Hamilton's principle: Why is the integrated difference of the kinetic and potential energy minimized?

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Why is the integrated difference of the kinetic and potential energies the quantity to be minimized in Hamilton's principle? I use simple arguments to convert the problem of finding the path of a particle connecting two points to that of finding the minimum potential energy of a string. The mapping implies that the configuration of a nonstretchable string of variable tension corresponds to the spatial path dictated by the principle of least action; that of a stretchable string in space–time is the one dictated by Hamilton's principle. This correspondence provides the answer to the question.
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I. INTRODUCTION

Soap films minimize their surface area and adopt a spherical shape; a large piece of matter maximizes the gravitational attraction between its parts, and hence planets also are spherical. Light rays refracting on a magnifying glass bend and follow the path of least time, and a relativistic particle follows the path between two events in space–time that maximizes the time measured by a clock on the particle.[1,2]

In 1744, Maupertuis proposed that "Nature, in the production of its effects, does so always by the simplest means,"[3] and in 1746, wrote "in Nature, the action (*la quantité d'action*) necessary for change is the smallest possible. Action is the product of the mass of a body times its velocity times the distance it moves."[4] For a light ray or a particle passing from one medium into another, both the minimization of time and minimization of action gives rise to angles of incidence and refraction in a fixed proportion to each other: the analog of Snel's law[5] for a light ray, $n_1 \sin \theta_1 = n_2 \sin \theta_2$, corresponds to the conservation of particle momentum along the interface, $mv_1 \sin \theta_1 = mv_2 \sin \theta_2$. Although Maupertuis' formulation was vague, and there was controversy over the priority over the idea,[6] his name remains attached to the principle arguably for two reasons: his metaphysical view that minimum action expresses God's wisdom in the form of an economy principle,[8] and Euler's role in settling the controversy in his favor.[7]

Hamilton's principle, formulated almost a century later, is similar to the principle of least action and is based on the optical-mechanical analogy as well.[9] Although the trajectory followed by a particle of fixed energy E connecting two points in space is given by the principle of least action, Hamilton's principle determines the trajectory for which the particle will spend a given time t traveling between the same points. The optimal path is the one for which the sum of the products $(K-U)\Delta t$ along the path is a minimum (K and U are the kinetic and potential energies and Δt is the time interval). Hamilton's method was mentioned throughout the nineteenth century, but was rarely used because simpler methods were as effective in most cases.[10]

The situation changed in 1926 when Schrödinger used Hamilton's analogy between mechanics and geometrical optics, and arrived at his famous equation for the dynamics of a quantum mechanical particle.[11] In 1948 Feynman[12] offered a new perspective on Hamilton's principle: a quantum particle "explores" all paths between two points. This "democracy of histories"[13] becomes the principle of least action for a classical particle, which due to destructive interference eliminates those paths that differ significantly from the classical (or extremal) path.

A lesser known approach to the principle of least action was taken by John Bernoulli,[14] who showed that Snel's law can be obtained from the condition of mechanical equilibrium of a tense, nonstretchable string (see Fig. 1). This analogy also was noted by Möbius[15,16] and discussed by Ernst Mach.[17] Reference 18 considers an inextensible string and is the only article I found on this analogy.

In this paper I show that a simple extension of this analogy to paths that are covered in fixed time can be used to prove the equivalence of Hamilton's principle to the static equilibrium of a *stretchable* string. My goal is to provide insight, in the spirit of Refs. 19–23, into why it is the difference between the kinetic and potential energy that appears in Hamilton's principle. I review Bernoulli's approach in Sec. II, and present a simplified derivation of Hamilton's principle in Sec. III. In Sec. IV, I present a somewhat more elaborate derivation using elementary calculus. Given the importance of the principle of least action in many areas of physics, I hope that this paper will contribute to its presentation in introductory courses, rather than it being postponed to advanced mechanics courses.

II. THE LEAST ACTION PRINCIPLE AND NONSTRETCHABLE STRINGS

Figure 1 shows the diagram used by Bernoulli to derive Snel's law for a light ray traveling from point A to point B using the analogy with the static equilibrium of a string under tension. The following is a rephrasing of Bernoulli's argument, which is based on the assumption that, for any system in mechanical equilibrium, it is equivalent to say that the net force on each point of the system is zero, and the system is in the state of minimum potential energy. I will assume knowledge of Newton's law $\mathbf{F}=\Delta\mathbf{p}/\Delta t$ relating the force on a particle with the rate of change of its momentum.

Call T_1 and T_2 the weights hanging from points A and B in Fig. 1. The point of contact between the upper and lower portions of the string slides horizontally without friction along the line CD. The pulleys at A and B are frictionless and have zero inertia; therefore the tensions of the different portions of the string will be T_1 and T_2. Compare the potential energy of the configurations for which the point of contact is

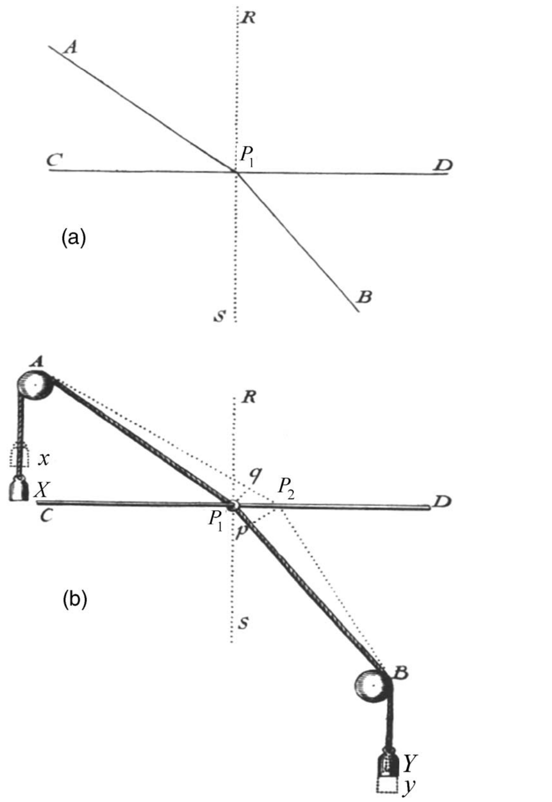

Fig. 1. (John Bernoulli's proof of Snel's law using the mechanical equilibrium of a string under tension.) Two weights T_1 and T_2 hang from frictionless pulleys at A and B, meaning that the tension on the portions AP_1 and P_1B is T_1 and T_2. The point of contact, P_1, slides without friction along CD. The potential energy of the system, $U=T_1\overline{AP_1}+T_2\overline{P_1B}$, is minimized at equilibrium, where the horizontal components of the tensions T_1 and T_2 cancel, giving Snel's law: $T_1 \sin \angle AP_1R = T_2 \sin \angle SP_1B$. (Figure reproduced from Ref. 14.)

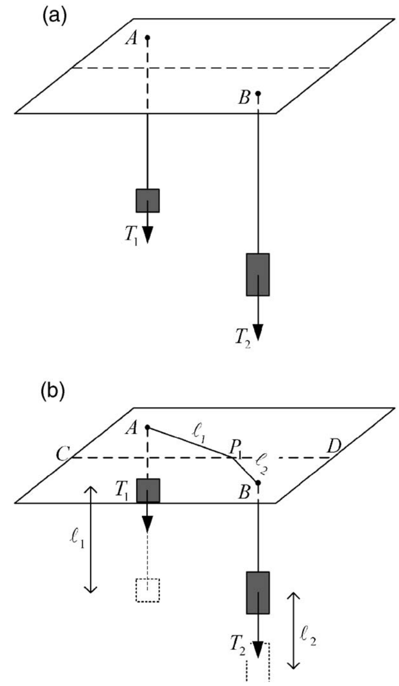

Fig. 2. An alternative version of Bernoulli's setup: (a) Two weights hanging from frictionless points A and B are assigned zero potential energy. (b) The two weights are then lifted, and the ends of the strings are joined at point P_1 along the line CD. The work done is equal to the increase in the potential energy: $U=T_1\ell_1+T_2\ell_2$.

P_1 and P_2. There is a potential energy change between the two configurations because in going from P_1 to P_2, mass 1 will rise from X to x and mass 2 will decrease its height from Y to y. The tensions are determined by the weights and are equal to T_1 and T_2, respectively. We let $\ell_1=\overline{AP_1}$ and $\ell_2=\overline{P_1B}$, and write the change in the potential energy ΔU of the system as

$$\Delta U = T_1 \Delta \ell_1 + T_2 \Delta \ell_2, \tag{1}$$

where $\Delta\ell_1=\overline{Xx}=\overline{qP_2}$ and $\Delta\ell_2=-\overline{Yy}=\overline{P_2p}$. Because T_1 and T_2 are constant, Eq. (1) implies that, up to an additive constant, the potential energy of the system can be expressed as

$$U = T_1 \ell_1 + T_2 \ell_2. \tag{2}$$

Another way of visualizing Eq. (2) is offered in Fig. 2. Because the configuration of mechanical equilibrium corresponds to the minimum of potential energy, the minimum U is attained when the components of the forces from the different portions of the string along CD cancel. In terms of the angles $\theta_1=\angle AP_1R$ and $\theta_2=\angle SP_1B$, the minimum potential energy is attained when

$$T_1 \sin \theta_1 = T_2 \sin \theta_2. \tag{3}$$

Equation (3) is equivalent to Snel's law if the indices of refraction n_1 and n_2 in the different regions are identified with the tension of the strings. If the time t or, equivalently, the optical length ct given by

$$ct = n_1 \ell_1 + n_2 \ell_2, \tag{4}$$

is minimized, then $n_1 \sin \theta_1 = n_2 \sin \theta_2$.

To make the analogy between the configuration of the strings and the trajectory of a particle with energy E between points A and B in Fig. 2, I reason in reverse. Conservation of momentum at the interface requires that

$$mv_1 \sin \theta_1 = mv_2 \sin \theta_2. \tag{5}$$

What is the magnitude that should be minimized to obtain Eq. (5)? Because Snel's law results from minimization of Eq. (4), the string version of Snel's law [Eq. (5)] will result by minimizing the quantity A given by

$$A = mv_1 \ell_1 + mv_2 \ell_2, \tag{6}$$

which is Maupertuis' action. The correspondence between Eqs. (4) and (6) expresses the analogy between mechanics and geometric optics, which has been the subject of many recent expositions.[24,25]

The tension T_i of the string is identified with the velocity v_i in each region, which is given by $v_i=\sqrt{2m(E-U_i)}$, where E is the total energy and U_i is the corresponding potential energy. For paths that traverse many regions where the particle velocities are different, the trajectory has to be divided into many straight segments. The principle of least action states that for a given total energy, the trajectory of the particle between two fixed points is the one that minimizes the sum of the products $mv_i\ell_i$ in each segment. To use the analogy with the string, the corresponding arrangement for a

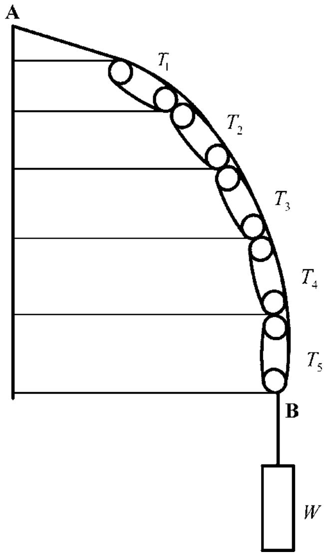

Fig. 3. Frictionless pulleys that can slide in horizontal lines with a string passing through them a sufficient number of times gives the trajectory of the particle if T_i is identified with mv_i at each segment. Because the string can only pass through each pulley an integer number of times, the ratios of the velocities are approximated by the ratio of the times the rope passes through each segment.

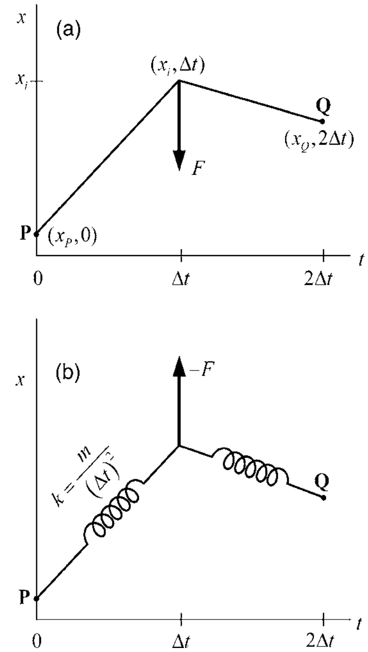

Fig. 4. (a) Space–time trajectory of an otherwise free one-dimensional particle acted on by an impulsive force F at $t_i = \Delta t$. (b) Equivalent equilibrium configuration of two segments of a stretchable string with spring constant $k = m/(\Delta t)^2$ and an external force $-F$.

string of N segments consists of frictionless pulleys that can slide on rods, with the string passing through them a sufficient number of times[17] (see Fig. 3). Table I summarizes the analogies between the quantities discussed in this section.

III. HAMILTON'S PRINCIPLE AND STRETCHABLE STRINGS

The principle of least action as stated in Sec. II gives the trajectory for a particle of a given energy between two fixed points. It does not say anything about the time it takes to travel from one point to the other. Now consider the problem of finding a path that will connect point $P = (x_P, 0)$ to point $Q = (x_Q, t)$ in a fixed time t. To extend the treatment to paths that go between two fixed space–time points, it is useful to treat t as a new dimension. To simplify the analysis, and to retain the two-dimensional picture of Sec. II, I will consider motion in one spatial dimension.

Consider the continuous path $x(t)$, which is broken into small straight segments connecting points separated by a fixed time interval Δt. The fact that the segments are straight means that the velocity is constant during each interval, and changes due to an impulsive force. This force will be nonzero if the potential changes as a function of x at the particle position.

Consider two segments as in Fig. 4(a). Before the force F acts on the particle, the velocity is given by the slope of $x(t)$:

$$v_P = \frac{x_i - x_P}{\Delta t}. \tag{7}$$

The effect of the force is to change the velocity. Therefore, in the (x, t) plot (the world line), the slope of the line changes at the intermediate time. The velocity after the force has acted on the particle is

$$v_Q = \frac{x_Q - x_i}{\Delta t}. \tag{8}$$

Because the force is down, the slope decreases: a downward force means that at x_i, the potential is increasing as a function of x.

The path in space–time (x, t) is a solution of Newton's second law such that the rate of change of the velocity times the particle mass is the force acting on it:

Table I. Analogies used in the principle of least action between mechanics, geometric optics, and the equilibrium of a nonstretchable string.

Particle	Light ray	Nonstretchable string
mv (momentum)	n (refractive index)	T (tension)
$mv_1 \sin\theta_1 = mv_2 \sin\theta_2$ (momentum conservation)	$n_1 \sin\theta_1 = n_2 \sin\theta_2$ (Snel's law)	$T_1 \sin\theta_1 = T_2 \sin\theta_2$ (static energy)
$\Delta A = mv\Delta\ell$ (action)	$c\Delta t = n\Delta\ell$ (optical length)	$\Delta U = T\Delta\ell$ (potential energy)

$$F = m\frac{v_Q - v_P}{\Delta t}. \tag{9}$$

We replace v_Q and v_P in Eq. (9) by the velocities from Equations (7) and (8):

$$F = \frac{m}{(\Delta t)^2}(x_Q - x_i) - \frac{m}{(\Delta t)^2}(x_i - x_P). \tag{10}$$

At this point we become more abstract and forget the space–time picture for a moment. Equation (10) gives the force of a system of two springs with identical spring constants, the first spring connects point $(x_i, \Delta t)$ with $(x_P, 0)$, the second connects $(x_Q, 2\Delta t)$ with $(x_i, \Delta t)$. For the system to be in equilibrium, that is, for the intermediate coordinate to have the value x_i (the other two are fixed), there has to be a force of magnitude F but of opposite sign.

The path given by Newton's law is given by the equilibrium condition for a mechanical model of two springs in the presence of a potential of opposite sign to $U(x)$. The equilibrium condition is the one that minimizes the potential energy of the entire system, springs plus "external" potential $U(x)$. Because the potential energy for a spring of spring constant k connecting two points separated by a distance δ is $k\delta^2/2$, the total potential energy of the system $(\widetilde{S})$ is given by

$$\widetilde{S} = \frac{m}{2}\left(\frac{x_i - x_P}{\Delta t}\right)^2 + \frac{m}{2}\left(\frac{x_Q - x_i}{\Delta t}\right)^2 - U(x_i). \tag{11}$$

We return to the original world line picture in which the variable t in Eq. (11) is time and we see that the optimum path in space–time is the one that minimizes the difference between the kinetic and potential energy. The fact that the quantity to be minimized is $\widetilde{S}$ is Hamilton's principle. In this section I obtained this result using a mechanical analogy similar to the principle of least action in the sense that there is a correspondence between the kinetic energy and the potential energy of fictitious springs of spring constant $k = m/(\Delta t)^2$. In other words, the stretchable string is in equilibrium due to two types of forces in space–time: the external force due to (minus) the real external potential, and the elastic force of the fictitious springs, which plays the role of the kinetic energy.

For a longer path with N straight segments each of them traversed by the particle in a time Δt, the velocity at the ith segment will be $v_i = (x_{i+1} - x_i)/\Delta t$, and the equivalent potential energy will be given by

$$\widetilde{S} = \left[\frac{mv_1^2}{2} - U(x_1)\right] + \left[\frac{mv_2^2}{2} - U(x_2)\right] + \cdots + \frac{mv_N^2}{2}. \tag{12}$$

For a large number of segments, corresponding to a continuously varying path, the last term in Eq. (12) can be ignored; the total "potential" energy (of the fictitious springs plus the real external potential) to be minimized corresponds to the sum of the differences between the (real) kinetic and (real) potential energies.

Notice that the potential energy for the fictitious springs corresponds to springs of zero length. Also, Eqs. (11) and (12) omit the potential energy associated with the "horizontal displacement" Δt of each spring. I ignore this contribution because the horizontal forces due to the springs cancel, and therefore, the potential energy associated with this displacement is the same for all configurations of the string.

IV. HAMILTON'S PRINCIPLE USING ELEMENTARY CALCULUS

I now derive Hamilton's principle using a slightly more sophisticated but still elementary approach. The principle of least action gives the path of a particle of fixed energy E in going from A to B [see Fig. 1(a)]. Call U_1 and U_2 the potential energies in the upper and lower parts of the line CD, and $v_1 = \sqrt{2m(E - U_1)}$ and $v_2 = \sqrt{2m(E - U_2)}$ the corresponding velocities. Now consider paths with different energies and ask for which of these paths will the particle satisfy Newton's laws and spend a fixed time t going from A to B. Following the principle of least action, we want to find a function that will give the desired path upon minimization. For the case under consideration, the path consists of two straight segments and the function has to be such that, of all paths that take a time t in going from A to B, it chooses the one that satisfies Eq. (5).

Call a and b the perpendicular distances of A and B to the interface CD, L the horizontal distance between A and B, and x the distance CP_1. The Maupertius action of Eq. (6) can be thought of as a function of x and the energy E:

$$A(x,E) = mv_1(E)\sqrt{x^2 + a^2} + mv_2(E)\sqrt{(L-x)^2 + a^2}. \tag{13}$$

To explore whether $A(x,E)$ is the desired function, we calculate the variations of A with respect to x and E, assuming knowledge of the ratios of dE and dx that will keep the time t constant:

$$dA = (mv_1 \sin\theta_1 - mv_2 \sin\theta_2)dx + \frac{\partial A}{\partial E}dE. \tag{14}$$

It is clear that minimizing A (or equivalently setting $dA = 0$) does not give us the desired Eq. (5) because of the second term in Eq. (14). However, notice that $dv_i/dE = 1/mv_i$, and

$$\frac{\partial A}{\partial E} = \frac{\sqrt{x^2 + a^2}}{v_1} + \frac{\sqrt{(L-x)^2 + a^2}}{v_2} = t_1 + t_2 = t, \tag{15}$$

where $t_1 = \ell_1/v_1$ and $t_2 = \ell_2/v_2$ are the times it takes the particle to go from A to P_1 and from P_1 to B, respectively.

Thus, if Et is subtracted from A, the desired quantity is obtained: $S = A - Et$. [Note that $d(Et) = tdE$ because the paths take a constant time.] Therefore,

$$\begin{aligned} S &= (mv_1\ell_1 - Et_1) + (mv_2\ell_1 - Et_2) \\ &= (mv_1^2 - E)t_1 + (mv_2^2 - E)t_2 \\ &= (K_1 - U_1)t_1 + (K_2 - U_2)t_2, \end{aligned} \tag{16}$$

which is the quantity to be minimized according to Hamilton's principle.

V. COULD HAMILTON HAVE DISCOVERED QUANTUM MECHANICS?

If we write $p = mv$ and $\ell = x$, the action S in Eq. (16) has the same form as the phase change of a wave

$$\phi \sim px - Et, \tag{17}$$

(up to a multiplicative constant that renders ϕ dimensionless) with momentum and energy playing the role of wave number k and frequency ω. The path of least action could hence be regarded as the stationary phase limit of a wave. Could Hamilton have discovered quantum mechanics in 1834? The answer is probably no, because Hamilton did not have any experimental motivation to think of particles as waves.[26] However, the close analogy between geometric optics and mechanics could have motivated him to ask the following: what would be the structure of a wave equation for particles that, in the limit of small wavelength, gives the trajectories of particles just as the wave equation for light in the same limit gives the trajectories of light rays?

Let us follow the analogy provided by the principle of least action and consider trajectories of constant energy, corresponding to light rays of constant frequency. Because the principle of least action establishes an equivalence between the geometry of these trajectories, I seek an equivalence between stationary states of the corresponding wave equations. The wavelength of a monochromatic light wave in a region in which the index of refraction $n(x)$ is varying slowly is given by

$$\lambda(x) = \frac{\lambda_0}{n(x)}. \tag{18}$$

Because Eqs. (4) and (6) imply that the trajectories of particles and light rays are equivalent if $n(x)$ is identified with $mv(x)$, a natural choice for the spatial dependence of the particle wavelength λ_P is

$$\lambda_P(x) = \frac{K}{mv(x)} = \frac{K}{\sqrt{2m[E - U(x)]}}, \tag{19}$$

with K a constant with units of angular momentum; K is the constant needed to make ϕ in Eq. (17) dimensionless.

Now consider the wave equation for the amplitude $\phi(x,t)$ describing a light wave in one dimension (we ignore the polarization):[27]

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{n^2(x)}{c^2}\frac{\partial^2 \phi}{\partial t^2}. \tag{20}$$

To compare Eq. (20) with the stationary wave equation for particles, I substitute $\phi(x,t) = \phi(x)e^{i\omega t}$ so that Eq. (20) becomes

$$-\left(\frac{\lambda_0}{2\pi}\right)^2 \frac{\partial^2 \phi}{\partial x^2} = n^2(x)\phi. \tag{21}$$

From the equivalence of Eqs. (18) and (19), the structure of the wave equation for the stationary states Ψ for particles is

$$-\frac{(K/2\pi)^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = [E - U(x)]\Psi. \tag{22}$$

If we compare Eq. (22) with Eq. (21) and treat K as a free parameter, the limit $K \to 0$ (which corresponds to the limit $\lambda_0 \to 0$) gives the trajectories for particles of energy E, and Eq. (22) could be used as a wave equation for particles. Of course, now we can identify K with h, Planck's constant. Given the identification of the energy with the frequency, the time dependence of the stationary states is $\Psi(x)e^{-iEt/\hbar}$ and the implied time dependence for Eq. (22) is Schrödinger's equation:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right]\Psi = i\hbar\frac{\partial}{\partial t}\Psi. \tag{23}$$

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[1]E. F. Taylor and J. A. Wheeler, *Exploring Black Holes* (Addison–Wesley, San Francisco, 2000), p. 5.

[2]Strictly speaking, extremal (maxima or minima) are a subset of stationary paths (or states). However, the term principle of least action is more frequently used than the more precise principle of stationary action.

[3]P. L. Moreau de Maupertuis, "Accord de différentes loix de la nature, qui avaient jusqu'ici paru incompatible," *Memoires de l'Académie Royale de Sciences* (Paris, 1744), pp. 417–426, reprinted in Oeuvres, **4**, 1–23 Reprografischer Nachdruck der Ausg. Lyon (1768).

[4]P. L. Moreau de Maupertuis, "Recherche des lois du Mouvement," Oeuvres **4**, 36–38 (1768).

[5]The English spelling (Snell) found in most textbooks of the Dutch astronomer and mathematician Willebrord Snel (1580–1626) derives from its Latinized version Willebrodus Snellius. See for example K. Hentschel, "The law of refraction according to Snellius–Reconstruction of his path of discovery and a translation of his Latin manuscript along with additional documents," Arch. Hist. Exact Sci. **55**, 297–344 (2001).

[6]Several years before, Clairaut in "Sur les explications Cartésienne et Newtonienne de la réfraction de la lumière," *Memoires de l'Académie Royale de Sciences* (Paris, 1739), pp. 259–275 showed that Newtonian attraction could be applied to refraction. The controversy with Samuel Koenig, who accused Maupertuis of plagiarizing Gottfried Wilhelm Leibniz's work, is detailed in Ref. 7.

[7]P. Brunet, *Etude Historique sur le Principe de La Moindre Action* (Herman et Cie, Paris, 1938), pp. 49–60.

[8]See, for example, M. Terrall, *The Man Who Flattened the Earth: Maupertuis and the Sciences of Enlightenment* (University of Chicago Press, Chicago, 2002), pp. 178–179.

[9]W. R. Hamilton, "On a general method of expressing the paths of light, and of the planets, by the coefficients of a characteristic function," Dublin University Review and Quarterly Magazine **1**, 795–826 (1833).

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[14]J. Bernoulli, "Disquisitio Catoptico-Dioptrica," *Opera Omnia* (Geneve, sumptibus Marci-Michaelis Bousquet & sociorum, 1742), Vol. 1, pp. 369–376.

[15]A. F. Möbius, *Lehrbuch der Statik*, Sweiter Theil (Liepzig, 1837), pp. 217–313.

[16]J. Gray, "Möbius's geometrical mechanics," included in *Möbius and his Band*, edited by J. Fauvel, R. Flood, and R. Wilson (Oxford U.P., Oxford, 1993), pp. 79–103.

[17]E. Mach, *The Science of Mechanics: Account of its Development*, translated by Thomas J. McCormack (Open Court, IL, 1960), 6th ed., pp.

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[25]D. S. Lemons, *Perfect Form* (Princeton U.P., Princeton, 1997).
[26]The title of this section was taken from Ref. 10, p. 209.
[27]It could be objected that Hamilton did not know the wave equation for light, but the argument applies to any scalar wave in a nonhomogeneous medium. For example, Eq. (20) could represent a sound wave in a medium with varying density.