror} & \left\{\begin{array}{l} x(t) = 0 \\ y(t) = L \end{array}\right. \\
\\
\text{A's longitudinal mirror} & \left\{\begin{array}{l} x(t) = L \\ y(t) = 0. \end{array}\right.
\end{array} \tag{1}
$$

Special relativity tells us that, in *all* inertial frames, light travels through the vacuum with speed c in all spatial directions, where the speed c has the value[20] of 2.99792458×10^8 m/s.

Let event O, with coordinates $(0, 0, 0)$, mark the emission of a flash of light from the source. One light ray emitted at event O reaches the transverse mirror at event

$$
Y_A : \qquad \left(0, \quad L, \quad \frac{L}{c}\right)
$$

since light travels with speed c for a time L/c in order to reach the transverse mirror a distance L away. Its
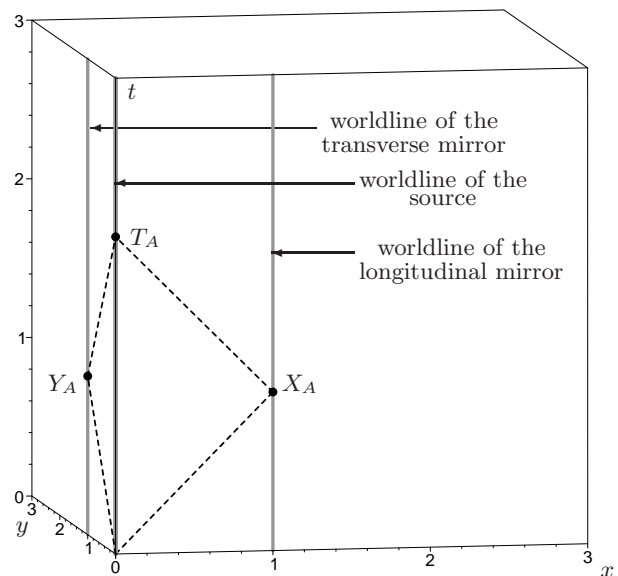

FIG. 2: This is the spacetime diagram of a simplified Michelson-Morley apparatus in its rest-frame. (The x- and y-axes are marked in units of L. The t-axis is marked in units of L/c.)

reflection is received back at the source at event

$$
T_A : \qquad \left(0, \quad 0, \quad 2\frac{L}{c}\right).
$$

Similarly, one light ray reaches the longitudinal mirror at event

$$
X_A : \qquad \left(L, \quad 0, \quad \frac{L}{c}\right),
$$

and its reflection is also received at event T_A. Hence, the two rays, emitted at event O and directed in different directions, are received at a common event T_A, whose coordinates are $(0, 0, 2L/c)$. (See Figure 2.)

If one arranges another light ray to be emitted upon reception (for example, by placing suitably oriented mirrors at the source), then this apparatus can serve as a simple clock—the light clock.

The reception event T_A marks one "tick" of this clock. This tick will be chosen to be the "standard tick." The elapsed time logged by an observer sitting at the source is equal to the number of ticks multiplied by $2L/c$, the duration of one round trip of a light ray.[21] If a finer scale of time is required, one can increase the resolution by choosing a smaller separation L. In addition, the reflection events X_A and Y_A, which are on mirrors equidistant from the source, can be regarded as "half-ticks" of this clock. We define these half-ticks to be "simultaneous events for this clock."

B. An apparatus in motion

Now, suppose an identical apparatus moves with spatial-velocity v parallel to the x-axis of the A-frame. This moving inertial-frame will be called the "B-frame," and the coordinates (x', y', t') will be used to describe events from this frame. For simplicity, the origins of the primed and unprimed coordinate systems are taken to coincide at the emission event O. In addition, the corresponding spatial axes are assumed to be spatially-parallel[22] within each inertial-frame.

Since B's apparatus is identical to A's, the worldlines of B's light source and mirrors are described parametrically by

$$\text{B's light source} \qquad \left\{ \begin{array}{l} x'(t') = 0 \\ y'(t') = 0 \end{array} \right.$$

$$\text{B's transverse mirror} \qquad \left\{ \begin{array}{l} x'(t') = 0 \\ y'(t') = L \end{array} \right. \qquad (2)$$

$$\text{B's longitudinal mirror} \qquad \left\{ \begin{array}{l} x'(t') = L \\ y'(t') = 0. \end{array} \right.$$

What does the spacetime diagram of this moving apparatus look like in the A-frame? In particular, what are the (x, y, t) coordinates of the worldlines of the moving apparatus and of the events X_B, Y_B, and T_B?

Since B's light source and transverse mirror move with velocity v in the x-direction, they are described as:[23]

$$\text{B's light source} \qquad \left\{ \begin{array}{l} x(t) = vt \\ y(t) = 0. \end{array} \right.$$

$$\text{B's transverse mirror} \qquad \left\{ \begin{array}{l} x(t) = vt \\ y(t) = L. \end{array} \right.$$

Due to this mirror's motion, the light ray from event O that meets this mirror must travel a longer distance in the A-frame. (See Figure 3.) Using the Pythagorean theorem, we find this distance is $L(1-(v/c)^2)^{-1/2}$, which is traveled by light in time $(L/c)(1 - (v/c)^2)^{-1/2}$. Thus, the reflection event on B's transverse mirror is

$$Y_B: \quad \left(v\frac{L}{c}\frac{1}{\sqrt{1-(v/c)^2}}, \quad L, \quad \frac{L}{c}\frac{1}{\sqrt{1-(v/c)^2}} \right).$$

Similarly, the reflected ray is received by B's source at:

$$T_B: \quad \left(v\frac{2L}{c}\frac{1}{\sqrt{1-(v/c)^2}}, \quad 0, \quad \frac{2L}{c}\frac{1}{\sqrt{1-(v/c)^2}} \right).$$

Note that, in the A-frame, event T_B ("the first tick of the moving clock") occurs later than T_A ("the first tick of the stationary clock"). This is the "time dilation" effect.

Now, let us consider the longitudinal mirror. (See Figure 4.) Recall that the reflected light rays for A's apparatus were received by A's source at a common event T_A. According to the principle of relativity, the apparatuses cannot distinguish their states of inertial motion. Thus, the reflected light rays for B's apparatus must also be received by B's source at a common event, here, T_B.
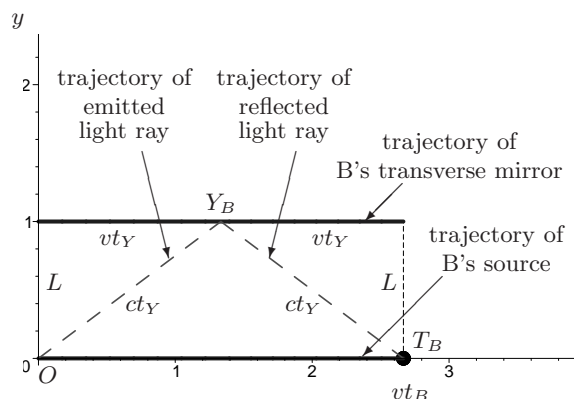

FIG. 3: On A's xy-plane, the spatial trajectories of B's transverse arm and its associated light rays are drawn. The marked dot corresponds to the spatial coordinates of B's first "tick," which occurs after an elapsed time t_B in the A-frame. Let t_Y be the elapsed time for a light ray from event O to reach B's transverse mirror. Using the Pythagorean theorem, it can be shown that $t_Y = (L/c)(1 - (v/c)^2)^{-1/2}$. By symmetry, it follows that $t_B = (2L/c)(1 - (v/c)^2)^{-1/2}$, which is longer than the duration of A's tick, $(2L/c)$. This is called the "time dilation" effect. (The x- and y-axes are marked in units of L.)
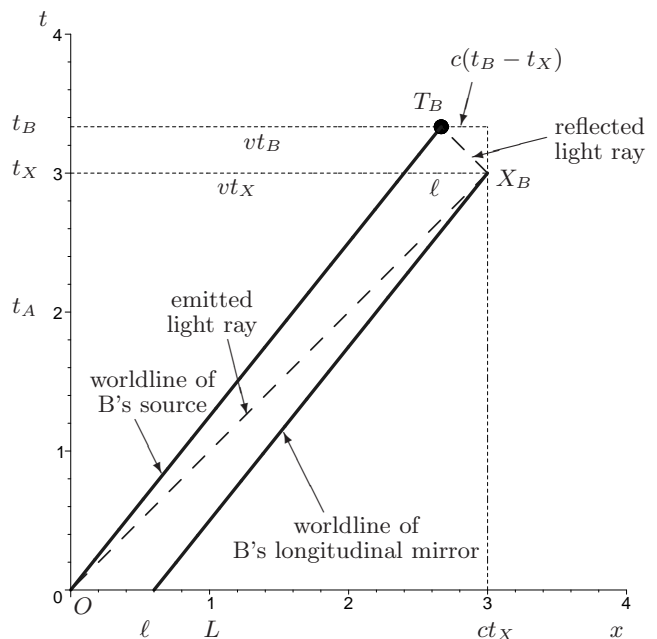

FIG. 4: On A's xt-plane, the worldlines of B's longitudinal arm and its associated light rays are drawn. The marked dot corresponds to B's first "tick." Let ℓ be the apparent length of B's longitudinal arm. Let t_X be the elapsed time for a light ray from event O to reach B's longitudinal mirror. Since $ct_X = vt_B + c(t_B - t_X)$ and $t_B = (2L/c)(1 - (v/c)^2)^{-1/2}$, it can be shown that $t_X = (L/c)((1 + v/c)/(1 - v/c))^{1/2}$. Furthermore, since $ct_X = vt_X + \ell$, it follows that $\ell = L(1 - (v/c)^2)^{1/2}$, which is shorter than the proper length L of A's identical apparatus. This is the "length contraction" effect. (The x-axis is marked in units of L. The t-axis is marked in units of L/c.)

For light rays to be received at event T_B, what is the required reflection event X_B on B's longitudinal mirror? Event X_B is the intersection on the xt-plane of the forward-directed light ray from event O and the backward-directed light ray toward event T_B. (Refer again to Figure 4.) After a little algebra, the reflection event X_B is determined to be

$$X_B: \quad \left(c\frac{L\sqrt{1-(v/c)^2}}{c-v}, \quad 0, \quad \frac{L\sqrt{1-(v/c)^2}}{c-v} \right).$$

Thus, B's longitudinal mirror is described by:

$$\text{B's longitudinal mirror} \begin{cases} x(t) = vt + L\sqrt{1-(v/c)^2} \\ y(t) = 0. \end{cases}$$

Note that, in the A-frame, the length of B's longitudinal arm is $L(1-(v/c)^2)^{1/2}$, which is shorter than its proper length L. This is the "length contraction" effect.

This completes the construction of B's apparatus.

As a check, these results can be obtained directly from the Lorentz transformation:

$$\left. \begin{aligned} t' &= \frac{t - vx/c^2}{\sqrt{1-(v/c)^2}} \\ x' &= \frac{x - vt}{\sqrt{1-(v/c)^2}} \\ y' &= y \end{aligned} \right\} \quad (3)$$

For instance, given the worldlines for B's apparatus in (x', y', t')-coordinates (Equation 2), expressions for x and y as functions of t can be obtained:

$$\left. \begin{array}{l} \text{B's light source} \quad \begin{cases} x(t) = vt \\ y(t) = 0 \end{cases} \\ \text{B's transverse mirror} \begin{cases} x(t) = vt \\ y(t) = L \end{cases} \\ \text{B's longitudinal mirror} \begin{cases} x(t) = vt + L\sqrt{1-(v/c)^2} \\ y(t) = 0. \end{cases} \end{array} \right\} (4)$$

Similarly, given the (x', y', t')-coordinates of B's tick and half-ticks, the (x, y, t)-coordinates of X_B, Y_B, and T_B can be obtained.

These results are summarized in Figure 5.

In addition to the time-dilation and length-contraction effects, note that the events X_B and Y_B, which are defined to be simultaneous according to B's clock, are not simultaneous according to A's clock. This is the "relativity of simultaneity."

In passing, we observe that the length-contraction factor $(1 - (v/c)^2)^{1/2}$, not found in the Galilean transformations,[24] enforces the requirement that the reflection occur at event X_B so that the reflected light ray is received at event T_B. Without length contraction, the reflection occurs at event X_{Gal}, and the reception event U_{Gal} occurs at the source *after* event T_B. (See Figure 6.) Such a result violates the principle of relativity since one's inertial state of motion could now be detected. Indeed, the Michelson-Morley apparatus was used to measure the time difference between events U_{Gal} and T_B, as predicted by the Galilean transformations. However, no time difference was experimentally observed.[12,25]
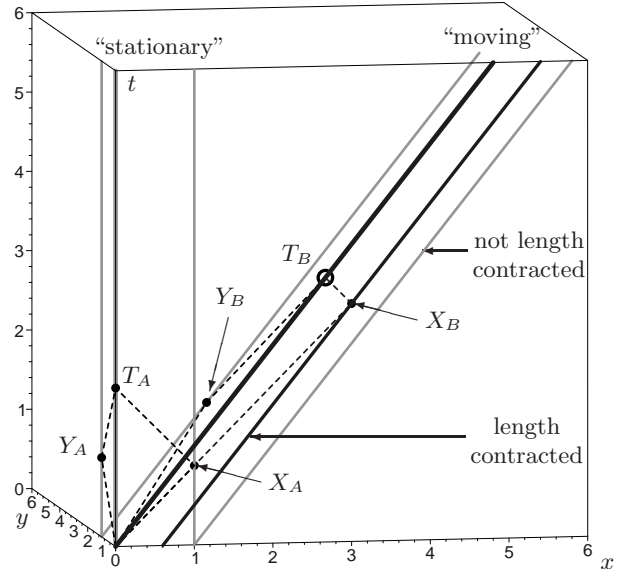

FIG. 5: This is A's spacetime diagram of B's identical apparatus, which moves with velocity $v = 0.8c$ along A's x-axis.
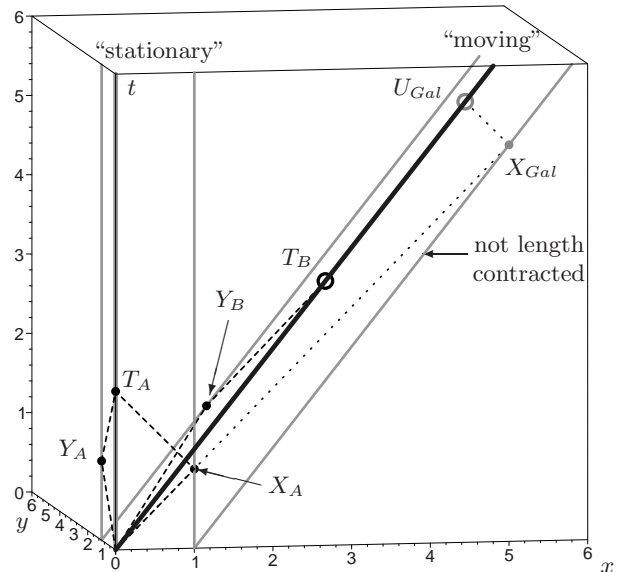

FIG. 6: This is A's spacetime diagram of B's identical apparatus *without* length contraction. Observe that without length contraction, the light rays reflected by the moving mirrors are not received simultaneously (at T_B) by the moving source.

For clarity, it is useful to introduce the standard abbreviations. Let γ denote the time-dilation factor

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}, \qquad (5)$$

and let k denote the Doppler-Bondi factor[19,26]

$$k = \sqrt{\frac{1+(v/c)}{1-(v/c)}}. \qquad (6)$$

With these abbreviations, the coordinates of X_B, Y_B, and T_B can be expressed as

$$
\begin{aligned}
X_B: & \quad \left(kL, \quad 0, \quad k\frac{L}{c}\right) \\
Y_B: & \quad \left(\gamma v\frac{L}{c}, \quad L, \quad \gamma\frac{L}{c}\right) \\
T_B: & \quad \left(2\gamma v\frac{L}{c}, \quad 0, \quad 2\gamma\frac{L}{c}\right).
\end{aligned}
$$

For the examples used throughout this paper, the B-frame moves with velocity $v = 0.8c$ relative to the A-frame. For this choice, we have $\gamma = 5/3$ and $k = 3$.

III. CIRCULAR LIGHT CLOCKS

A. A generalized apparatus

Generalizing the analysis of the last section, it is easy to see that:

> With *any* relative orientation of the arms one would obtain the same results:
>
> 1. Light rays emitted by the source at event O to mirrors a distance L away would be received back at the source at a time $2L/c$ later.
>
> 2. The reflection events at the mirrors are simultaneous according to that source.

So, instead of a *pair* of equidistant mirrors, consider a whole collection of mirrors placed inside a circle [generally, a sphere] of radius L. Henceforth, this will be called the "circular light clock."[27] What would the spacetime diagram of this light clock look like?

In this case, one would have a hollow world*tube* to describe the collection of mirrors for each clock. In addition, for each tick of a given clock, one would draw the portion of its light cone contained inside the clock's worldtube. These cones represent events in spacetime traced out by the collection of light rays that reflect off the mirrors from tick to tick. (See Figure 7.)

In particular, the stationary light clock will be drawn with a circular cross-section. The moving one will be drawn tilted with an elliptical cross-section since it is length-contracted in the direction of relative motion.

Then, given a starting emission event on the axis of each worldtube, one traces out the paths of the light rays and their reflections, which are drawn upward with a slope of 45 degrees in this diagram. For simplicity, the starting emission event O is taken to be the intersection of the two sources. (Refer again to Figure 7.)

We now make a series of observations.

The original pair of reflection events X_A and Y_A (and X_B and Y_B, respectively) are among the events on the circle of mutual intersections of the worldtube of the light clock and the light cones of its zeroth and first ticks. We extend our definition of simultaneity according to this clock to that circle of events. In fact, one can extend this simultaneity to events on the unique [hyper]plane that contains this circle [respectively, sphere]. Physically, this [hyper]plane represents "all of space at a particular instant for this light clock." As before, the events that are simultaneous to this light clock are generally different from those events determined to be simultaneous by the other light clock.

With this notion of simultaneity, the light cones can be interpreted in a complementary way. For each light clock, a simultaneous "slice" of its cones represents a circular [respectively, spherical] wavefront traveling at the speed
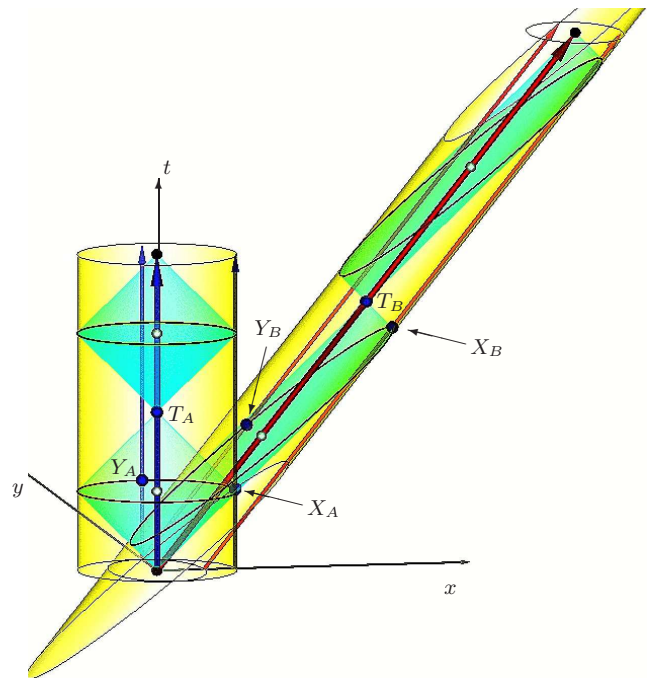

FIG. 7: Two circular light clocks in relative motion. For each light clock, the intersection of the worldtube and the light cones from two consecutive ticks is a circle of simultaneous events for that clock. The white dots represent events at the source that are simultaneous with the corresponding circle of intersection. Note that the "moving" circular mirror is length-contracted in this inertial frame.

of light. Hence, the light cone can also be interpreted as a sequence of wavefronts traveling at the speed of light.[28]

B. Visualizing proper-time

By continuing this light-clock construction along each inertial observer's worldtube, an accurate visual representation of the proper time elapsed for each observer is obtained. From such a diagram, however, it may not be evident that the two observers are equivalent.

First, we demonstrate their symmetry with the Doppler Effect. (See Figure 8.) Suppose these inertial observers emit light signals at one-tick intervals. From the diagram, each observer receives those signals from the other observer at three-tick intervals. In other words, the received frequency is one-third of the original frequency, in accordance with the Doppler effect for two observers separating with speed $v = 0.8c$. In general, a light ray emitted by a source at its first tick after separation reaches the receiver at the receiver's kth tick after separation, where k is the Doppler-Bondi factor defined in equation 6. This is the basis of the Bondi k-calculus.[19,26,29,30]

Next, we demonstrate their symmetry with the time-dilation effect. (Refer again to Figure 8.) Consider the signal emitted at the first tick. As just noted, this signal is received by the receiver at his third tick. According to the source of that signal, that distant reception event is simultaneous with his fifth tick. In other words, the apparent elapsed time assigned to a distant event is five-thirds as long as the proper elapsed time measured by the inertial observer who visits that distant event. In general, the apparent elapsed time assigned to a distant event is γ times as long as the proper elapsed time measured by the inertial observer who meets that distant event, where γ is the time-dilation factor defined in equation 5.

In addition, the source measures the apparent distance to that reception event to be $vt_{apparent} = v \cdot (\gamma k \text{ ticks})$. For $v = 0.8c$, this is

$$v\gamma k \text{ ticks} = (0.8\text{c})\left(\frac{5}{3}\right)(3)(1 \text{ tick}) = 4 \text{ ticks} \cdot c$$
$$= 4 \text{ "light-ticks"}.$$

Since L is the radius of the worldtube and 1 tick $= (2L/c)$, this distance can be expressed as "four worldtube *diameters*." This suggests the diagrams in Figure 9. Of course, this is just the calculation of the square-interval in terms of the temporal and spatial coordinates

$$\left(\frac{\text{proper}}{\text{time}}\right)^2 = \left(\frac{\text{apparent}}{\text{time}}\right)^2 - \frac{1}{c^2}\left(\frac{\text{apparent}}{\text{distance}}\right)^2$$
$$(3 \text{ ticks})^2 = (5 \text{ ticks})^2 - (4 \text{ ticks})^2,$$

which can be regarded as the spacetime version of the Pythagorean theorem. Observe that, for a constant value of the proper-time, the admissible pairs of temporal and spatial coordinates locate events on a hyperbola.
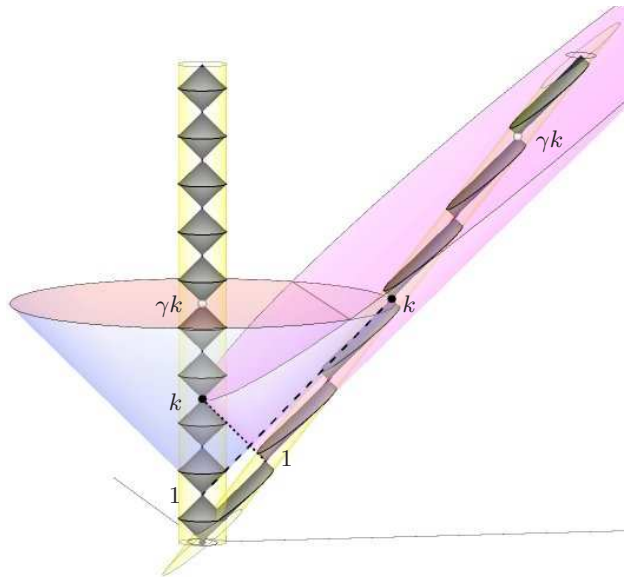

FIG. 8: Symmetry of the observers: The Doppler and Time-Dilation effects. For a relative speed of $v = 0.8c$, we have $\gamma = 5/3$ and $k = 3$.
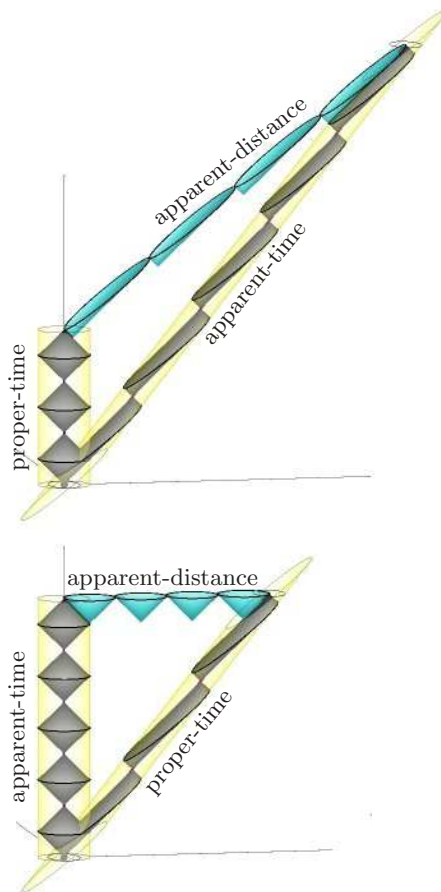

FIG. 9: Symmetry of the observers: The spacetime analogue of the Pythagorean Theorem: $(\text{proper-time})^2 = (\text{apparent-time})^2 - (\text{apparent-distance}/c)^2$.

C. The Clock Effect

With this pictorial device, we present a visual representation of the Clock Effect. (See Figure 10.)
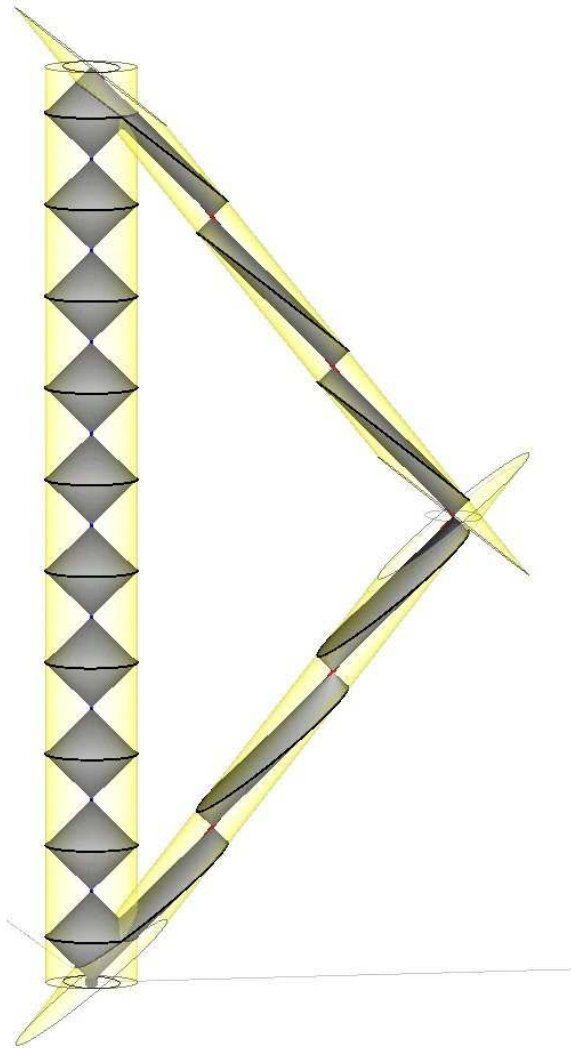

FIG. 10: The Clock Effect. A non-inertial observer travels away with velocity $v = 0.8c$ for 3 ticks, then returns with velocity $v = -0.8c$ for another 3 ticks. Between the departure and reunion events, he has logged 6 ticks for his entire trip, whereas the inertial observer has logged 10 ticks.

From the diagram, the non-inertial observer travels away with velocity $v = 0.8c$ for 3 ticks, then returns with velocity $v = -0.8c$ for another 3 ticks. Between the departure and reunion events, he has logged 6 ticks for his entire trip. On the other hand, the inertial clock logs

$$\frac{\text{(first 3 ticks)}}{\sqrt{1 - (0.8c/c)^2}} + \frac{\text{(second 3 ticks)}}{\sqrt{1 - (-0.8c/c)^2}} = 10 \text{ ticks}$$

between the same departure and reunion events. Clearly, the diagram reveals that more time elapses for the inertial observer that meets both events.

In addition, the two observers are certainly inequivalent. The kink in the non-inertial worldline causes the sequence of simultaneous events to change discontinuously, leading to the apparent break in the non-inertial worldtube. This is not to say that the non-inertial worldtube actually breaks. Rather, it is an artifact of how the diagram was drawn. In order to draw the true worldtube, a more careful analysis with a detailed model of the apparatus is needed. We refer the reader to some articles on the Clock Effect that discuss this kink in the non-inertial observer's worldline.[10,19,30]

D. A brief summary

Let us summarize the logical development up to this point.

Given the simplified Michelson-Morley apparatus in relative motion, the invariance of the speed of light (so that *all* light rays are drawn at an angle of 45 degrees) is used to draw the light rays associated with the perpendicular arm. Invoking the principle of relativity (so that the duration of the round-trip defines the same standard tick), we deduce the effect of time dilation. Again using the invariance of the speed of light, we draw the light rays associated with the parallel arm. Again invoking the principle of relativity, we deduce the effects of length contraction and the relativity of simultaneity.

Generalizing these results to arbitrary directions, we obtain the circular light clock. By continuing this construction along a piecewise-inertial worldline, we obtain a visual representation of the proper-time elapsed along that worldline.

IV. LONGITUDINAL LIGHT CLOCKS

Let us now consider a simplified two-dimensional version of the light clock diagram. Consider the worldlines of the source and of one longitudinal mirror, that is, the longitudinal light clock. As before, the moving light clock appears length-contracted in the direction of relative motion. In this case, we have the following diagrams. (See Figures 11 and 12.)

Although these figures are much easier to draw, it is unfortunate that the role of length contraction appears here so prominently. Recall that event T_B was determined using the invariance of the speed of light and the principle of relativity, which required that the transverse and longitudinal reflections be received simultaneously at the source. However, without the transverse direction, the role of invariance may not be evident.

In this section, we will draw attention to a certain geometric property of this diagram and use it to emphasize instead the "invariance of the spacetime interval".
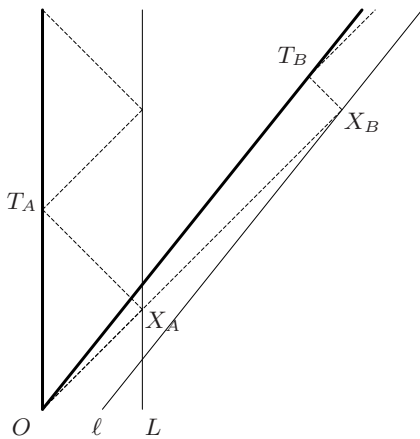
FIG. 11: Two longitudinal light clocks. The apparent length ℓ of the moving apparatus was shown to be $L(1-(v/c)^2)^{1/2}$.
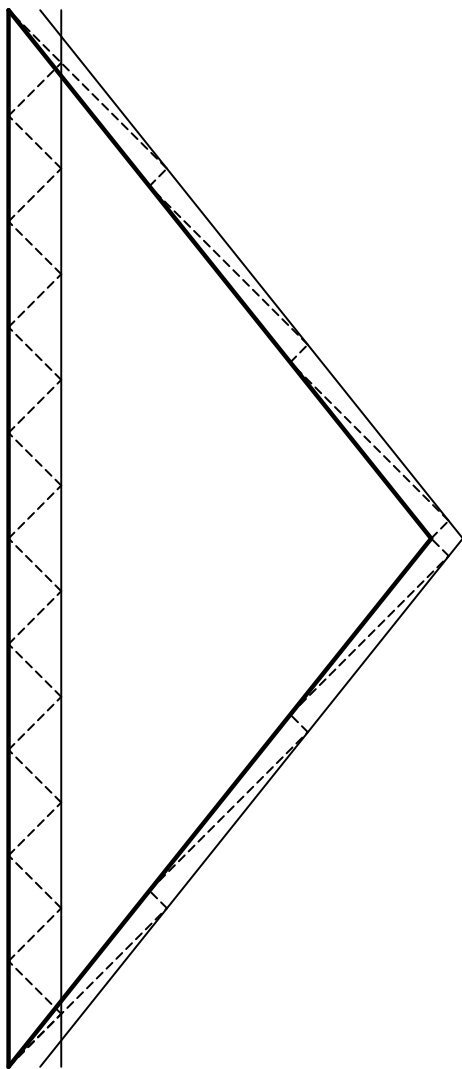

FIG. 12: The Clock Effect with the longitudinal light clocks.

A. A simple construction

Refer to the diagram of two longitudinal light clocks. (See Figure 13.)

Consider the triangles $\triangle OX_AT_A$ and $\triangle OX_BT_B$, which are formed from the timelike intervals from event O to the first ticks and their associated light rays. Since, on a spacetime diagram, events T_A and T_B are at equal intervals from event O, they lie on a rectangular hyperbola asymptotic to the light cone of event O. From this, it can be shown that these triangles (which are related by Lorentz transformations) have the same area. In fact, we will show in Section IV B that this area is proportional to the square-interval of one tick.[31]

Using the similarity of triangles $\triangle OX_AF$ and $\triangle OX_BT_B$ and the relation $k = (OX_B)/(OX_A)$, we obtain the useful corollary that

$$\frac{(X_AF)}{(X_AT_A)} = 1/k^2.$$

With this property, we can now draw the longitudinal light clock with the emphasis on the invariance of the interval, rather than on length contraction.

Given the standard tick for the stationary observer (events O, T_A, and X_A) and the worldline of a moving observer (line OF), one can determine the standard tick for the moving observer (events O, X_B and T_B) as follows.

- Measure $(X_AF)/(X_AT_A)$. [For a classroom activity, one might use a sheet of graph paper with its axes aligned with the future light cone of event O.]

- Calculate k using the corollary $(X_AF)/(X_AT_A) = 1/k^2$. (In the next section, we show that k is equal to the Doppler-Bondi factor.)
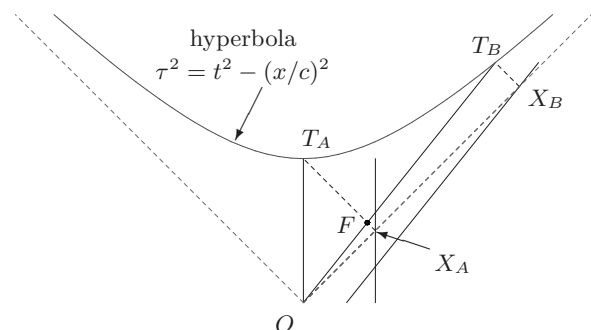

FIG. 13: Since T_A and T_B are points of this hyperbola, the triangles $\triangle OX_AT_A$ and $\triangle OX_BT_B$ have the same area. Let $k = (OX_B)/(OX_A)$. Using the similarity of the triangles $\triangle OX_AF$ and $\triangle OX_BT_B$, it follows that $(X_AF)/(X_AT_A) = 1/k^2$. With these facts, given O, T_A, X_A, and F, one can easily determine X_B and T_B. (It will be shown that k is precisely the Doppler-Bondi factor.)

- Determine the reflection event X_B along the outgoing light ray using the relation $k = (OX_B)/(OX_A)$.

- Determine the reception event T_B by tracing the reflected light ray back onto the moving worldline. This displays the time dilation effect.

- Determine the worldline of the longitudinal mirror by drawing through X_B the parallel to OF. This displays the length contraction effect.

- Finally, determine a set of simultaneous events for this clock by first completing the parallelogram with sides OX_B and $X_B T_B$ and then drawing the diagonal through X_B. This displays the relativity of simultaneity.

B. An invariant area

The following calculation reveals that this area of the triangles used in the previous section is proportional to the square-interval of one tick.[31]

Since we will be discussing an aspect of the *geometry* of the spacetime diagram, it is convenient to work with a more natural set of coordinates $(x/c, t)$, where now each coordinate has the same units.

Consider the segment drawn from the emission event O to any event T with coordinates $(x/c, t)$. (See Figure 14.) Regard that segment as the hypotenuse of a Euclidean right triangle whose sides are parallel to the light cone of
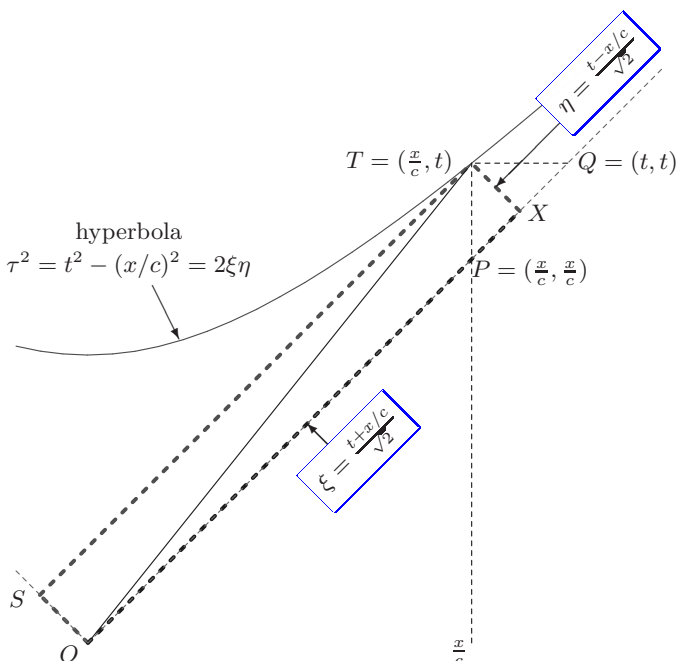

FIG. 14: The legs of triangle $\triangle OXT$ have measure $\xi = (2)^{-1/2}(t + x/c)$ and $\eta = (2)^{-1/2}(t - x/c)$.

event O. The legs of this triangle have measure

$$\xi = \frac{t + x/c}{\sqrt{2}} \qquad \eta = \frac{t - x/c}{\sqrt{2}}.$$

These are called the Dirac light-cone coordinates[31,32] of event T in the A-frame. In these coordinates, the Euclidean area of this triangle is simply $\xi\eta/2$, which is equal to $(t^2 - (x/c)^2)/4$. That is, the Euclidean area of triangle $\triangle OXT$ is equal to one-fourth of the square-interval from event O to event T.

Let us explicitly verify that this area is invariant under Lorentz transformations.[33] Using Equation 3, the light-cone coordinates of event T in the B-frame are

$$\xi' = \frac{t' + x'/c}{\sqrt{2}} = \gamma\left(1 - \frac{v}{c}\right)\left(\frac{t + x/c}{\sqrt{2}}\right)$$
$$= k^{-1}\left(\frac{t + x/c}{\sqrt{2}}\right) = k^{-1}\xi$$

$$\eta' = \frac{t' - x'/c}{\sqrt{2}} = \gamma\left(1 + \frac{v}{c}\right)\left(\frac{t - x/c}{\sqrt{2}}\right)$$
$$= k\left(\frac{t - x/c}{\sqrt{2}}\right) = k\eta,$$

where k is Doppler-Bondi factor defined in Equation 6. Thus, the quantity $\xi\eta/2$ is Lorentz invariant.

It is instructive to interpret this geometrically. For concreteness, let us start with $T = T_B$ and $X = X_B$. In this case, we seek the Lorentz transformation that sets OT_B to be at rest in the B-frame. Imagine sliding T_B down along the hyperbola until OT_B is vertical. As that happens, X_B slides down the light cone, scaling the ξ-leg down by a factor k and scaling the η-leg up by a factor k. Thus, the area of triangle $\triangle OX_B T_B$ is invariant.

We extend this result to the parallelogram formed with light rays OX_B and $X_B T_B$, *i.e.*, the *"area of intersection between the interior of O's future light-cone and the interior of T_B's past light-cone."* Clearly, this area is invariant and is proportional to the square-interval of one tick.

In fact, this result generalizes to higher-dimensions: the analogous *volume* of intersection between the light-cone interiors is also invariant and is proportional to the square-interval of one tick. We show this for the 3- and 4-dimensional case. The Euclidean volume of a cone with an elliptical base is $(\pi ab)h/3$, where a and b are the semi-major and semi-minor axes of the elliptical base, and h is the altitude of the cone. In four dimensions, the hyper-volume of a cone with an ellipsoidal base is $(4\pi ab^2/3)h/4$. Since we orient the base so that b is a length along the transverse direction, it is unchanged under a Lorentz transformation. Regarding $\triangle OSX$ in Figure 14 as the xt-cross-section of the light cone of event O, observe that its area is $(2a)h/2 = ah$. Using the symmetries of the parallelogram, the area of $\triangle OSX$ is equal to the area of $\triangle OXT$. Thus, ah is invariant. It follows that the volume of intersection between the light-cone interiors is invariant.

V. FINAL REMARKS

By drawing the spacetime diagram of a Michelson-Morley apparatus, we have obtained an accurate visualization of the proper-time elapsed along a piecewise-inertial observer's worldline. Measurements of spacetime intervals have been reduced to the counting of ticks, emphasizing the relativistic view that "time is measured by an observer's clock." We believe that the resulting diagram can be used to discuss special relativity in a qualitative way which emphasizes the physics first and the algebra second. We feel this could easily be incorporated into the standard textbook treatments of special relativity, which often discuss the Michelson-Morley experiment.

The ideas presented in this paper are being implemented in a series of interactive computer programs which will be posted to our website.[34]

VI. ACKNOWLEDGMENTS

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[21] Implicitly, it is assumed that the impulses transferred to the mirrors upon reflection are negligible so that these mirrors maintain their separation from the source.

[22] This is the usual convention for "aligning" the spatial axes. In special relativity, the x- and x'- axes are *not* parallel in *spacetime*. However, in the A-frame, the spatial-projection of the x'-axis *is parallel* to the x-axis, and conversely for the B-frame. By contrast, in Galilean relativity, the corresponding spatial axes are parallel in both the spatial and spacetime senses.

[23] Strictly speaking, we should prove that, for a moving object, lengths along its transverse directions are unchanged. We refer the reader to Feynman[2], Mermin[5], and Arons[11] for a symmetry argument.

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